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STRATEGIES FOR PROVING THE VOLUME OF SPHERES IN HIGH SCHOOL

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Abstract: In this paper, we present two strategies for proving the relationship for calculating the volume of a sphere in high school: Cavalieri's principle and Archimedes' lever law. In Cavalieri's principle, we use the free software GeoGebra 3D to dynamically prove the relationship for the volume of a sphere; in the lever law or principle of equilibrium, we construct an Archimedes' balance with geometric solids made on a 3D printer to experimentally prove the relationship for the volume of a sphere. We conclude that the dynamic geometry application GeoGebra 3D is an effective tool for constructing two-dimensional and three-dimensional figures, as well as for comparing the areas and volumes of these figures, and that dynamic and experimental proofs enable the organization of classroom activities that address the competencies and skills outlined in the National Common Core Curriculum (BNCC) for mathematics teaching in high school.
Keywords: Archimedes' lever law. Cavalieri's principle. Mathematics teaching. GeoGebra 3D. BNCC.

INTRODUCTION

In recent decades, educational reforms proposed for mathematics teaching in basic education have highlighted the importance of teaching plane geometry and spatial geometry. The National Common Core Curriculum (BNCC, Brazil, 2018) for mathematics in elementary school emphasizes the development of skills through five related thematic units, one of which is geometry. For high school, the objective is even broader, as it seeks to build an integrated view of the the subject with reality. In addition, the BNCC for Mathematics and its Technologies proposes the use of technological tools and computer programs.

> It should also be noted that the use of technologies provides students with alternative experiences that facilitate learning and reinforce their ability to reason logically, formulate and test conjectures, evaluate the validity of reasoning, and construct arguments (Brazil, 2018, p. 536).

Geometry is present in all documents that guide the planning and development of mathematics at various educational levels (Brazil, 2018), being applied both directly and transversally, in order to contribute to the development of students' spatial vision. Regarding geometry and measurement skills, the BNCC establishes the following for high school:

> (EM13MAT309) Solve and develop problems involving the calculation of total areas and volumes of prisms, pyramids, and round bodies in real situations (such as calculating the amount of material needed to cover or paint objects whose shapes are compositions of the solids studied), with or without the support of digital technologies (Brazil, 2018, p. 545).

In addition, geometry is the basis of knowledge for other areas of science and technology, such as physics and engineering, reinforcing its multidisciplinary nature in the educational process. This importance is evidenced by the significant number of questions on plane geometry and spatial geometry in the National High School Exam (ENEM). In the more than twenty years of this exam's existence (INEP, 2025), we can list many questions involving the calculation of areas and volumes in the Mathematics and Technology test (Nós; Fernandes, 2019, 2018; Tavares, 2019).

In the spatial geometry questions of the ENEM, we highlight several that address the sphere (Tavares, 2019). These questions are both applied, such as volume calculation – Figure 1(a), and conceptual – Figure 1(b). The question illustrated in Figure 1(b) addresses an aspect of demonstrating the relationship for calculating the volume of a sphere using Cavalieri's principle.

The volume of a sphere is commonly addressed in mathematics textbooks for high school through the simple presentation of the relationship $\frac{4}{3}\pi r^3$. Students are not encouraged to prove/justify this relationship. However, the BNCC establishes as specific competency 5 for Mathematics and its Technologies:

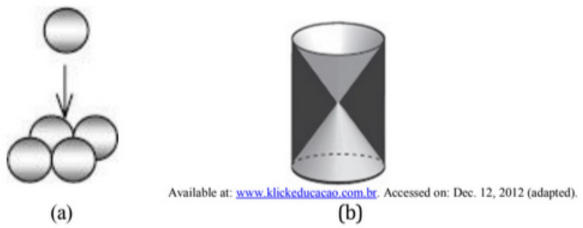

Figure 1 – ENEM questions involving spheres: (a) question 02 from 1998; (b) question 170 from the 2018 Yellow Test

Source: INEP (2025).

> Investigate and establish conjectures about different mathematical concepts and properties, using strategies and resources such as pattern observation, experimentation, and different technologies, identifying the need, or not, for an increasingly formal demonstration in the validation of these conjectures (Brazil, 2018, p. 540).

Thus, in this paper, we present strategies to prove the relationship for calculating the volume of a sphere in high school (Nós; Tavares, 2021), and we use the dynamic geometry application GeoGebra 3D (GeoGebra, 2025; Nós; Silva, 2022, 2020, 2019, 2018), thus aligning the planning of classroom activities with the BNCC.

THE VOLUME OF THE SPHERE

We can define a sphere as the geometric locus of points in three-dimensional space that are a distance *r* from a given point O. Figure 2 illustrates a sphere ε with center O and radius *r*, whose volume is defined by Theorem 1.

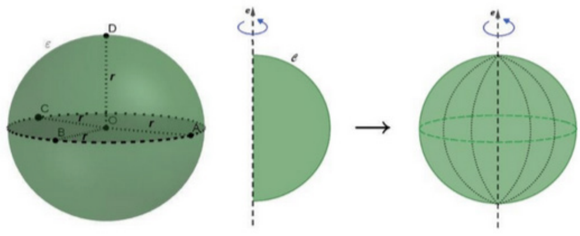

Figure 2 – Sphere with center O and radius r and its conception as a solid of revolution

Source: Tavares (2019, p. 39).

Theorem 1*. The volume V of the sphere ε with radius r is given by*

$$V(\varepsilon) = \frac{4}{3}\pi r^3.$$

According to Lima (2011), among the strategies that high school mathematics teachers can use to justify the relationship determined by Theorem 1 are Cavalieri's principle and the classic presentation by Euclid and Archimedes.

CAVALIERI'S PRINCIPLE

Principle 1 or Cavalieri's principle [1](Lima, 2011; Lima *et al.*, 2006; Paterlini, 2010), illustrated in Figure 3, can be presented to high school students as a postulate, emphasizing the need to prove the equivalence of the sections. According to Lima (2011), the use of Cavalieri's principle "allows for a remarkable simplification in the arguments that lead to the classic volume formulas" (Lima, 2011, p. 96).

Principle 1*. If every plane parallel to the plane of the bases of two solids, with equivalent bases and congruent heights, determines equivalent*

1. Bonaventura Francesco Cavalieri (1598-1647): Italian priest and mathematician, disciple of Galileo. He is considered one of the precursors of integral calculus.

sections in both solids, then the two solids are equivalent, that is, they have the same volume.

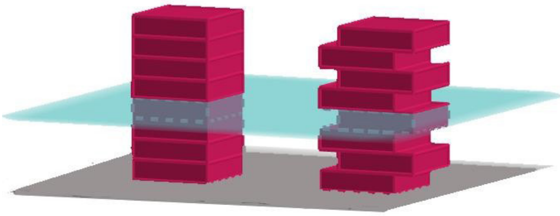

Figure 3 – Solids with equivalent bases and congruent heights intersected by a plane parallel to the bases that determines equivalent sections in both

Source: Tavares (2019, p. 40).

Thus, to use Cavalieri's principle in calculating the volume of a solid, we need to compare this solid with a solid of known volume. In the case of a sphere, this solid is the anticlepsydra - Figure 4: an equilateral cylinder with a base congruent to the maximum circle of the sphere, from which two straight cones with bases congruent to the base of the cylinder and of heights equal to the radius of the sphere. Let us then show, using Cavalieri's principle, that the sphere and the anticlepsydra, or the hemisphere and the semianticlepsydra, of the same radius have the same volume.

Figure 4 – Anticlepsydra and one of the cones that form the clepsydra made of carbon steel

Source: Nós (2019).

DEMONSTRATION

Let the semianticlepsydra be A_s and the hemisphere be E_s , both with radius r, and β a plane parallel to the plane α containing the bases of A_s and E_s . The plane β intersects A_s and E_s at a distance d , $d< r$, from the centers of the bases of the two solids, as illustrated in Figure 5.

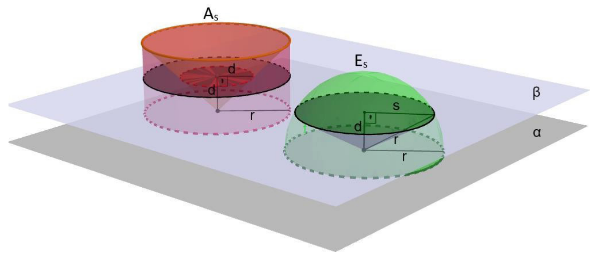

Figure 5 – Equivalent semianticlepsydra and hemisphere

Source: Tavares (2019).

In the hemisphere E_s , the plane β determines a circle of radius s . As, by the Pythagorean theorem (Silva; Nós; Sano, 2023) $r^2= d^2+ s^2$, or equivalently, $s^2= r^2- d^2$, the area of the circular section is equal to:

$$A(\text{circular section})=\pi s^2= \pi\ (r^2-d^2). \quad (1)$$

In the semianticlepsydra A_s, the β plane determines a circular crown with an external radius r and an internal radius d, as proven by the similarity of triangles (angle-angle case). Thus, the area of the section in the semianticlepsydra is equal to:

$$A(\text{circular crown})=\pi s^2= \pi\ (r^2-d^2). \quad (2)$$

The areas (1) and (2) of the sections that the plane β determines at E_s and A_s, respectively, are equal regardless of the distance d , provided that β is parallel to α. Thus, by Cavalieri's principle, we conclude that E_s and A_s are equivalent. Therefore:

$V(E_s) = V(A_s) = V(\text{semicylinder}) - V(cone)$;

$V(E_s) = \pi r^2 r - \frac{1}{3}\pi r^2 r = \frac{2}{3}\pi r^3$.

Since the volume of the sphere ε is equal to twice the volume of the hemisphere E_s, multiplying the result (3) by two, we have that

$$V(\varepsilon) = \frac{4}{3}\pi r^3,$$

which confirms the thesis of Theorem 1.

CAVALIERI'S PRINCIPLE IN GEOGEBRA 3D

We constructed an animation in GeoGebra 3D to prove the volume of the sphere using Cavalieri's principle.

Steps

Construct a hemisphere with radius *3 cm* through the function $z = f(x, y) = \sqrt{9 - (x - x_A)^2 - (y - y_A)^2}$. Mark the center A of the maximum circle of the hemisphere ($z_A = 0$) and the radius $\overline{AB}$ perpendicular to the plane of this circle. Finish by drawing two lines, the first passing through point B and the second parallel to the radius $\overline{AB}$ and perpendicular to the first, as shown in Figure 6.

Construct a straight circular cylinder with height $\overline{AB}$ and radius *3 cm*, whose axis is the second line drawn in the first step, as shown in Figure 7. The axis of the cylinder must be more than *6 cm* away from the support line $\overline{AB}$.

Construct a right cone with height $\overline{AB}$ and radius *3 cm*, whose vertex is the center of the base lower part of the cylinder constructed in the second stage, as shown in Figure 8. The solid formed by the cylinder minus the cone is the semianticlepsydra.

Determine the midpoint of $\overline{AB}$ and draw the plane that intersects the semianticlepsydra and the hemisphere that passes through the midpoint and is perpendicular to the axis of the cylinder, as shown in Figure 9.

Highlight the intersection of the secant plane with the semianticlepsydra, a circular crown, and with the hemisphere, a circle, as shown in Figure 10.

Calculate the area of the sections in the semianticlepsydra and the hemisphere and conclude that the sections are equivalent, as shown in Figure 11.

Calculate the volume of the semianticlepsydra and the hemisphere, as shown in Figure 12.

The animation created according to the step-by-step instructions described above is available at https://www.geogebra.org/3d/dd4eh7dz.

ARCHIMEDES' SECOND THEOREM

Cavalieri's principle, although intuitive, cannot be demonstrated in an elementary way (Lima, 2011). Thus, Archimedes' theorem[2] is another approach that can be adopted in high school.

In their book *O Método*, Assis and Magnaghi (2014) describe Archimedes' mechanical strategy for investigating volumes such as that of a sphere – Theorem 2.

Theorem 2*. The volume of any sphere is equal to four times the cone whose base is equal to the maximum circle of the sphere and whose height is equal to the radius of the sphere, while the volume of the cylinder with a base equal to a maximum circle of the sphere and a height equal to the diameter of the sphere is one and a half times the volume of the sphere.*

To prove the volume of the sphere according to Archimedes (Aaboe, 2013; Archimedes; Heath, 1953; Assis; Magnaghi, 2014; Ávila, 1986; Hellmeister, 2013; Tavares, 2019), we need the law of the lever or principle of equilibrium – Principle 2, proposed by Archimedes in his work *On the Equilibrium of Planes*.

2. Archimedes of Syracuse (287 BCE – 212 BCE): Greek mathematician, physicist, engineer, inventor, and astronomer. He is credited with the laws of buoyancy and the lever.

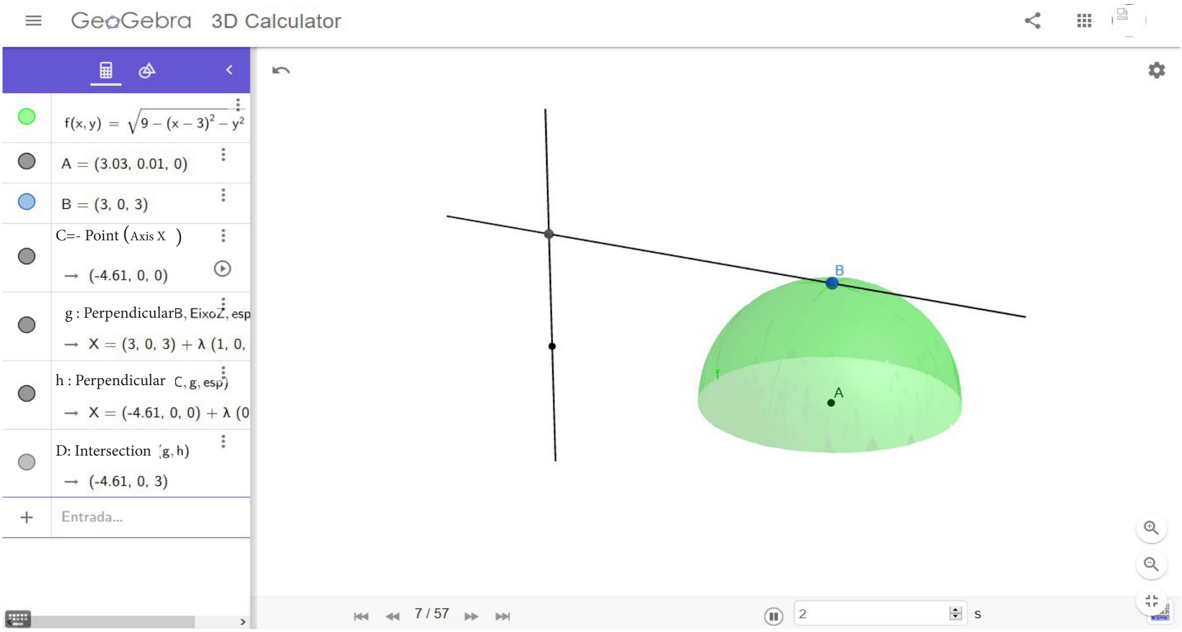

Figure 6 – Construction of the hemisphere with radius *AB*= 3 *cm*

Source: Tavares (2019, p. 112).

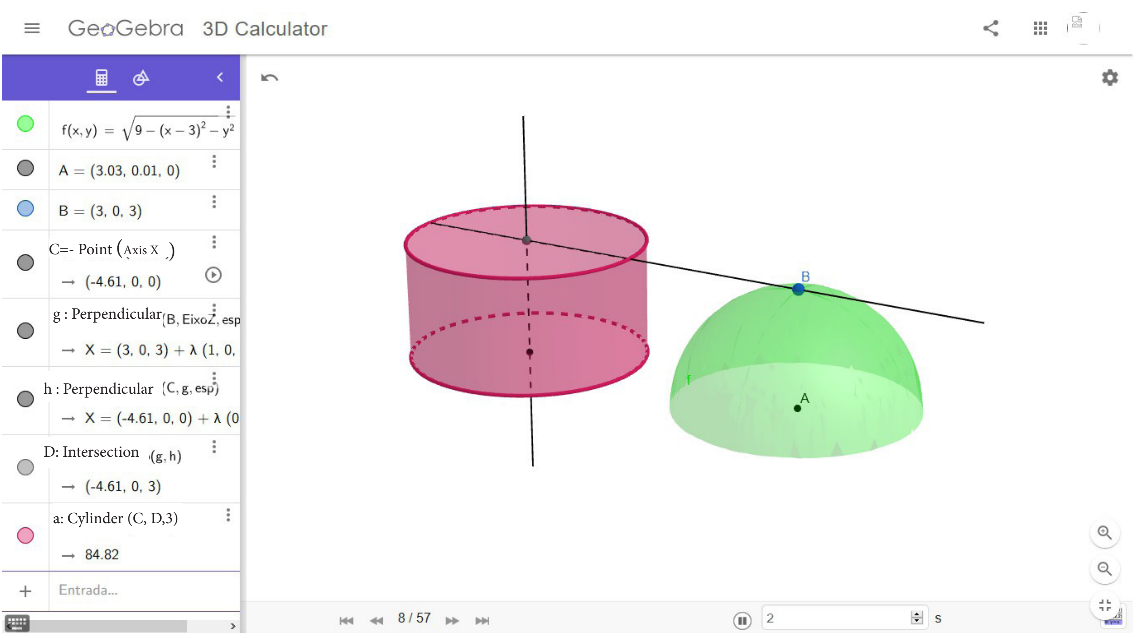

Figure 7 – Construction of a cylinder with height AB= 3*cm* and radius *r*= 3 *cm*

Source: Tavares (2019, p. 112).

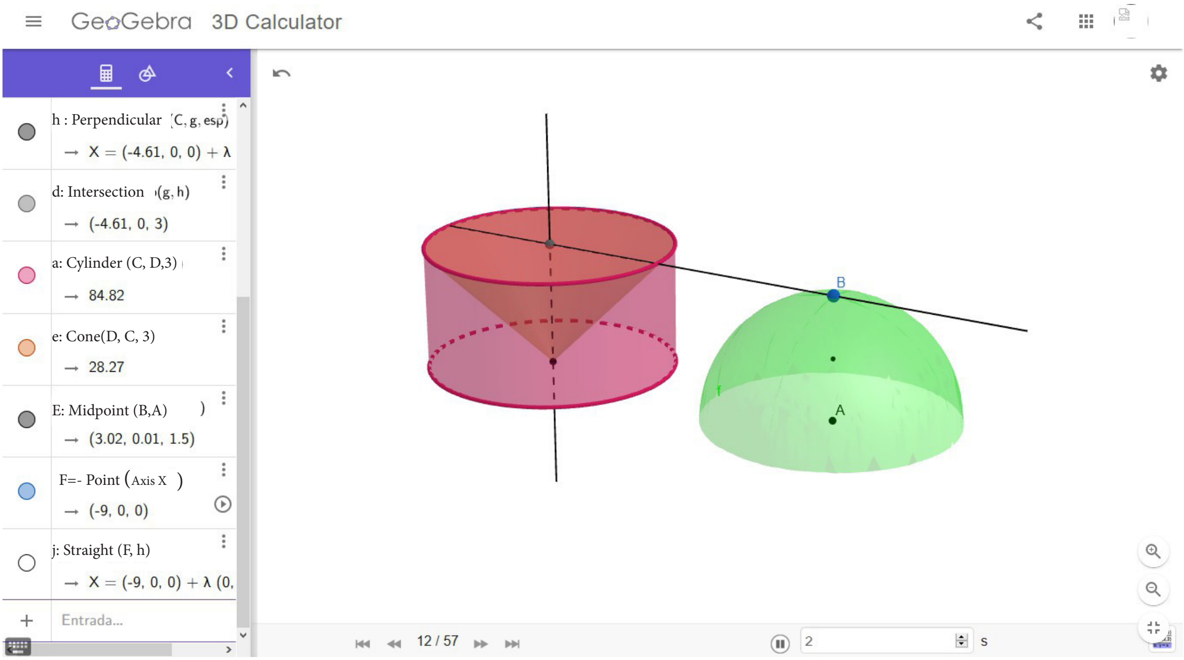

Figure 8 – Construction of the semianticlepsydra

Source: Tavares (2019, p. 113).

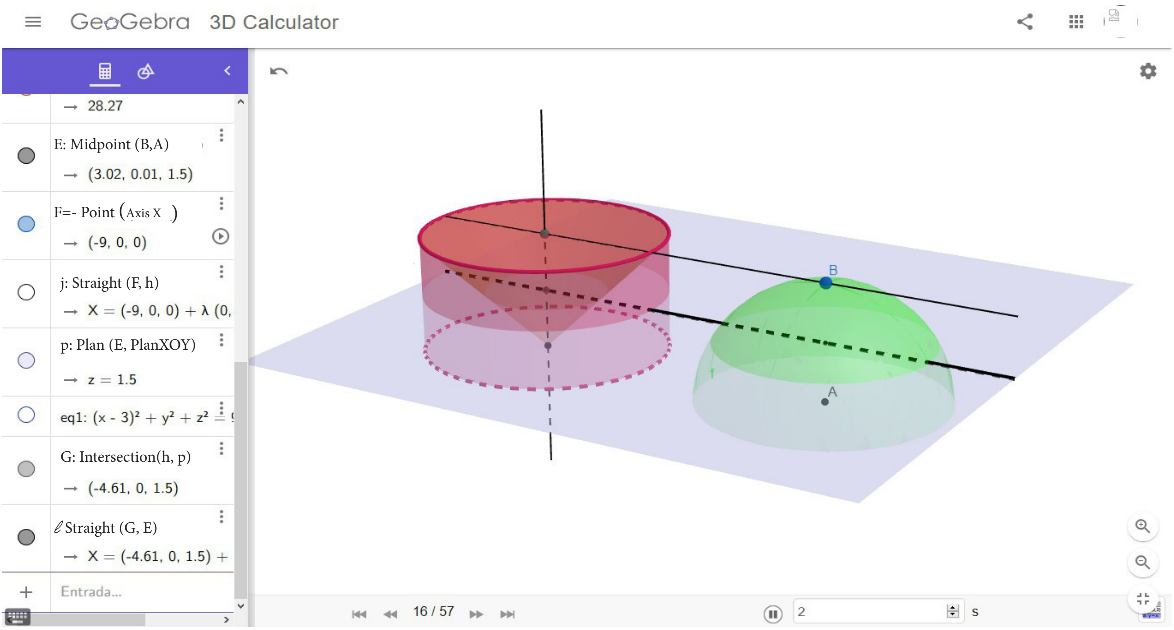

Figure 9 – Construction of the secant plane to the semianticlepsydra and the hemisphere

Source: Tavares (2019, p. 113).

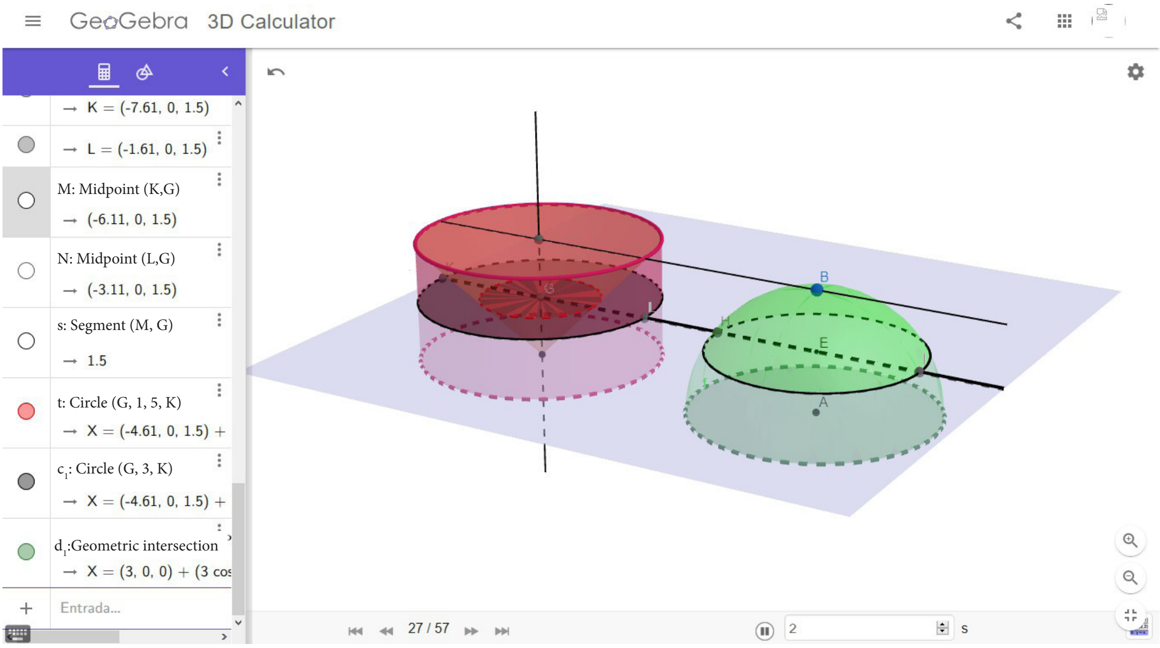

Figure 10 – Sections in the semianticlepsydra and the hemisphere

Source: Tavares (2019, p. 114).

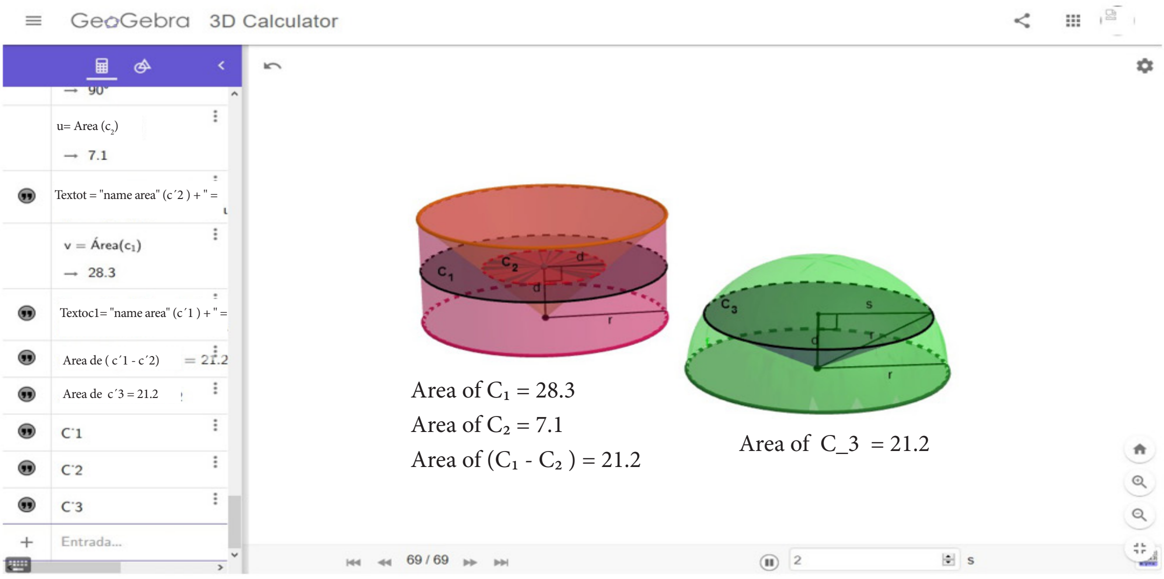

Figure 11 – Area of the sections in the semianticlepsydra and in the hemisphere

Source: Tavares (2019, p. 114).

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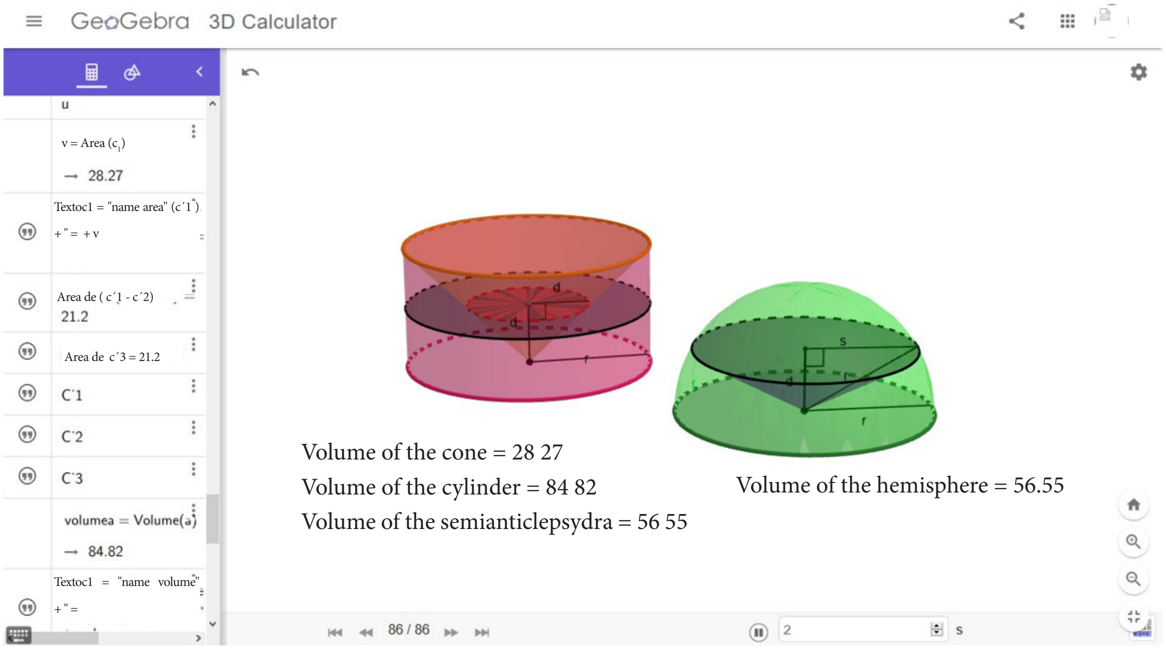

Figure 12 – Volume of the semianticlepsydra and the hemisphere

Source: Tavares (2019, p. 115).

Principle 2*. A lever is in equilibrium if the product of weight A and the distance a between the fulcrum* [3]*and the suspension point of A is equal to the product of weight B and its distance b from the fulcrum, that is,*

$$\frac{A}{B} = \frac{b}{a},$$

or equivalently

$A \bullet a = B \bullet b.$

Figure 13 shows a lever in equilibrium, that is, a lever where the relationship (4) is verified.

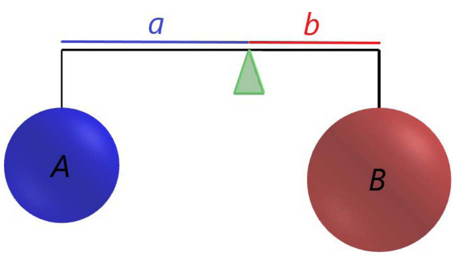

Figure 13 – Lever in equilibrium

Source: Tavares (2019).

Using Principle 2, we can show that the cylinder with radius and height *2r*, at a distance *d* from the fulcrum of the lever, balances the cone with radius and height *2r* and the sphere with radius *r*, both at a distance *2d* from the fulcrum of the lever, as illustrated in Figure 14.

Thus, using the solids illustrated in Figure 14(a) on the lever illustrated in Figure 14(b), we conclude that:

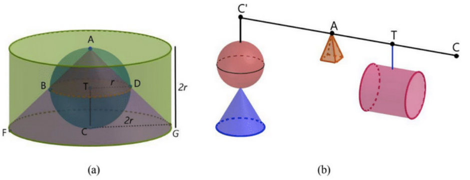

Figure 14 – (a) Sphere, cone, and cylinder; (b) Archimedes' lever in equilibrium

Source: (a) Tavares (2019, p. 54); (b) Tavares (2019, p. 53).

$$V(\text{cylinder}).\, d = [V\,(cone) + V\,(\text{sphere})].\, 2d;$$

$$V(\text{sphere}) = \frac{1}{2}\, V(\text{cylinder}) - V(cone) = \frac{1}{2}\pi(2r)^2 2r - \frac{1}{3}\pi(2r)^2 2r = \frac{4}{3}\pi r^3,$$

3. Fixed point.

which corroborates the thesis of Theorem 1.

ARCHIMEDES' PRINCIPLE OF EQUILIBRIUM

Archimedes' greatest contribution to geometry is found in his work *The Method*, in which he explores the determination of volumes through Principle 2 or the law of the lever, which defines a mechanical system of weight equilibrium on a lever (Archimedes; Heath, 1953; Assis; Magnaghi, 2014), illustrated in Figure 15.

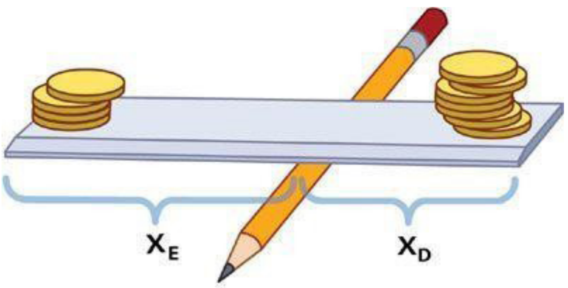

Figure 15 – Balance according to Archimedes' lever law

Source: Antônio (2015).

As Greek mathematics was essentially geometric, Archimedes did not determine the relationship for calculating the volume of a sphere. However, he demonstrated the proportionality between the masses of two solids through the law of the lever, which allows us to deduce the relationship. In his work *On the Sphere and the Cylinder* (Archimedes; Heath, 1953), Archimedes demonstrated Theorem 2 using a method similar to exhaustion (Nós; Sano; Tavares, 2021).

Previously, we showed that, for a sphere of radius *r*, a right cone of radius and height *2r*, and a right cylinder of radius and height *2r*, the law of the lever establishes that:

$$\frac{V(cone)+V(\text{sphere})}{V(\text{cylinder})}=\frac{1}{2};$$

$$2[V(cone)+V(\text{sphere})]=V(\text{cylinder}). \quad (5)$$

Thus, from relation (5) we have that the cone and the sphere, at a distance d from the fulcrum of the lever, balance the cylinder at a distance $\frac{d}{2}$ from the fulcrum of the lever. To prove this relationship experimentally, we built an Archimedes' balance using a straight cylinder, a straight cone, and a sphere made with a 3D printer. Figures 16 and 17 illustrate the relationship (5) for $r = 2.5cm$ and $d = 30\ cm$, on the Archimedes' balance in equilibrium and disequilibrium, respectively.

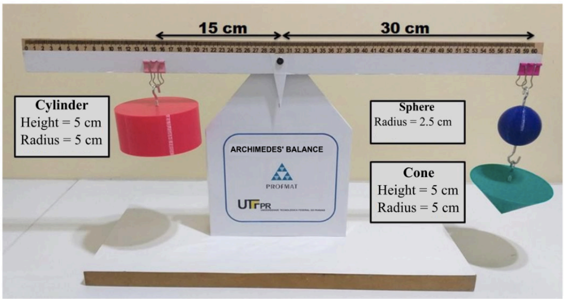

Figure 16 – Archimedes' lever law: the sphere and the cone balance the cylinder

Source: Tavares (2019, p. 111).

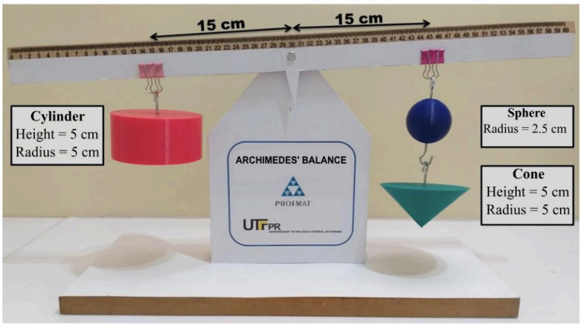

Figure 17 – Archimedes' lever law: the sphere and the cone do not balance the cylinder

Source: Tavares (2019, p. 111).

FINAL

The strategies for proving the relationship for calculating the volume of a sphere presented in this paper were presented in 2019 to third-year high school students at CPM, where the author is a mathematics teacher. We concluded that the activities were relevant for consolidating concepts related to calculating the volume of spheres, cones, and cylinders and contributed to preparing students for the ENEM.

We hope that this work will motivate elementary school math teachers to develop activities and experiments to prove/justify geometric relationships, rather than simply presenting them to students, as well as to use dynamic geometry applications, such as GeoGebra 3D, in geometry classes.

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