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# UNCERTAINTY QUANTIFICATION IN BEAM BENDING THEORIES BY STOCHASTIC SPECTRAL FINITE ELEMENT METHOD

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**ABSTRACT:** In stochastic structural mechanics, it is possible to associate uncertainties with material, geometric properties and external loading, whiles estimates response may be present in fields of displacement, stress, strain, frequency, and phase differences, among others. At work the stochastic formulation of elliptic differential equations associated with the beam bending problem is present. The propagation of uncertainty is explored from the perspective of the numerical methodology of asymptotic complexity Neumann-Monte Carlo (NMC). The variational solution is studied for the classical theory of Euler-Bernoulli and the higher order theories of Timoshenko and Levinson-Bickford using from the stochastic version of Galerkin's method. Numerical results present the expected value and variance of the stochastic displacement field for the three beam theories. Comments are present for the cases where the initial variability is associated with the modulus of elasticity and height beams, not simultaneously and simultaneously.

**KEYWORDS:** Stochastic Structural Mechanics, Bending Beam Theory, Uncertainty Quantification, Neumann-Monte Carlo methodology.

## 1. INTRODUCTION

One of the main concerns in dealing with the stochastic problem is to portray the irregular disparity of the mechanical properties so that an adequate formulation can be established. The model must represent the uncertainties of the random variables in a way that the response contain relevant information about the stochastic process.

The numerical results depend on the adopted model and, usually contain two terms: the expected value and the variability of the responses. The mathematical model identifies a set of relationships generally based on principles, conservation laws, and physical magnitude metrics related through differential equations.

In this regard, the Galerkin method is used to obtain approximate numerical solutions based on the lemma of Doob-Dynkin, Rao and Swift, 2006 ensure that to obtain the space of the approximate numerical solutions for all the realizations of the stochastic process displacement truncating a total system in the approximation space. In this method, the coefficients of the equation system are to be determined by minimizing the projection residue generated by the approximation function of the problem. Therefore by replacing the numerical approximation of the sampling function of the stochastic displacement process, a linear system of algebraic equations is generated.

The application of the Neumann series in stochastic problems arises from the works of Shinozuka, 1987 and Yamazaki et al. 1988. In these works, numerical results are obtained to estimate the uncertainty in the stress and strain fields for the beam bending problem. The Neumann-Monte Carlo methodology combines the Monte Carlo simulation methods and the Neumann series, with the objective of obtaining the results of the realizations of the stochastic process. The series acts to obtain the approximation of the inverse of the matrix of random coefficients originated by the system of equations of the variational formulation.

The theories of bending stationary or vibrational beams are of fundamental importance in the solution of other structural elements, such as plates, frames, and membranes. The classical deterministic theory of Euler-Bernoulli beam bending, in terms of the elasticity equations, can be found in Reddy, 1984 taking into account the partial differential equation of motion, the external loading, and the boundary conditions.

In Timoshenko, 1921, and Timoshenko, 1922 the author adds the effect of shear and develops the equation of motion from principles of equilibrium. An asymptotic expansion procedure is presented in Goodie e Timoshenko, 1951, which shows second-order effects on beam curvature. However, the formulation requires the arbitrary insertion of the shear coefficient and the boundary conditions of the numerical solution require the estimation of the curvature of the displacement field.

The theory presented by Levinson e Stephens, 1979 includes shear deformation of the cross-section using a high-order approximation function that satisfies the condition of free shear on the side surfaces. Bickford, 1981 presents a variational formulation of the theory and develops the expressions of the primary and secondary variables associated with the boundary conditions. Heyliger and Reddy, 1988 derive equations of motion for a beam from Levinson's kinematic assumptions and Hamilton's principle. Karttunen and Hertezen, 2015 employ the principle of virtual displacements with the addition of external virtual work on the lateral surfaces of the beam and numerical examples are presented using the Finite Element method. Stationary and dynamic solutions to the high-order beam bending problem can be found in Reddy, 2010. Squarcio and Ávila, 2022 apply the Neumann-Monte Carlo methodology to the stationary beam bending problem based on the high-order theory of Levinson-Bickford.

The layout of a reminder of this paper is as follows. The uncertainty propagation analysis is presented for the Neumann-Monte Carlo (NMC) methodology. The following section is a technical and pragmatic review of the stochastic problem of elastic and stationary beam bending with numerical solutions obtained from three theories: Euler-Bernoulli, Timoshenko e Levinson-Bickford. The numerical simulations discuss the results for the statistical moments of the displacement field when the uncertainty is associated with stiffness matrix. The results obtained by SSFEM, for each of the beam theories, are compared.

## 2. STOCHASTIC FORMULATION OF ELLIPTIC DIFFERENTIAL EQUATIONS

This section is presented the problem of random bending beam through random stiffness coefficient or loading. The problem is formulated from the point of view of Monte Carlo simulation-based methods on the Hilbert space with functions in the domain $D \subset \mathbb{R}^n$ and random variables defined in $(\Omega, \mathscr{F}, \mathscr{P})$ such that, $\omega \in \Omega$, were $\Omega$ the sample space, $\mathscr{F}$ σ-álgebra, and $\mathscr{P}: \mathscr{F} \to \mathbb{R}^+$ a Probability with values in [0,1] such that $\mathscr{P}(A) \geq 0$ and $\mathscr{P}(\Omega) = 1, \forall A \in \mathscr{F}$.

Considering the problem of sample stochastic bending is presented, for the k-th structural samples, a finite set of random variables $\xi_k(\omega)$, respectively, for the stiffness and loading coefficient. The uncertainty model is mapped by spatial functions $\Omega \to \mathbb{R}$ denoted by $\Xi_k = \Xi(\xi_k(\omega))$ (Arnold, 1973). The family $\Xi_k = \{\xi(\omega_k) \in I\} \subset \mathbb{R}^n$ is the set of random vectors, where $I \in (\Omega, \mathscr{F}, \mathscr{P})$, a non-empty set of indices and $\Xi_k$ the parameter and state spaces, respectively.

In this space, the estimators, in general, are represented by Lebesgue integrals, and the inner product is defined in $H \times \Omega$, that this, $\left(h_i(x), h_j(x)\right) = \int_D h_i(x) \, . \, h_j(x) dx$, where $h$ is the orthogonal approximation functions, if and only if, $\left(h_i(x), h_j(x)\right) = 0$.

From the point of view of stochastic mechanics, most systems involve a differential equation with random coefficients. These coefficients represent the properties of the problem under investigation. Thus, the stochastic differential equations are written by,

$$\Lambda u = f, \tag{1}$$

where $\Lambda$ is stochastic differential operator, $u$ is a random response and $f$ the random loading.

Whereas $\Lambda(x, \Xi_k)$ is positive definite and uniformly bounded in probability, $f(x, \Xi_k)$ bounded and with finite variance then, for the set of n structural samples $\{\Lambda(.,\Xi_k)\}_{k=1}^{n}$ and $\{f(.,\Xi_k)\}_{k=1}^{n}$ the initial hypotheses are:

$$(\mathbf{H}1): \exists\ \underline{\Lambda}, \overline{\Lambda} \in \mathbb{R}^+ : P\left(\Xi_k \in \Omega / \Lambda(x,\Xi_k) \in \left[\underline{\Lambda}, \overline{\Lambda}\right], \forall x \in (0,l)\right) = 1,$$

$$(\mathbf{H}2): \Lambda(\,.\,,\Xi_k) \in H^2(0,l),\ \forall\, k \in \{1, \dots, n_s\},$$

$$(\mathbf{H}3): f(\,.\,,\Xi_k) \in L^2(0,l),\ \forall\, k \in \{1, \dots, n_s\}.$$

Another way of writing the stochastic differential equation, Eq. (1), considering the randomness in the differential operator is given by,

$$(L(x) + \Pi(x,\omega))u(x,\omega) = f(x,\omega), \tag{2}$$

where $L(x)$ is the deterministic differential operator $\Pi(x,\omega)$ is the differential operator with coefficients obtained for the zero mean random process.

The solution of Eq. (2) is proposed considering the result with deterministic and random behavior (Ghanem and Spanos, 1991), such that,

$$u_k(x,\omega) = \bar{u}_k(x) + \alpha_k(x,\omega), \tag{3}$$

where $\bar{u}_k(x)$ is the expected value of the linear differential operator and $\alpha_k(x,\omega)$ is the random process with zero mean. It is observed that $\bar{u}_k(x)$ and $\alpha_k(x,\omega)$ in the same covariance function.

## 2.1. Spectral Stochastic Finite Element Method

In the deterministic numerical method, a finite element of coordinates $x_i$ e $x_j$, the local mean or mean value $x_k$ is used, with interpolation functions $P_1$ and $P_2$, whose weighted average is given by $P_1 u(x_i) + P_2 u(x_j)$. The stochastic process $u(x,\omega)$ is defined by the arguments of a set of orthogonal functions $g(x)$ from a linear or non-linear functional, in the probability space $\mathscr{P}$, that is,

$$u(x,\omega) = \int g(x)\, d\mathscr{P}(\omega), \tag{4}$$

whose covariance function is $C(x_1, x_2) = \int g(x_1)g(x_2)\langle d\mathscr{P}_1(\omega)d\mathscr{P}_2(\omega)\rangle$.

To obtain the inner product of the orthogonal functions in discrete form, the random vector sets $\xi_i(\omega_k)$ is rewritten such that,

$$\left(\xi_i(\omega_k), \xi_j(\omega_k)\right) = \int_{H\times\Omega} \xi_i(\omega_k) \cdot \xi_j(\omega_k) d\mathscr{P} = \Xi_{ij}(\omega_k). \quad (5)$$

Furthermore, for numerical solution, the Fredholm equation is written as

$$\int_D C(x_1, x_2)\psi(x_2)dx_2 = \lambda\psi(x_1). \quad (6),$$

where $\psi_i(x)$ is a complete set of functions defined in Hilbert space H and $\lambda$ are eigenvalues.

The Kernel eigenfunctions $C(x_1, x_2)$ are obtained from,

$$\psi_k(x) = \sum_{i=0}^{N} d_i^{(k)} h_i(x). \quad (7),$$

where $d_i^{(k)}$ is a set of coefficients associated with the functions $\psi_i(x)$ for the *kth* realization of the stochastic process.

Substituting the approximate solution into the differential equation, we have the error, or residuals given by,

$$\epsilon_N = \sum_{i=0}^{N} d_i^{(k)} \int_D C(x_1, x_2)\psi(x_2)dx_2 - \lambda\psi(x_1). \quad (8)$$

In Galerkin's method, the weight functions are selected and it is imposed that the weighted average of the residue in relation to the weight functions is equal to zero. In mathematical terms, the error is made orthogonal to the weight functions, such that, $\left(\epsilon_N, h_j(x)\right) = 0, j = 1, \ldots, N$.

The solution of the linear system, is obtained solving for **D** and $\lambda_k$ and, replacing in Eq. (10) get the eigenfunctions, $\psi_k(x)$:

**CD = ABD** (9),

where $C_{ij} = \int_D \int_D C(x_1, x_2)h_i(x_2)dx_2 h_j(x_1)dx_1 dx_2$, $B_{ij} = \int_D h_i(x)h_j(x)dx$, $D_{ij} = d_i^{(j)}$ and $A_{ij} = \delta_{ij}\lambda_i$.

The columns of the matrix become the eigenvectors computed at the respective nodal point of the induced mesh, and the $ij^{th}$ element of the matrix **C** becomes the weighted correlation between the process at nodes *i* and *j*. **C** and **B** are symmetric, positive definite.

## 2.2. System of Linear Algebraic Equations

However, it should be noted that the mean and the covariance are inadequate to fully define a second-order general stochastic field. For the kth realization of the stochastic process $u = u(x, \omega_k)$ and defined by the random variables, $\Xi_k(\omega) = \left\{ \xi_i(\omega_k) \right\}_{i=1}^{n}$, the random response is given by,

$$u(x, \Xi_k(\omega)) = u\left(x, \xi_1(\omega_k), \ \ldots, \xi_n(\omega_k) \right). \quad (10)$$

Then the numerical approximation of the realization of the stochastic process has the following form:

$$u_m(x, \Xi_k(\omega)) = u_i(\Xi_k(\omega)) \,.\, \psi_i(x) = U(\Xi_k(\omega)) \,.\, \Phi(x) \quad (11),$$

where $u_i{}'s$ are the coefficients to be determined by minimizing the residual and $\Phi(x)$ are functions sets $\psi_i(x)$.

Substituting the numerical approximation of the sampling function in the Abstract Variational Problem, obtained from the stochastic version of the Lax-Milgram Lemma (Babuska et al., 2005), a system of linear algebraic equations is generated, that is,

$$(\mathbf{KU})(\Xi_k(\omega)) = \mathbf{F} \ \Rightarrow \ \mathbf{U}(\Xi_k(\omega)) = \mathbf{H}(\Xi_k(\omega)) \,.\, \mathbf{F} \quad (12),$$

where $\mathbf{K}(\Xi_k(\omega))$, is the matrix of random coefficients, $\mathbf{F} = [f_1, \ \ldots, f_m]^t$, is the force vector, $\mathbf{H}(\Xi_k(\omega)) = \left[h_{ij}(\Xi_k(\omega))\right]_{mxm} = \left(\mathbf{K}(\Xi_k(\omega))\right)^{-1}$, and displacement vector, $\mathbf{U}(\Xi_k(\omega)) = \left[u_1(\Xi_k(\omega)), \ \ldots, \ u_m(\Xi_k(\omega))\right]^t$.

The k-th response vector is given by,

$$u_i(\Xi_k(\omega)) \ = \ \sum_{j=1}^{m} h_{ij}(\Xi_k(\omega)) f_i \quad (13).$$

Of Eqs (11) and (12) the numerical approximation to the sampling function of the stochastic process of the response is given by,

$$u_m(x,\Xi_k(\omega)) = \sum_{i=1}^{m}\sum_{j=1}^{m}\left(h_{ij}(\Xi_k(\omega))f_j\right)\psi_i = \mathbf{F}\,.\left(\mathbf{H}(\Xi_k(\omega))\right)\Phi(x) \quad (14).$$

For the k-th realization, it is necessary to obtain the solution of the linear system expressed in Eq. (14). Given the conditions of the hypotheses (**H1-H3**), an alternative to reduce the computational effort is the use of the Neumann series. To obtain the variance, the Neumann series can be applied intrusively or non-intrusively, with considerable gain in computational processing time.

## 2.3. The Neumann Series

The Neumann series is a convergent series as long as the adopted variation around the expected value is small. (spectral radius less than 1). In this case, the matrix of random coefficients admits the following decomposition:

$$\mathbf{K}(\Xi_k(\omega)) = \mathbf{K}_0(\Xi_k(\omega)) + \Delta\mathbf{K}(\Xi_k(\omega)), \quad (15)$$

where $\mathbf{K}_0(\Xi_k(\omega))$ is the matrix composed by the expected value of the coefficients and $\Delta\mathbf{K}(\Xi_k(\omega))$ is the uncertainty represented by its statistical moments.

To approximate the inverse of the matrix of random coefficients it is written in the form:

$$\mathbf{K}(\Xi_k(\omega)) = \mathbf{K}_0\left[\mathbf{I} - \mathbf{P}(\Xi_k(\omega))\right], \quad (16)$$

where $\mathbf{I} \in \mathbb{M}_n(\mathbb{R})$ is the identity matrix and $\mathbf{P}(\Xi_k(\omega))$ is the argument of the series.

The matrix $\mathbf{P}(\Xi_k(\omega))$ is a linear operator, continuous with normed space, such that $\mathbf{P}^0 = \mathbf{I}$. Substituting Eq. (16) in Eq. (12), the approximate random response of the inverse of the stochastic coefficient matrix is such that,

$$\mathbf{U}(x,\Xi_k(\omega)) = \left(\mathbf{I} - \mathbf{P}(\Xi_k(\omega))\right)^{-1}\mathbf{U}_0, \quad (17)$$

where $\mathbf{U}_0 = (\mathbf{K}_0)^{-1}\mathbf{F}$.

In particular, if $\|\mathbf{P}\| < 1$ the matrix $(\mathbf{I} - \mathbf{P})^{-1} \in \mathbb{M}_n(\mathbb{R})$ is approximated by, $(\mathbf{I} - \mathbf{P})^{-1}_{(q)} = \sum_{n=0}^{q}\mathbf{P}^n$, where $q$ is the order adopted for expanding the series. Substituting this property in Eq. (17) the system of algebraic equations is rewritten as:

$$\mathbf{U}\left(x,\Xi_k(\omega)\right)=\sum_{n=0}^{q}\mathbf{P}^n\left(\Xi_k(\omega)\right)\mathbf{U}_0, \tag{18}$$

The first-order or linear approximation for the Neumann expansion has reached an appropriate accuracy. The application of SSFEM and NMC to obtain responses in the form of statistical moments is widely used in elliptic differential equations that represent the beam bending problem.

## 3. STOCHASTIC FORMULATION FOR BEAM BENDING THEORIES

This section presents a variational problem for a stationary beam based on the theories of Euler-Bernoulli, Timoshenko and Levinson-Bickford. It is a synthesis to obtain the bilinear form and consequently the random response.

### 3.1. Euler-Bernoulli Stochastic Beam Bending

The classical Euler-Bernoulli theory is applied to prismatic beams with longitudinal length as the predominant dimension. The beam is made of material with homogeneous density, isotropic, linearly elastic, obeying Hooke's law. The material properties are represented by the estimator of the expected value of Young's modulus, $\hat{\mu}_{E(x,\omega_k)}$. The beam is assumed to be symmetric about the vertical axis and remains so after bending. This kinematic hypothesis consists of assuming that the sections remain flat, undisturbed and orthogonal to the longitudinal axis of the beam. The beam geometry is represented by the estimator of the expected value of the moment of inertia, $\hat{\mu}_{E(x,\omega_k)}$. The shear stresses are very small in relation to the normal stresses and can be ignored, and the effects of the mass moment of inertia are neglected for small vertical displacements $w(x,\omega_k)$ and angular displacements $\phi(x,\omega_k)$ remains constant.

The stationary beam differential equation is obtained from equilibrium conditions, with homogeneous boundary conditions and complying with the Lax-Milgram lemma. Observing hypotheses **(H1)-(H3)**, the problem can be expressed as follows:

$$\begin{cases} \text{Find } w(x,\omega_k),\phi(x,\omega_k)\in(0,1)\times(\Omega,\mathscr{F},\mathscr{P}), \text{ such that,} \\ \frac{d^2}{dx^2}\left(K\frac{d^2w}{dx^2}\right)(x,\omega)=q(x,\omega), \\ w(0,\omega)=w(l,\omega)=0, \phi(0,\omega)=\phi(l,\omega)=0, \end{cases} \tag{19}$$

where $K = EI$ is the bending stiffness.

For the solution using the deterministic Finite Element Method, a beam element consisting of two nodes is considered. The degrees of freedom are associated with the transverse displacement $w(x,\xi_k)$ and angular displacement $\phi(x,\xi_k)$. The goal becomes to interpolate the deflection at any point on the element in terms of the local degrees of freedom, $w_1^e, \phi_1^e, w_2^e$ e $\phi_2^e$. As the beam element has four nodal values the approximate functions are a generalization of Hermite cubics. Thus the stiffness matrix of the beam element is given by,

$$\left[K(x,\xi_k)\right]^e = \frac{EI}{L^3}(x,\xi_k)\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}. \quad (20)$$

And the loading vector of the element,

$$\left\{q(x,\xi_k)\right\}^e = q(x,\xi_k)\left\{\frac{L}{2} \quad \frac{q.L}{12} \quad \frac{L}{2} \quad -\frac{q.L}{12}\right\}^t \quad (21).$$

The assembly of a global stiffness matrix and a global displacement vector must combine information from all elements. From the values generated by the approximation polynomial at the nodes of each element, interpolation functions are used to obtain the approximate solution.

## 3.2. Timoshenko Stochastic Beam Bending

The assumption that the cross section remains flat and normal to the neutral axis means neglecting the shear strain. Timoshenko's theory of beam bending considers that, although the cross section remains flat, additional rotation also occurs due to shear and the section does not remain normal to the neutral axis. As the relationship between beam length and height increases, shear stresses in the height direction become important and can no longer be neglected.

The strong formulation of the stochastic problem for Timoshenko's stationary beam bending problem obtained from equilibrium principles is expressed by:

$$
\begin{cases}
\text{Find } w(x,\omega_k) \text{ and } \phi(x,\omega_k) \in L^2(\Omega, F, P; H^2), \text{ such that,} \\
\frac{d}{dx}\left[k_s AG\left(\frac{\partial w}{\partial x} - \phi\right)(x,\omega_k)\right] + q(x,\omega_k) = 0, \\
\frac{d}{dx}\left[\left(EI\frac{\partial \phi}{\partial x}\right)(x,\omega_k)\right] + k_s AG\left(\frac{\partial w}{\partial x} - \phi\right)(x,\omega_k) = 0, \\
w(0,\omega) = w(l,\omega) = 0, \phi(0,\omega) = \phi(l,\omega) = 0, \forall x \in (0,l) \; e \; \omega \in (\Omega, F, P),
\end{cases}
\tag{22}
$$

where $G(x,\omega_k)$ is the transverse modulus of elasticity, $A(x,\omega_k)$ is the cross-sectional area and $k_s$ is the shear factor.

Draw $k_f = EI$ and $k_c = k_s AG$ the bilinear forms are obtained, in a system of equations. The model adopts an element with two nodes and four degrees of freedom. Integration is performed using Gaussian quadrature approximation polynomials. The Jacobian is associated with the Lagrange interpolation functions. With this expedient the stiffness matrix of the element, given by:

$$
K = \begin{bmatrix}
\frac{EI}{l} + k_c Ghl & -\frac{EI}{l} + k_c Ghl & -\frac{k_c Gh}{2} & -\frac{k_c Gh}{2} \\
-\frac{EI}{l} + k_c Ghl & \frac{EI}{l} + k_c Ghl & \frac{k_c Gh}{2} & \frac{k_c Gh}{2} \\
-\frac{kGhl}{2} & \frac{k_c Ghl}{2} & \frac{k_c Gh}{4} & \frac{k_c Ghl}{4} \\
-\frac{k_c Ghl}{2} & \frac{k_c Ghl}{2} & \frac{k_c Ghl}{4} & \frac{k_c Ghl}{4}
\end{bmatrix}.
\tag{23}
$$

And the force vector becomes:

$$
q^e = \{q_1^e \quad q_2^e \quad q_3^e \quad q_4^e\}^t,
\tag{24}
$$

where,

$$
q_1^e = -K_c\left(\frac{\partial w}{\partial x} - \phi\right)(x)\Big|_{x=0}, \qquad q_1^e = -K_f\frac{\partial \phi}{\partial x}(x)\Big|_{x=0}
$$

$$
q_3^e = -K_c\left(\frac{\partial w}{\partial x} - \phi\right)(x)\Big|_{x=h}, \; q_4^e = -K_f\frac{\partial \phi}{\partial x}(x)\Big|_{x=h}.
$$

For different linear interpolation functions of vertical and angular displacement shear numerical locking can occur. An exposition of these elements is presented in Tiwari and Hyer, 2002. To avoid this occurrence, an alternative is to use higher order beam theories, e.g., the Levinson-Bickford.

## 3.3. Levinson-Bickford Stochastic Beam Bending

The Levinson-Bickford beam theory stands out for the relationship between the cross-sectional deformation and the mass moment of inertia. The high-order stationary beam bending problem, for the *k-th* realization of the random process, is expressed by the following differential equation:

$$\begin{cases} \text{Find } w(x,\omega_k) \text{ and } \phi(x,\omega_k) \in L^2\Big((\Omega,F,P)\times H^m(0,l)\Big), \text{ such that,} \\ \frac{\partial}{\partial x}\left[\alpha\left(\phi+\frac{\partial w}{dx}\right)\right](x,\omega_k) - \frac{\partial^2}{\partial x^2}\left[\beta\left(5\frac{\partial^2 w}{\partial x^2}-16\frac{\partial\phi}{\partial x}\right)\right](x,\omega_k) = q(x,\omega_k), \\ -\alpha\left(\frac{\partial w}{\partial x}+\phi\right)(x,\omega_k) - \frac{d}{dx}\left[\beta\left(16\frac{\partial^2 w}{\partial x^2}-68\frac{\partial\phi}{\partial x}\right)(x,\omega_k)\right] = 0, \\ w(0,\omega) = w(l,\omega) = 0, \phi(0,\omega) = \phi(l,\omega) = 0, \forall x \in (0,l) \ e \ \omega \in (\Omega,F,P), \end{cases} \tag{25}$$

where $\alpha = \kappa_c\big(\xi(\omega_k)\big) = \frac{8}{15}GA\big(\xi(\omega_k)\big)$ and $\beta = \kappa_f\big(\xi(\omega_k)\big) = \frac{1}{105}EI\big(\xi(\omega_k)\big)$.

It is verified in Karttunen and Hertezen, 2015 that the stiffness matrix of the element is expressed by,

$$\left[K(x,\omega_k)\right]^e = \frac{EI(x,\omega_k)}{(1+\Phi)L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & (4+\Phi)L^2 & -6L & (2-\Phi)L^2 \\ -12 & -6L & 12 & -6L \\ 6L & (2-\Phi)L^2 & -6L & (4+\Phi)L^2 \end{bmatrix}. \tag{26}$$

And loading by,

$$[q]^e = \frac{f}{2}\left\{ l \quad -\frac{l^2}{30}(\Phi-5) \quad l \quad \frac{l^2}{30}(\Phi-5) \right\}, \tag{27}$$

where, $\Phi = \dfrac{3h^2(1+\upsilon)}{l^2}$, the functional associated with the position and the random variable.

The similarity between the stiffness matrix and the loading vector for the Levinson-Bickford beam element and the Euler-Bernoulli beam element is observed when $\Phi = 0$. The similarity between the stiffness matrix and the loading vector for the Levinson-Bickford beam element and the Timoshenko beam element is observed when $k_s = \frac{2}{3}$.

## 4. NUMERICAL RESULTS

In this section the stochastic beam bending problem is presented for the fixed boundary condition as shown in Figure 1 and statistical results are obtained for the transverse and angular displacement field. Are performed $n_s$ = 30.000 samples of the stochastic response process, $\left\{ u\left(x, \Xi(\xi_k)\right) \right\}_{k=1}^{n_s}, \forall x \in D$. The problem is discussed for a beam of rectangular cross section defined in the domain $D = \{x \in \mathbb{R} \mid 0 < x < L\}$, width b = 5 cm and uniformly distributed loading $q(x)$ = 1 KN/m. Initially, the relationship between the length L and the height of the beam h is discussed and, for subsequent analyzes, the height assumes the expected value $\mu_h$ = 15.24 cm. The beam is made of isotropic material, linearly elastic, obeying Hooke's law with an expected value of the Young modulus $\mu_E$ = 205x10$^9$ PA. The Neumann series is used with first order approximation and the domain is discretized with 100 finite elements of equal length.

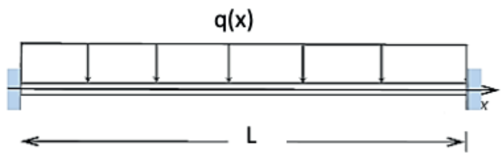


Figure 1. Fixed beam

Expected value estimators $\left(\hat{\mu}_u\right)$ and variance $\left(\hat{\sigma}_u^2\right)$ of the stochastic displacement process are given, respectively, by the following expressions:

$$\hat{\mu}_u(x) = \frac{1}{n_s} \sum_{k=1}^{n_s} u\left(x, \Xi(\xi_k)\right), \quad \forall x \in \bar{D}, \tag{28}$$

$$\hat{\sigma}_u^2(x) = \left(\frac{1}{n_s - 1}\right) \sum_{k=1}^{n_s} \left(u\left(x, \Xi(\xi_k)\right) - \mu_u(x)\right)^2, \quad \forall x \in \bar{D}. \tag{29}$$

Since Monte Carlo simulation is involved, the convergence of the solution is studied to verify that the number of simulations is appropriate for the reference accuracy. The behavior of the estimator of the expected value and the variance of the vertical displacement field, in relation to the number of realizations of the stochastic process $n_s$ is obtained by fixing $x = \frac{1}{2}l$, and coefficient of variation $\delta_E$ = 1/10. Convergence is analyzed for the uncertainty associated with the stiffness matrix.

Figure 2 and Figure 3 show the convergence of expected value and variance, respectively, with the number of realizations of the stochastic process, for the beam bending problem based on the Euler-Bernoulli theory.

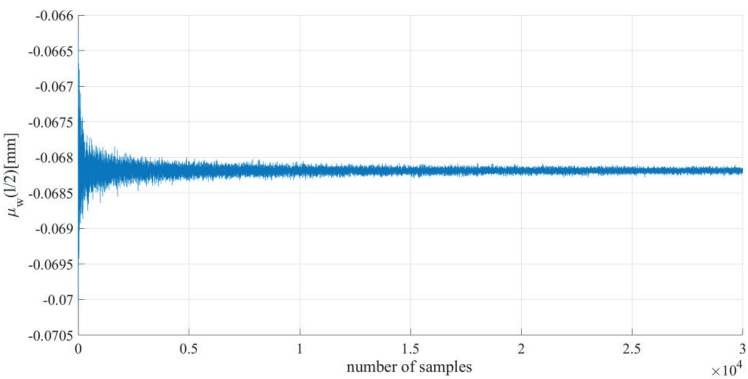


Figure 2. Convergence of the expected value of the vertical displacement field.

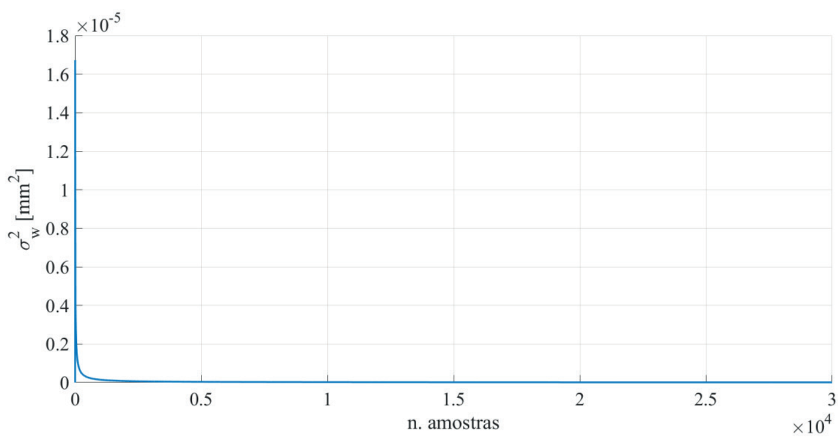


Figure 3. Convergence of displacement field variance.

In both figures, verify that for $n_s \geq 1.000$, the estimators stabilize for an adequate resolution adopted in the work.

A preliminary result for the section evaluates the expected value of the vertical displacement as a function of the geometric properties of the beam and verifying the differences between the beam theories. The boundary condition considered for a beam fixed at both ends. Thus, Figure 4, Figure 5 and Figure 6 show the graph of the estimator of the expected value of the vertical displacement for the different beam theories, and considering the beam length varying between L = 0.25, 0.5 and 1 m, respectively with coefficient of variation $\delta_E = 1/10$, using NMC methodology.

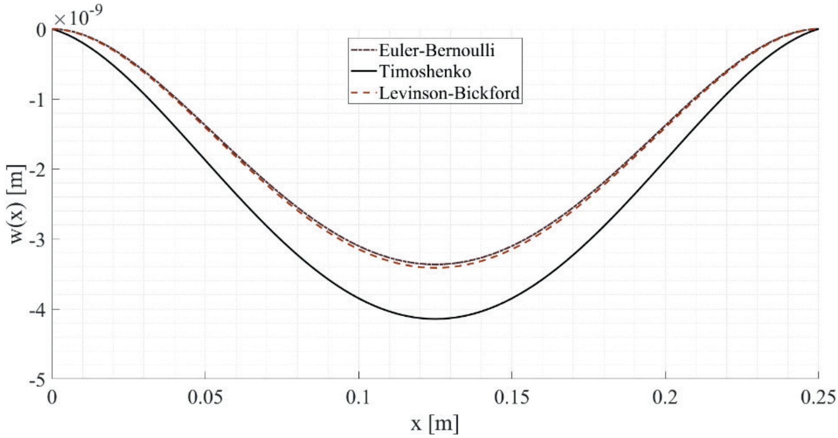


Figure 4. Expected value of vertical displacement for L = 0.25 m.

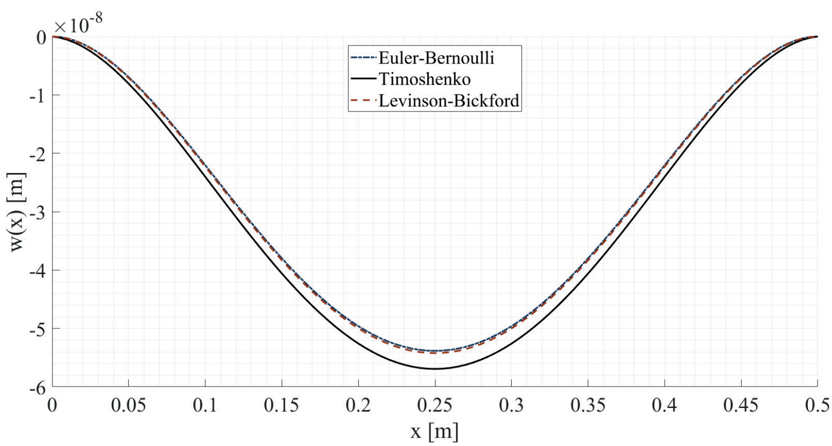


Figure 5. Expected value of angular displacement for L = 0.5 m.

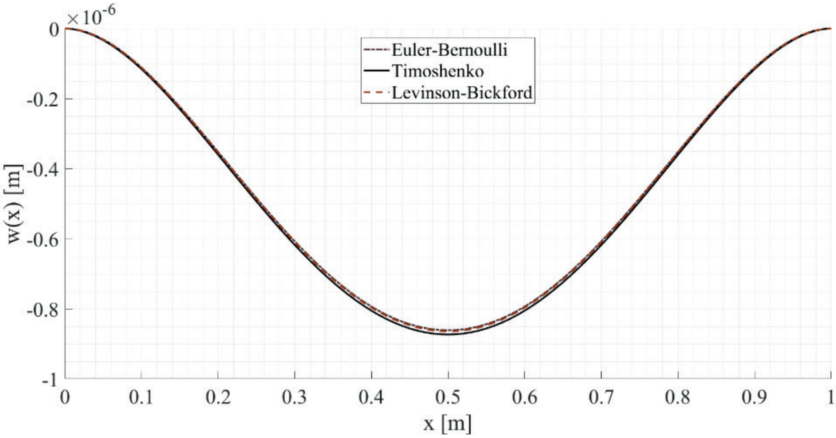


Figure 6. Expected value of angular displacement for L = 1m.

It is verified that, with the increase of the beam length (L) the difference between the beam theories becomes more accentuated. In this case, the Euler-Bernoulli and Levinson-Bickford theories remain close to each other, with a distancing of the vertical displacement for Timoshenko's theory.

Figure 7 shows the graph of the variance estimator of the vertical displacement, with δ = 1/10, for the three beam theories using the length L = 1m. This length was chosen because it presents the smallest difference between the expected value of the beam theories, allowing evaluate the effect of the deviations of each of the stochastic processes.

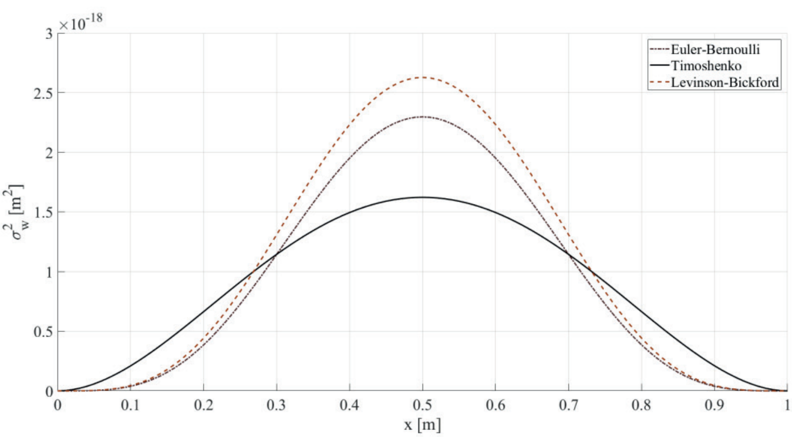


Figure 7. Vertical displacement variance.

Figure 8 shows the graph of the angular displacement variance estimator, with δ = 1/10, for the three beam theories using the length L = 1m. It is observed that, in relation to the variance estimator, the results obtained for the Euler-Bernoulli and Levinson-Bickford theories differ considerably from those obtained for the Timoshenko theory.

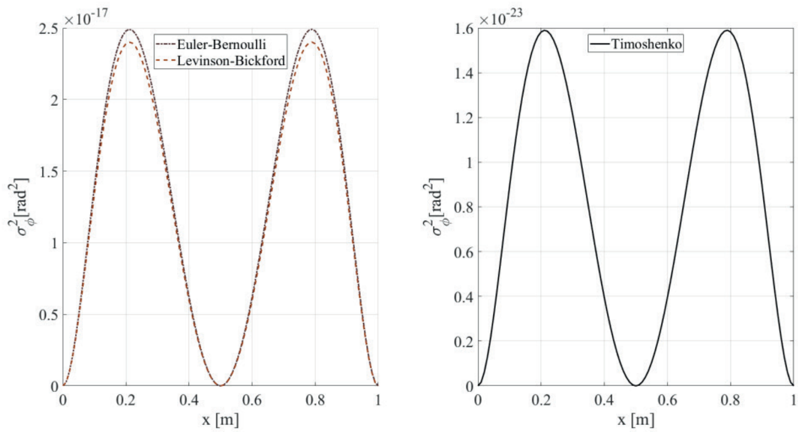


Figure 8. Angular displacement variance.

Table 1 presents the processing times for each beam theory and different uncertainty quantification methodologies.

Table 1. Processing time (seg)

| Methodology | Euler-Bernoulli | Levinson-Bickford | Timoshenko |
|---|---|---|---|
| SMC | 1,472 | 1,783 | 12,792 |
| NMC | 0,040 | 0,055 | 1,472 |
| SSFEM | 0,448 | 0,960 | 6,803 |

The computational times are close for the Euler-Bernoulli and Levinson-Bickford theories, but significantly higher for the Timoshenko theory. Similarly, the time is considerably longer when using the pure Monte Carlo simulation in relation to the Neumann-Monte Carlo methodology and the Spectral Finite Element method.

## 5. CONCLUSION

In this work, results are obtained for the quantification of the uncertainty of the stochastic bending problem of an elastic and stationary beam by Euler-Bernoulli, Timoshenko and Levinson-Bickford. The proposed object consists of applying the Spectral Finite Element Method and the Neumann series as a numerical strategy to obtain the approximate solution of the system of linear equations resulting from the variational procedure. The weak formulation of the problem is obtained from the sampling Galerkin method.

In the numerical simulations, initially, the convergence of the expected value is verified as a function of the number of simulations. For  samples, the estimator of expected value and variance does not present a sensitive or significant difference between theories. Then it is confirmed that, when the length and cross-section ratio is increased, the shear effect becomes more pronounced for Timoshenko's theory.

It turns out that the variance estimator is considerably larger for the Levinson-Bickford and Euler-Bernoulli theory. The interpretation of this result is associated with the uncertainty model adopted, that is, by trigonometric series. Regarding estimator error estimates the expected value is less than 1% for all beam theories. The Levinson-Bickford beam solution presents the greatest difference between the results, while the processing time is relatively higher when using the pure Monte Carlo simulation.

## BIBLIOGRAPHIC REFERENCES

Arnold, L.: Stochastic Differential Equations: Theory and Applications. Wiley Interscience, New York (1973).

Babuska, I., Tempone, R., Zouraris, G.E.: Solving elliptic boundary value problems with uncertainty coefficients by the Finite Element method: The stochastic formulation. Computer Methods in Applied Mechanics and Engineering, 194(12-16), 1251-1294 (2005).

Bickford, W.B.: A consistent higher order beam theory. Developments in Theoretical and Applied Mechanics, 11, 137 (1981).

Ghanem, R.G, Spanos, P.D.: Stochastic Finite Elements: A Spectral Approach. Springer-Verlag, New York (1991).

Goodier, J.N., Timoshenko S.P.: Theory of Elasticity. McGraw-Hill Book Company, New York (1951).

Heyliger, P.R., Reddy, J.N.: A higher order beam finite element for bending and vibration problems, Journal of Sound and Vibration, 126(2), 309-326 (1988).

Karttunen, A., Von Hertzen, R.: Variational formulation of the static Levinson beam theory. Mechanical Research Communications, 66, 15-19, (2015).

Levinson, M., Stephens N.G.: A second order beam theory. Journal of Sound and Vibration, 67(3), 293-305 (1979).

Rao M.M., Swift, R.J.: Probability Theory with Applications, Mathematics and Its Applications. Springer Verlag, New York (2006).

Reddy, J.N.: An Introduction to the Finite Element Method. Mc-Graw-Hill Book Company, New York (1984).

Reddy, J.N.: Nonlocal nonlinear formulations for bending of classical and shear deformation theories of beams and plates. International Journal of Engineering Science, 48(11), 1507-1518 (2010).

Shinozuka, M.: Structural response variability. Journal of Engineering Mechanics, ASCE, 113(6), 825-842, (1987).

Squarcio, R.M.F., Ávila, C.R.S.J.: Uncertainty quantification via λ-Neumann methodology of the stochastic bending problem of the Levinson-Bickford beam. Acta Mechanica, 233, 12-16 (2022).

Timoshenko S.P.: On the correction for shear of the differential equation for transverse vibrations of prismatic bars. Philosophical Magazine, 41, 744-746 (1921).

Timoshenko S.P.: On the transverse vibration of bars of uniform cross-section. Philosophical Magazine, 43, 125-131 (1922).

Tiwari N., Hyer M.W.: Secondary buckling of compression-loaded composite plates, AIAA journal, 40(10), 2120-2126, (2002).

Yamazaki, F., Shinozuka, M., Dasgupta, G.: Neumann expansion for stochastic Finite Element analysis. Journal of Engineering Mechanics, 114(8), (1988).