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BESSEL FUNCTION APPLIED TO THE VIBRATIONS OF A CIRCULAR MEMBRANE

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Abstract: The objective is to analyze and apply the Bessel function to systematize the movement that occurs due to the restoration of forces created by deformations. The analysis uses an exhaustive review of complex situations, where the theoretical and practical aspects are articulated with a set of theorems that are guides in different stages of the investigation. The results demonstrate that as we complicate the equation with boundary data, the need for new mathematical concepts arises for its correct interpretation. It is a problem of circular geometry of the limit with analysis of physical parameters, where factors such as the thickness, radial of a membrane and density intervene; Its importance is the interpretation of vibratory motion using Bessel functions.
Keywords: Bessel equation, Bessel functions, circular membrane, boundary problems.

INTRODUCTION

The vibrations of a circular membrane, which is fixed to a rigid circular frame and is mainly elastic of uniform thickness, are formulated as a wave equation, which satisfies boundary conditions. A membrane can vibrate in different ways depending on the shape of the membrane. given deformation in an initial time interval, by means of the method of separation of variables, it is possible to find a set of simple vibration modes [Asmar, 2005; Arfken, 2005).

Vibrations are the propagation of classical waves, which cause deformations and tensions in a continuous medium; that is, vibrations can be considered as repetitive movements around an equilibrium position. The equilibrium position is the place reached when the force acting on it is zero; this movement does not necessarily cause an internal deformation of the entire body (Levedev, 1972; Jackson, 1998).

For Asmar (2005) it is convenient to separate the terms vibration and oscillation because the amplitude of the oscillation is much greater. Vibrations produce movements of smaller amplitude than oscillations around the equilibrium point, vibratory movements can be easily linearized, while in oscillations kinetic energy is usually converted into gravitational potential and vice versa, while in vibrations there is an exchange between kinetic energy and elastic potential energy.

On the other hand, differential equations model many real-life processes, it is linked to science and engineering; while, Bessel functions are in the group of special functions with multiple applications in mathematical physics, it appears as a result of solving an ordinary second-order differential equation with variable coefficients (Malley, 1997; Davis, 1992; Arnol´d, 1992).

The mathematician Daniel Bernoulli (1700–1782) was the first to arrive at the Bessel functions on the basis of the study of the vibrations of a hanging chain, later it appears again in the studies of Euler (1707 – 1782) regarding the vibrations of a circular membrane; in the same way when Bessel studies the motion of planets. The Bessel equation arises naturally when dealing with boundary problems in potential theory for cylindrical domains (Braun, 1983; Brauer, 1967; Kreyszig, 2016).

In Spiegel (2011), he analyzes the singularity and the analyticity of its coefficients; but, he is not very explicit in the foundation regarding the conditions of its coefficients to obtain the Bessel function of order and does not emphasize the role played by the gamma function, the order reduction theorem in the face of the need to find two linearly independent solutions, while, in the works of Dehestani (2020), the substantial thing is the study of the spherical Bessel functions, associated with the wave equation in spherical coordinates;

however, Liouville He managed to show that $J_{p+\frac{1}{2}}, p \in \mathbb{Z}$ is elementary, and that they are the only elementary Bessel functions. Another problem is the zeros of Bessel functions, although tables can be found in many books, especially for, it is necessary to formulate affine theorems to determine zeros of any non-trivial function and to formulate the existence of zeros (Cruz, 2020); above all, pay attention to the orthogonality properties using change of variable, formulate new propositions related to the integration of Bessel functions (Barceló et al., 1997).

From the review of Muller (2001), it can be noted that the Bessel functions are initiated through a power series. In general, an expression of this type provides us only local information, but given the scarce global information on $J_p(t)$, the question arises, would a representation by means of an oscillatory integral be better? The answer puts on alert aspects of its form of representation. For Córdova (1989) there are some uniform estimates in p, this due to the concern of the dependence of this parameter, although it is not enough to study the behavior of the Bessel functions for large values; according to Kreyszig (2016), in attention to the Bessel functions in their various species and order, we make specific modifications that promote their application to the interpretation of the circular membrane with adjustment to the boundary conditions.

METHODOLOGY

It corresponds to a fundamental investigation, which is supported by other investigations presented in reliable references, at an exploratory level, the objective being to apply Bessel -type functions that require establishing relationships between formulas and concepts that are interconnected, theorems associated with definitions, throughout the entire process the deductive-inductive is manifested.

The design corresponds to a qualitative investigation, it is framed to detect a problem, interpret, apply theories, particularize and deepen its understanding. Its application revolves around the exhaustive analysis of new situations, where the theoretical aspects and the practical situation are articulated with a set of theorems being guides at different moments of the work. The arguments used in this work are: demonstrable; in addition, applicable within the research context; the assessment of criteria, transfer and dependency are part of a set of premises and concepts.

RESULTS

BESSEL DIFFERENTIAL EQUATION

In limited cases, the solution of a linear equation with second-order variable coefficients can be determined by means of elementary functions. Before solving a boundary problem, the Bessel differential equation is analyzed, which is presented in the canonical form

$$x^2y'' + xy' + (x^2 - p^2)y = 0\,, \qquad (1)$$

in its self-adjoint form, $(xy')' + \left(x - \frac{p^2}{x}\right)y = 0$.

One of the most commonly used methods to solve the Bessel equation (1) is that of Frobenius (Simmons, 1993, Simmons, 2018), using a series of the form $y(x) = \sum_{n=0}^{\infty} b_n x^{n+r}$ surrounding a point 0 that is a regular singular. The solution is sought in the form of a power series $y = b_o + b_1x + b_2x^2 + \cdots + b_nx^n + \cdots$, It is clear that there is no certainty that this series starts with the independent term. We consider the first non-zero coefficient, if $a_0=b_m$. Therefore, $y = a_0x^m + a_1x^{m+1} + a_2x^{m+2} + \cdots a_n x^{m+n} + \cdots$, being $m\geq 0$, $a_0\neq 0$, when replacing we obtain, $a_0m^2x^{m-1} + a_1(m+1)^2x^m + a_2(m+2)^2x^{m+1} + \cdots + a_n(m+n)^2x^{m+n+1} + \cdots$; $-a_0p^2x^{m-1} - a_1(p)^2x^m - a_2(p)^2x^{m+1} - \cdots - a_n(p)^2x^{m+n+1} + \cdots$. Equating coefficients to the same power for a zero polynomial, all of its coefficients must be zero,

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$a_0(m^2-p^2)=0$

$a_1[(m+1)^2-p^2]=0$

$a_2[(m+2)^2-p^2]+a_0=0$

$\dots$

$a_n[(m+n)^2-p^2]+a_{n-2}=0.$

From this part a recurrence formula is obtained and they are

$m^2=p^2,\ m=p$

$a_1[(p+1)^2-p^2]=0$

$a_2[(p+2)^2-p^2]+a_0=0$

$\dots$

$a_n[(p+n)^2-p^2]+a_{n-2}=0,$

where from,

$a_1(2p+1)=0, 2p+1\neq 0, a_1=0$

$a_2(2p+4)+a_0=0,\ a_2=-\frac{a_0}{2(p+2)}$

$a_3(6p+9)+a_1=0,\ a_3=-\frac{a_1}{3(2p+3)}=0$

$\dots$

$a_n(2np+n^2)+a_{n-2}=0,\ a_n=-\frac{a_{n-2}}{n(2p+n)}$

In this construction $p\in\mathbb{Z}^+$, then $a_1=0$, $a_3=0$, $a_5=0$, ..., $a_{2k+1}=0$, only the pairs remain $a_{2k}=\frac{a_0p!(-1)^k}{2^{2k}k!(p+k)!}$, it is the generic term, with some adjustments it is written,

$$y=a_02^pp!\left[\frac{\left(\frac{x}{2}\right)^p}{p!}-\frac{\left(\frac{x}{2}\right)^{p+2}}{1!(p+1)!}+\frac{\left(\frac{x}{2}\right)^{p+4}}{2!(p+2)!}-\dots+\frac{(-1)^k\left(\frac{x}{2}\right)^{p+2k}}{k!(p+k)!}-\dots\right]$$

$$J_p(x)=\frac{\left(\frac{x}{2}\right)^p}{p!}-\frac{\left(\frac{x}{2}\right)^{p+2}}{1!(p+1)!}+\frac{\left(\frac{x}{2}\right)^{p+4}}{2!(p+2)!}-\dots+\frac{(-1)^k\left(\frac{x}{2}\right)^{p+2k}}{k!(p+k)!}-\dots$$

It is clear that the solution to the problem for the Bessel equation is any function of the form $AJ_p(x)$, where A is reflecting a factor $A=a_02^pp!$, being arbitrary, one could even choose $a_0=1$, then the Bessel function is bounded for values of x, $x\to 0^+$, positive on the right, without losing sight of the fact that this comes from the problem of the circular membrane that has axial or cylindrical symmetry. There are many problems that are solved in mathematical physics that have a pre-supposition of this symmetry. Therefore, the solution to the Bessel equation is

$$y=c_1J_p(x)+c_2N_p(x). \quad (2)$$

In (2), $N_p(x)$ it is the Weber function or also known as the Newmann function, this is not used in the work, since this function tends to infinity, that is $N_p(x)\to-\infty$, when $x\to 0^+$, we only work with $J_p(x)$, in the case where is not $x\to-\infty$, it can be if we consider the perforated membrane and we can work as a superposition of both.

For the normal form of the Bessel equation, we first write

$$y''+\frac{1}{x}y'+\frac{(x^2-p^2)}{x^2}y=0, \quad (3)$$

in (3), making an appropriate change of variable $u(x)=\sqrt{x}y(x)$, $u''+\left(1+\frac{1-4p^2}{4x^2}\right)u=0.$

Therefore, the Bessel function of the first kind, $J_p(x)$, of order p, is

$$J_p(x)=\frac{x^p}{2^p\Gamma(p+1)}\left[1-\frac{x^2}{2(2p+2)}+\frac{x^4}{2.4(2p+2)(2p+4)}+\cdots\right]$$

$$J_0(x)=1-\frac{x^2}{2^2}+\frac{x^4}{2^2.4^2}-\frac{x^6}{2^2.4^2.6^2}+\frac{x^8}{2^2.4^2.6^2.8^2}-\frac{x^8}{2^2.4^2.6^2.8^2.10^2}+\cdots$$

If $p\notin\mathbb{N}$, then $J_p(x)$ and are linearly independent and form a basis of the solution space of $J_{-p}(x)$ Bessel 's equation , the general solution is the linear combination of both. If $p\in\mathbb{N}$, in this case $J_p(x)$ and, $J_{-p}(x)$ are linearly dependent, then $J_p(x)=(-1)^p J_{-p}(x)$. If $p\notin\mathbb{N}$ and $x\to 0$, then $J_p(x)\approx\frac{x^p}{2^p\Gamma(p+1)}$ and $J_{-p}(x)\approx\frac{x^{-p}}{2^p\Gamma(-p+1)}$.

Bessel function of the first kind, $I_p(x)$, of order p, is

$$I_p(x)=i^{-p}J_p(ix)=\frac{\left(\frac{x}{2}\right)^p}{\Gamma(p+1)}+\frac{\left(\frac{x}{2}\right)^{p+2}}{1!\Gamma(p+2)}+\frac{\left(\frac{x}{2}\right)^{p+4}}{2!\Gamma(p+3)}+\frac{\left(\frac{x}{2}\right)^{p+6}}{3!\Gamma(p+4)}+\cdots$$

The Bessel function of the second kind, $N_p(x)$, of order p, is

$$N_p(x)=\begin{cases}\frac{J_p(x)cos(p\pi)-J_{-p}(x)}{sen(p\pi)}, & p\neq 0,1,2,3,\dots\\ \lim_{\lambda\to p}\frac{J_\lambda(x)cos(\lambda\pi)-J_{-\lambda}(x)}{sen(\lambda\pi)}=N_\lambda(x), & p=0,1,2,3,\dots\end{cases},$$

The modified Bessel function of the second kind of order $K_p(x)$, is

$$K_p(x)=\begin{cases}\frac{\pi}{2}\left[\frac{I_{-p}(x)-I_p(x)}{sen(p\pi)}\right], p\neq 0,1,2,3,\dots\\ \lim_{\lambda\to p}\frac{\pi}{2}\left[\frac{I_{-\lambda}(x)-I_\lambda(x)}{sen(\lambda\pi)}\right], p=0,1,2,3,\dots\end{cases}.$$

The functions $J_p(x)$ and $N_p(x)$ are linearly independent, so the general solution of the Bessel equation in the case $p=\lambda\in\mathbb{N}$ is a linear combination of $J_p(x)$ and $N_p(x)$. If $p\in\mathbb{N}$ and $x\to 0$, then

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$N_p(x) \approx \frac{(p-1)!}{\pi}\left(\frac{x}{2}\right)^{-p}$, $p \geq 1$and $N_0(x) \approx \frac{2}{\pi} log\left(\frac{x}{2}\right)$.

BESSEL FUNCTIONS

There are differential equations that generate new functions, such as the differential equations of Bessel, the peculiarity is that these functions are applied to the field of physics. It is written in a generic way $J_p(x) = \sum_{n=0}^{\infty}(-1)^n\left(\frac{x}{2}\right)^{2n+p}\frac{1}{n!(p+n)!}$, this is the Bessel function of order $p \in \mathbb{Z}^+$. This expression is generalized using the gamma function defined as, $\Gamma(z) = \int_0^{\infty} t^{z-1}e^{-t}dt$, z is the complex argument, in this case not only in the real numbers, it is also extended to the complex numbers, previously there was the presence of the factorial, which is defined for positive integers. In the case of generalizing for complex z, it is highlighted that $Re(z)>0$ (Leighton, 1996; Zill, 2003). Then, $J_\lambda(x) = \sum_{n=0}^{\infty}(-1)^n\left(\frac{x}{2}\right)^{2n+\lambda}\frac{1}{\Gamma(n+1)\Gamma(\lambda+n+1)}$.

The definition given about the Bessel function for p positive integers is generalized to any real number, $\lambda \in \mathbb{R}$ even complex ones, so that it is understood, taking into account some results,

(i) $\Gamma(1) = 1, \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$,

(ii) $\Gamma(z+1) = z\Gamma(z)$, Yeah $z = n \in \mathbb{N}, \Gamma(n+1) = n!$

(iii) $\Gamma(z)\Gamma(1-z) = \frac{\pi}{sen(\pi z)}$

(iv) An integral representation of the function on the boundary called the Riemann-Hanke contour $\Gamma(z) = \frac{1}{e^{2\pi i z}_{-1}}\int_\gamma e^{-t}t^{z-1}dt$ we have that γ it is any contour of the complex plane that surrounds the point t=0, in the counterclockwise direction and that tends to infinity along the real axis. That is, we have moved on to a complex integral, it is also useful $\frac{1}{\Gamma(z+1)} = \frac{e^{i\pi z}}{2\pi i}\int_\Gamma e^{-t}t^{-z-1}dt$, where the border is any one that ends at the point t=0, in the counter clockwise direction, if the integration is in the opposite direction we obtain negative, the integration can be in polar coordinates or any more comfortable coordinates in the complex variable.

Bessel function, in particular if $\lambda=-m$ and change the sign to $n'=n-m= 0,1,2,...$, the sum will start at n', then $J_{-m}(x)=(-1)^m J_n(x)$, which leads to a series of expressions or formulas for the Bessel functions, associated with the ways in which they were constructed, formulas associated with the Bessel functions are written and many are called recurrence because it puts us in terms of the Bessel functions (Dehestani, 2020),

$$\frac{d}{dx}\left(\frac{J_\lambda(x)}{x^\lambda}\right) = -x^{-\lambda}J_{\lambda+1}(x),$$

$$\frac{d}{d\lambda}\left(x^\lambda J_\lambda\right) = x^\lambda J_{\lambda-1}(x)$$

$$J_{\lambda+1}(x) = \frac{2\lambda}{x}J_\lambda(x) + J_{\lambda-1}(x)$$

$$J_{\lambda-1}(x) - J_{\lambda+1}(x) = 2J'_\lambda(x)$$

$$\int x^\lambda J_{\lambda-1}(x)dx = x^\lambda J_\lambda(x) + c$$

$$\int x^{-\lambda} J_{\lambda+1}(x)dx = -x^{-\lambda} J_\lambda(x) + c$$

Although some integrals end up being improper, it is infinite along the X axis, when applying derivatives it is for $J_p(x) = \sum_{n=0}^{\infty}(-1)^n\left(\frac{x}{2}\right)^{2n+p}\frac{1}{n!(p+n)!}$, $J_\lambda(x) = \sum_{n=0}^{\infty}(-1)^n\left(\frac{x}{2}\right)^{2n+\lambda}\frac{1}{\Gamma(n+1)\Gamma(\lambda+n+1)}$.

BESSEL FUNCTIONS

The mathematical complexity of these requires analysis when applying concepts to specific situations. In the Bessel equation $x^2y''+xy'+(x^2-p^2)y=0$ the solution, being of second order, is the superposition of two functions that are linearly independent of each other, $y=c_1J_p(x)+c_2N_p(x)$.

For now we are interested in the Bessel functions with some properties, the interest of this function responds to a problem that comes from a second order differential equation, in which it is required that the solution is essentially finite, and the Newmann functions are not finite (they are infinite), so they cannot be solutions of our physical problem, the reason for their interest is their consistency. In case a Newmann function is required, it can be applied to another problem,

$$J_P(x) = \frac{\left(\frac{x}{2}\right)^p}{p!} - \frac{\left(\frac{x}{2}\right)^{p+2}}{1!(p+1)!} + \frac{\left(\frac{x}{2}\right)^{p+4}}{2!(p+2)!} - \cdots + \frac{(-1)^k\left(\frac{x}{2}\right)^{p+2k}}{k!(p+k)!} - \cdots,$$

It is observed that functions of even order are even and those of odd order are odd,

$$J_P(x) = x^p\left[\frac{1}{p!2^p} - \frac{x^2}{1!(p+1)!2^{p+2}} + \frac{x^4}{2!(p+2)!2^{p+4}} - \cdots + \frac{(-1)^k x^{2k}}{k!(p+k)!2^{p+2k}} - \cdots\right].$$

Bessel functions are continuous and defined on the number line, the continuity comes from the fact that it is a power series and is derivable infinitely many times, then it is continuous and has derivatives of any order, its analyticity comes from the analytical extension of the real function.

Bessel function has a set of real roots, only the positive ones on the positive semi-axis are considered, they can be numbered with the natural numbers in ascending order. If $J_0(x)=0$, the roots are on the X axis, these are u_1, u_2, ..., in the same way they can be numbered for $J_1(x)$ and $J_2(x)$. The asymptotic equality and others are important results and are very used expressions,

$$\frac{d}{dx}\left[x^{-p}J_p(x)\right] = -x^{-p}J_{p+1}(x), p = 0,1,2,\ldots$$

$$\frac{d}{dx}\left[x^{p}J_p(x)\right] = x^{p}J_{p-1}(x), p = 1,2,3,\ldots$$

For all $p\in\mathbb{R}$, there exists a constant $c_p>0$ such that if $x\geq 1$, then

$$J_p(x) \cong \sqrt{\frac{2}{\pi x}}\cos\left(x - \frac{p\pi}{2} - \frac{\pi}{4}\right) + E_p(x), \left|E_p(x)\right| \leq \frac{c_p}{x^{3/2}}$$

$$J_{-p}(x) \cong \sqrt{\frac{2}{\pi x}}\cos\left(x + \frac{p\pi}{2} - \frac{\pi}{4}\right) \text{and } N_p(x) \cong \sqrt{\frac{2}{\pi x}}sen\left(x - \frac{p\pi}{2} - \frac{\pi}{4}\right), x \to \infty.$$

As for the damped trigonometric function, this equality is increasingly closer as it tends to x, which is more precise the larger is x, its periodicity helps to explain the problem, it is asymptotic if $x\to\infty$, the vibration is smaller towards the axis x, for x negative it is fulfilled by being even or odd, the roots are separated by the period π, however, the closer to infinity it is π with more decimals.

Bessel functions are functions of the form $J_{p+\frac{1}{2}}(x)$. These appear when solving the wave equation in spherical coordinates, from the above, the formulas,

$$J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}}sen(x) \text{and } J_{-\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}}cos(x),$$

even obtaining others like $J_{-\frac{3}{2}}(x) = \sqrt{\frac{2}{\pi x}}\left(-\frac{cosx}{x} - senx\right)$, it can be seen that this process of obtaining waves can be continued indefinitely; therefore, any $J_{p+\frac{1}{2}}(x)$ integer p is elementary. Liouville showed that these were the only elementary Bessel functions.

BESSEL FUNCTIONS, THE ZEROS OF SOME

Bessel functions are found in many references, the theorem is useful.

Theorem 1. Let be $u(x)$ any nontrivial solution of $u''+p(x)u = 0$, with $p(x)>0$, for all $x>0$. If $\int_0^1 p(x)dx = \infty$, then $u(x)$ it has infinitely many zeros on the positive semiaxis.

Theorem 2. Let $y(x)$ and be $v(x)$ nontrivial solutions of the equations $y''+p(x)y=0$ and $v''+q(x)v=0$, where $p(x)$, $q(x)$ are positive functions such that $p(x)> q(x)$. Then $y(x)$ vanishes at least once between two successive zeros of $v(x)$.

Sturm's comparison theorem, now we analyze the behavior of the zeros of the Bessel functions.

Bessel 's equation has infinitely many positive zeros.

Indeed, let, be a non-trivial solution of $u(x)$ Bessel's equation, in the normal form $u''+\left(1+\frac{1-4p^2}{4x^2}\right)u = 0,$, when x is large $p(x) = 1 + \frac{1-4p^2}{4x^2}$ is greater than zero, when calculating the integral, and by theorem 1, $u(x)$ has infinitely many zeros on the positive semi-axis,

$$\int_0^1 p(x)dx == \lim_{b\to\infty}\int_1^b\left(1 + \frac{1-4p^2}{4x^2}\right)dx = \infty$$

Theorem 4. Let be a non-trivial solution of the $u(x)$ Bessel equation $x^2y''+xy'+(x^2-p^2)=0$, on the positive semi-axis. The following is true: (a) If $0 \leq p < \frac{1}{2}$, then every interval of length π contains at least one zero of $u(x)$, (b) if $p = \frac{1}{2}$, the distance between two successive zeros of $u(x)$ is exactly π, (c) if $p = \frac{1}{2}$, then every closed interval of length π contains at most one zero of $u(x)$.

Indeed, from the normal Bessel equation $u'' + \left(1 + \frac{1-4p^2}{4x^2}\right)u = 0$; (a) if $0 \leq p < \frac{1}{2}$, then $p(x) = 1 + \frac{1-4p^2}{4x^2} > 1$ let be $u(x)$ a nontrivial solution of $u''+p(x)u=0$ and $v(x)$ be a nontrivial solution of $v''+v=0$, since $p(x)> q(x)=1$, by Theorem 2, $u(x)$ vanishes at least once between two successive zeros of $v(x)$. While the solutions of $v''+v=0$ are *senx* and *cosx*, such that the distance between two zeros of, *senx* and the distance between two zeros of *cosx* is π, therefore, $u(x)$ vanishes at least once in an interval of length π.

On the other hand (b), when $p = \frac{1}{2}$, the equation is $u''+u=0$, whose solutions are the functions *senx* and *cosx*. Then $y_h(x)$ it vanishes exactly once in an interval of length π. However (c), if $p = \frac{1}{2}$, $p(x) = 1 + \frac{1-4p^2}{4x^2} < 1$. Let be $u(x)$ a non-trivial solution of $u''+p(x)u=0$ and $v(x)$ a non-trivial solution of $v''+v=0$, since $1=q(x)>p(x)$, by Theorem 2, we have that $v(x)$ it vanishes at least once between two successive zeros of $u(x)$, it is specified that the distance between two successive zeros of $v(x)$ is π, then $v(x)$ it vanishes at least once between two successive zeros of $u(x)$. In physics, we have a limit associated with measurement, we cannot measure more than the accuracy of the measurement of our instruments; therefore, from the fourth root onwards, the real value of the difference in π, is not significant but is frequently used. In fact, for x large, the difference between two consecutive roots is approximately equal to π, with data from Table 1, these approximations can be seen, $\Delta_1=5{,}52-2{,}40=3{,}12\cong\pi$, in general if u_n they are the roots, then we have $u_{n+1}\approx u_n+\pi$.

	Values of x when $J_p(x)=0$						
X	$J_0(x)$	$J_1(x)$	$J_2(x)$	$J_3(x)$	$J_4(x)$	$J_5(x)$	$J_6(x)$
First zero	2.40	3.83	5.14	6.38	7.59	8.87	9.93
Second zero	5.52	7.02	8.42	9.76	11.06	12.34	13.59
Third zero	8,65	10,17	11,62	13.02	14.37	15.70	17.00
Quarter zero	11.79	13.32	14.80	16,22	17,62	18.98	20.32
Fifth zero	14.93	16.47	17.96	19.41	20.83	22,21	25.59
Sixth zero	18.07	19,61	21,12	22.58	24.02	25.43	26.82
Seventh zero	21,21	22.76	24,27	25.75	27.20	28,63	30.03
Eighth zero	24,35	25.90	27.42	28.91	30.37	31.81	33.23

Table 1: Bessel functions

Source: Author's adaptation.

Figures 1 and 2 show a damped oscillation. Comparing both figures, the figure of the Newmann function, when $x\to0^+$, means that there is a break on the left, while on the right both are similar.

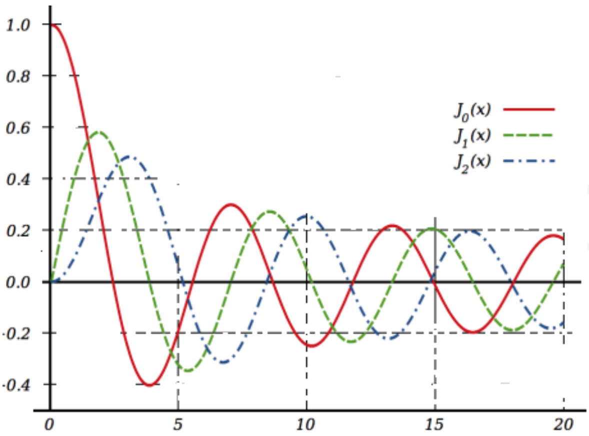

Figure 1: Bessel functions of the first kind for p=0,1,2.

Source: Author's adaptation of Simmons (2018).

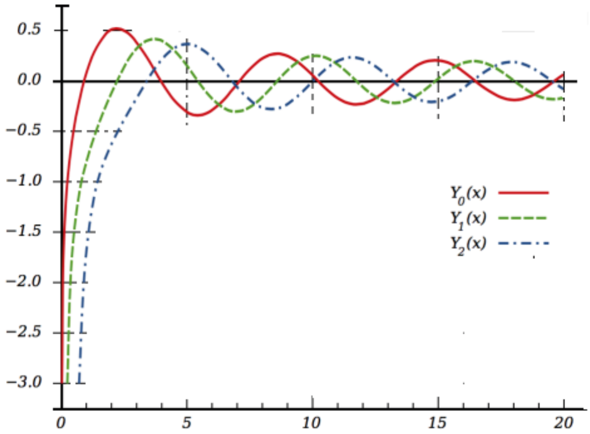

Figure 2: Second Bessel functions for p=0,1,2.

Source: Author's adaptation of Simmons (2018).

Proposition 5. Between two zeros of $J_p(x)$ there is at least one zero of $J_{p+1}(x)$.

Indeed, by proposition 3, $J_p(x)$ has infinite positive zeros, on the other hand, the positive zeros of $x^{-p}J_p(x)$ and the positive zeros of $J_p(x)$ are the same. It is assumed that u_1 and u_2 are two positive zeros of $x^{-p}J_p(x)$, where $u_1<u_2$; the function $x^{-p}J_p(x)$ is continuous on the interval $[u_1, u_2]$, derivable on $]u_1, u_2[$ and we have $u_1^{-p}J_p(u_1) = u_2^{-p}J_p(u_2)$. Rolle's theorem states that there exists a point $r\in]u_1, u_2[$ that satisfies, $\left(\frac{d}{dx}[x^{-p}J_p(x)]\right)_{x=r} = 0$, however, $\left(\frac{d}{dx}[x^{-p}J_p(x)]\right)_{x=r} = -r^{-p}J_{p+1}(r)$.

Therefore, $J_{p+1}(r)$=0. Consequently between two successive positive zeros of $J_p(x)$ there exists at least one positive zero of $J_{p+1}(x)$.

Proposition 6. Between two zeros $J_{p+1}(x)$ there exists at least one zero of $J_p(x)$.

Indeed, according to proposition 3, $J_{p+1}(x)$ has infinite positive zeros, moreover, the infinite zeros of $J_{p+1}(x)$ are the same positive zeros of $x^{p+1}J_{p+1}(x)$. Let u_1 and u_2 be two positive zeros of $x^{p+1}J_{p+1}(x)$, where $u_1<u_2$; the function $x^{p+1}J_{p+1}(x)$ is continuous on the interval $[u_1, u_2]$, derivable on $]u_1, u_2[$ and we have $u_1^{p+1}J_{p+1}(u_1) = u_2^{p+1}J_{p+1}(u_2)$. Rolle's theorem states that there exists a point $s\in]u_1, u_2[$ that satisfies, $\left(\frac{d}{dx}[x^{p+1}J_{p+1}(x)]\right)_{x=s} = 0$, however, $\left(\frac{d}{dx}[x^{p+1}J_{p+1}(x)]\right)_{x=s} = s^{p+1}J_p(s)$.

Therefore, $J_p(s)$=0. Consequently between two successive positive zeros of $J_{p+1}(x)$ there is at least one positive zero of $J_p(x)$, then between two successive positive zeros of $J_{p+1}(x)$ there is at least one positive zero of $J_p(x)$. From propositions 5 and 6, the results are immediate, the positive zeros $J_p(x)$ and of $J_{p+1}(x)$ appear alternately, that is, between each pair of successive zeros of one of them there is exactly one zero of the other. This can be seen in figure 1, where the function $J_0(x)$ and is shown represented $J_1(x)$.

Orthogonality: The eigenvalues and eigenfunctions that we have are inherited from Sturm's problems. Liouville and satisfy properties. Each eigenvalue $\lambda_k = \frac{u_k^2}{L}$ corresponds to only one eigenfunction $J_p\left(\frac{u_k x}{L}\right)$,, two different eigenfunctions are orthogonal with weight x.

Proposition 7. Let $x\in[0, L]$, u_k and be the positive zeros of some u_m fixed $J_n(x)$ Bessel function with $p\geq 0$, it holds, $\int_0^L J_P\left(\frac{u_k x}{L}\right) J_P\left(\frac{u_m x}{L}\right) x dx = \begin{cases} 0 & si\ k \neq m \\ \frac{L^2}{2}[J'_P(u_k)]^2 & si\ k = m \end{cases}$.

The statement is almost an inheritance from the Sturm problem. Liouville, however, we will demonstrate, first we have, for L=1, this makes us lose generality, then we can make a change of variable in the Bessel equation, if we replace the Bessel function, we obtain the identity

$$\left[x\frac{d}{dx}J_p(\varepsilon x)\right]' + \left(\varepsilon^2 x^2 - \frac{p^2}{x}\right)J_p(\varepsilon x) \equiv 0 \qquad (4)$$

Now if $\varepsilon=u_k$ you have it, $\left[x\frac{d}{dx}J_p(u_k x)\right]' + \left(u_k^2 x^2 - \frac{p^2}{x}\right)J_p(u_k x) \equiv 0$ (5), being a generic value, u_k it is already specific. Now, equation (4) by $xJ_p(u_k x)$, equation (5) by $xJ_p(\varepsilon x)$, consequently by subtracting both we obtain, $J_p(\varepsilon x)\left[x\frac{d}{dx}J_p(u_k x)\right]' - J_p(u_k x)\left[x\frac{d}{dx}J_p(\varepsilon x)\right]' + (u_k^2 - \varepsilon^2)xJ_p(\varepsilon x) = 0$, then it is integrated from 0 to 1, the integration is by parts

$$J_p(\varepsilon x)\left[x\frac{d}{dx}J_p(u_k x)\right]' = \left[J_p(\varepsilon x)x\frac{d}{dx}J_p(u_k x)\right]' - x\frac{dJ_p(\varepsilon x)}{dx}\frac{dJ_p(u_k x)}{dx}$$

consequently,

$$= \left[J_p(\varepsilon x)x\frac{d}{dx}J_p(u_kx) - J_p(u_kx)x\frac{d}{dx}J_p(\varepsilon x)\right]_0^1$$

$$= (\varepsilon^2 - u_k^2)\int_0^1 x\,J_p(u_kx)J_p(\varepsilon x)dx,$$

that is to say,

$[J_p(\varepsilon x)xJ'_p(u_kx)u_k - J_p(u_kx)xJ'_p(\varepsilon x)\varepsilon]_0^1 = (\varepsilon^2 - u_k^2)$ $\int_0^1 x\,J_p(u_kx)J_p(\varepsilon x)dx$ evaluating limits, $u_kJ_p(\varepsilon)J'_p(u_k) = (\varepsilon^2 - u_k^2)\int_0^1 x\,J_p(u_kx)J_p(\varepsilon x)dx$ where it comes from,

$$\int_0^1 x\,J_p(u_kx)J_p(\varepsilon x)dx = \frac{u_kJ_p(\varepsilon)J'_p(u_k)}{(\varepsilon^2-u_k^2)},$$

L' Hópital applies, $\int_0^1 x[J_p(u_kx)]^2\,dx = \frac{1}{2}[J'_p(u_kx)]^2$ for a change of variable $\frac{x}{L} = z$, we arrive at the orthogonality of the Bessel function, $\int_0^L \left[J_p\left(\frac{u_kx}{L}\right)\right]^2 x\,dx = \frac{L^2}{2}[J'_p(u_k)]^2$.

Bessel series: For many investigations in mathematical physics it is appropriate to obtain a function in terms of Bessel. A series of the form is considered (Brito et al, 2003; Pinsk, 2003),

$$b(x) = \sum_{n=1}^{\infty} a_nJ_p(u_nx) = a_1J_p(u_1x) + a_2J_p(u_2x) + \cdots \qquad (6)$$

where $b(x)$ is defined on the interval [0,1] and u_n are the zeros of a fixed Bessel function $J_p(x)$, with $p \geq 0$, we analyze on the interval [0,1] and by changing variables it is possible to extend to the interval $[0,L]$.

In the case that it is possible to write the function $b(x)$ as an expansion of the form (6), then we multiply in both members by $xb(x)J_p(u_mx)$ the result, $xb(x)J_p(u_mx) = \sum_{n=1}^{\infty} xa_nJ_p(u_nx)$.

Integrating each term in [0,1], applying orthogonality, we obtain,

$$\int_0^1 xb(x)J_p(u_mx)dx = \int_0^1\left[\sum_{n=1}^{\infty} xa_nJ_p(u_mx)J_p(u_nx)\right]dx$$

$$= \int_0^1 xa_m\left(J_p(u_mx)\right)^2dx = \frac{1}{2}a_mJ_{p+1}(u_m)^2.$$

We change it m by n, we obtain the formula for the a_n and properly it is the Fourier- Bessel series $a_n = \frac{2}{J_{p+1}(u_m)^2}\int_0^1 xa_m\left(J_p(u_mx)\right)^2dx$.

Regarding convergence, there are conditions under which the Bessel series with sum converges $b(x)$.

Theorem 8. If $b(x)$ and $b'(x)$ have at most finitely many jump discontinuities on the interval [0,1]. If, then the $x\in]0,1[$ Bessel series converges to $b(x)$ when x is a point of continuity of this function, and to $\frac{1}{2}[b(x^-) + b(x^+)]$ when x is a point of discontinuity. However, in the extremum $x=1$ the series converges to zero independently of any function $J_p(u_n)$, it also converges to zero on $x=0$, if $p>0$ and $b(0^+)$ if $p=0$.

Bessel integral: It is about deducing the Bessel integral, finding the functions $J_p(x)$ in the integral formula. From the expression, $\frac{x}{2}(e^{i\varphi} - e^{-i\varphi}) = ixsen\varphi$, with the generation of the generating function becomes the sum (Muller, 2001), $e^{ixsen\varphi} = \sum_{n=-\infty}^{\infty} J_n(x)e^{in\varphi}$, we equalize the real parts, we have $\cos(xsen\varphi) = \sum_{n=-\infty}^{\infty} J_n(x)\cos(n\varphi)$, by the relation $J_{-n}(x) = (-1)^nJ_n(x)$, considering the parity of the cosine, we write

$$\cos(xsen\varphi) = J_0(x) + \sum_{n=1}^{\infty} J_n(x)\cos(n\varphi) + \sum_{n=1}^{\infty}(-1)^n\cos(n\varphi) = J_0(x) + 2\sum_{n=1}^{\infty} J_{2n}(x)\cos(2n\varphi).$$

As for $\varphi=0$, the series remains $1 = J_0(x) + 2\sum_{n=1}^{\infty} J_{2n}(x) = J_0(x) + 2J_2(x) + \cdots$,while when $\varphi = \frac{\pi}{2}$, is the series $cosx = J_0(x) + 2\sum_{n=1}^{\infty} J_{2n}(x)\cos(n\pi) = J_0(x) - 2J_2(x) + \cdots$. When the imaginary part is equalized, we have $\text{sen}(xsen\varphi) = \sum_{n=-\infty}^{\infty} J_n(x)\text{sen}(n\varphi)$, by the relation $J_{-n}(x) = (-1)^nJ_n(x)$, considering that sine is odd, we write

$$\text{sen}(xsen\varphi) = J_0(x) + \sum_{n=1}^{\infty} J_n(x)\,\text{sen}(n\varphi) + \sum_{n=1}^{\infty}(-1)^{n+1}\text{sen}(-n\varphi) = 2\sum_{n=1}^{\infty} J_{2n-1}(x)\,\text{sen}((2n-1)\varphi).$$

When $\varphi = \frac{\pi}{2}$ is the series $senx = 2\sum_{n=1}^{\infty} J_{2n-1}(x)\cos(n\pi) = 2J_1(x) - 2J_3(x) + \cdots$.

These results show the close relationship of the Bessel functions with the trigonometric functions sine and cosine.

Proposition 9. Let the p Bessel function of order p y be $\varphi\in[0,\pi]$ an integer $J_n(x)$, then the integral is fulfilled $J_p(x) = \frac{1}{\pi}\int_0^{\pi} cos(p\varphi - xsen\varphi)d\varphi$. Indeed, by multiplying $\cos(xsen\varphi)$ by $\cos(m\varphi)$

and sen($xsen\varphi$) by sen($m\varphi$), we obtain, $\cos(m\varphi)\cos(xsen\varphi) = \sum_{p=-\infty}^{\infty} J_p(x)cos\,(p\varphi)\cos(m\varphi)$, $\text{sen}(m\varphi)\,\text{sen}(xsen\varphi) = \sum_{p=-\infty}^{\infty} J_p(x)sen\,(p\varphi)\,\text{sen}(m\varphi)$.

Applying the sum member by member, with application of trigonometric identities to obtain, $cos(m\varphi - xsen\varphi) = \sum_{p=-\infty}^{\infty} J_p(x)\cos(p-m)\varphi$, the term is $p = m$ is $J_m(x)$. Therefore, we can write in the form, $cos(m\varphi - xsen\varphi) = J_m(x) + \sum_{\substack{p=-\infty \\ p\neq m}}^{\infty} J_p(x)\cos(p-m)\varphi$, Integrating between φ=0 and $\varphi=\pi$ in both members, we have, $\int_0^{\pi} cos(m\varphi - xsen\varphi)d\varphi = J_m(x)\pi$, Changing m to p, gives the Bessel integral, $J_p(x) = \frac{1}{\pi}\int_0^{\pi} cos(p\varphi - xsen\varphi)d\varphi$.

THE EQUATION OF MOTION

For this investigation, a membrane is a thin uniform sheet of a flexible material in a state of uniform tension and clamped along a closed curve in the xy plane. The membrane is displaced slightly from its equilibrium position and released.

It is a matter of formulating a partial differential equation that describes the motion. In this case, only small oscillations of a free membrane are considered. Assuming that the vibrations are very small and that the membrane only moves in the direction u with displacement given in time t by a function $u=u(x,y,t)$. A small piece of membrane is taken, limited by the vertical planes that pass through the points (x,y), $(x+\Delta x,y)$, $(x+\Delta x,y+\Delta y)$, of the $(x,y+\Delta y)$ xy plane. Then, the mass of the piece considered is $m\Delta x\Delta y$, being the mass per unit area. By Newton's second law, the force acting on it in the direction of u is

$$m\Delta x\Delta y\,\frac{\partial^2 u}{\partial t^2} \qquad (7)$$

If the membrane is in the equilibrium position, the physical meaning of the tension T is: along any segment of length Δs, the material on one side exerts a force on the material on the other side. This force has magnitude $T\Delta s$ and is normal to the segment Δs. Thus, the forces on opposite edges in the small piece of membrane are parallel to the xy plane, so they cancel each other out.

On the other hand, if the membrane is slightly curved, although the deformation is very small and the tension is still T, it does not act parallel to the xy plane, but rather acts parallel to the tangent plane; therefore, it presents an appreciable vertical component. The curvature of our piece of membrane produces different magnitudes for the vertical components on opposite edges and is responsible for the restoring forces that produce the movement (Brito, 2011).

Let a fragment of membrane be noted by PQRS as being only slightly curved. On the edges PQ and SR the forces are perpendicular to the x-axis and nearly parallel to the y-axis, their small components at the points (x,y), $(x,y+\Delta y)$ are approximately $-T\Delta x\left(\frac{\partial u}{\partial y}\right)_y$ and $T\Delta x\left(\frac{\partial u}{\partial y}\right)_{y+\Delta y}$, whose sum is, $T\Delta x\left[\left(\frac{\partial u}{\partial y}\right)_{y+\Delta y} - \left(\frac{\partial u}{\partial y}\right)_y\right]$, similarly for edges PS and QR, the components at points (x,y), $(x+\Delta x+y)$ are approximately $-T\Delta y\left(\frac{\partial u}{\partial x}\right)_x$ and $T\Delta y\left(\frac{\partial u}{\partial x}\right)_{x+\Delta x}$, whose sum is, $T\Delta y\left[\left(\frac{\partial u}{\partial x}\right)_{x+\Delta y} - \left(\frac{\partial u}{\partial x}\right)_x\right]$.

Therefore, the total force in the direction of u, neglecting all external forces, is approximately $F = T\Delta y\left[\left(\frac{\partial u}{\partial x}\right)_{x+\Delta x} - \left(\frac{\partial u}{\partial x}\right)_x\right] + T\Delta x\left[\left(\frac{\partial u}{\partial y}\right)_{y+\Delta y} - \left(\frac{\partial u}{\partial y}\right)_y\right]$, so that (7) can be expressed, $m\,\frac{\partial^2 u}{\partial t^2} = T\,\frac{\left(\frac{\partial u}{\partial x}\right)_{x+\Delta x} - \left(\frac{\partial u}{\partial x}\right)_x}{\Delta x} + T\,\frac{\left(\frac{\partial u}{\partial y}\right)_{y+\Delta y} - \left(\frac{\partial u}{\partial y}\right)_y}{\Delta y}$, tending to the limit, $\Delta x\to 0$, $\Delta y\to 0$ we obtain the derivatives, $\frac{\partial^2 u}{\partial x^2} = \lim_{\Delta x\to 0}\frac{\left(\frac{\partial u}{\partial x}\right)_{x+\Delta x} - \left(\frac{\partial u}{\partial x}\right)_x}{\Delta x}$ and $\frac{\partial^2 u}{\partial y^2} = \lim_{\Delta y\to 0}\frac{\left(\frac{\partial u}{\partial y}\right)_{y+\Delta y} - \left(\frac{\partial u}{\partial y}\right)_y}{\Delta y}$.

Finally, if we denote $c^2 = \frac{T}{m}$, we arrive at the two-dimensional wave equation, $c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = \frac{\partial^2 u}{\partial t^2}$.

CIRCULAR MEMBRANE PROBLEM

To solve it, you have a circle, the imagination is the following, you have a drum, it is attached to that leather membrane, attached to one of its two ends, analyze what will happen with that drum when you hit it in the center, as a result it will begin to oscillate, with radius of the drum r. The problem of small oscillations of a circular membrane of radius

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with fixed ends, since the edge is fixed and is expressed as $\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} - \nabla^2 u = 0$, with the condition on the edge $u(r,\varphi,t)=0$, another additional condition the value that forms u with the center $|u(r,\varphi,t)|<M$, the absolute value because it is a vibration up and down; that is, it forms a finite value, the vibration can be tiny. The problem together with the boundary conditions is formulated as follows (Cruz, 2020) $\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} - \nabla^2 u = 0, u(r,\varphi,t) = 0, |u(r,\varphi,t)| < M$.

In the problem, it is not yet being indicated what the coordinates are, although the reasonable thing would be to use cylindrical coordinates, since it is more comfortable when starting with the analysis. The membrane is seen from one side where the ends are fixed and it will have to be oscillating around those ends, it means that when choosing an axis u that depends on the distance to the center r, the angle φ, the time t. The formulated equation is when the distance to the center or at any point of the membrane, that depends on the way the problem is posed, normally indicating where we touch it, will be associated with the initial conditions, at that instant a pulse is given, imagine for a moment that you have a pot lid, then you hit it with a spoon, those are the initial conditions, we can give it speed, an initial deformation, that is relative.

On the other hand, the equation is zero, $\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} - \nabla^2 u = 0$ because there is no vibrating source, i.e. external force. The initial conditions part will not be worked on, rather the second order boundary conditions and that leads to a Sturm problem. Liouville. We want to see where some data come from, let's say the magnitude that represents the speed of sound, because in a vibration it generates a sound and considering polar coordinates, so it will be useful to know the Laplacian in different coordinates, $\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2}$. We could point out that they are cylindrical coordinates, but only the two-dimensional part is taken. It is important that the differential operator ∇, there are many problems that result in coordinates other than the Cartesian, for our case the symmetry is axial, that is, everything that happens around this point on the vertical axis is the same around that axis, it is convenient to use a type of coordinate that has axial symmetry, in this case cylindrical or polar. In others.

Regarding the third condition, it must be limited. It can be stretched but not infinitely, because for it to have a solution it must be limited. If it is infinite it is assumed that it broke, in this case it is a different problem. For the drum problem, it is assumed that the blow is not very strong so that it does not break. The value of M can be very large, but we are looking at the case of small oscillations; for large oscillations the problem posed changes a little, it must be understood as small in the sense of the radius.

The interest is in the oscillations that have a harmonic behavior; therefore, a solution of the form is sought, $u(r,\varphi,t)=U(r,\varphi)e^{-iwt}$, since it has to do with sine and cosines it takes the exponential form, $e^{-iwt}=\cos(wt)+isen(wt)$, depending on the problem one could take sine or cosine, it is the representation of a harmonic oscillation in general. By replacing in the nabla operator, taking into account $e^{-iwt} \neq 0, \forall t, w$, we have, $\nabla^2 u = -k^2 U, k^2 = -\frac{w^2}{c^2}$ and $u(a,\varphi) = 0$, the other condition is the same $|u(a,\varphi)|<N$, it is not the same M or it could be, what is important is that it is bounded. We have obtained the boundary problem for the wave equation, based on the Helmholtz equation $\nabla^2 u=-k^2 U$, will be solved using the method of separation of variables. Let the equation of separation of variables be, $u(r,\varphi)=R(r)\Phi(\varphi)$, by substituting in $\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2}$ in the explicit form $\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \varphi^2} + k^2 u = 0$, $\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \varphi^2} + k^2 u = 0$ making a change of function $R''\varphi + \frac{1}{r}R'\Phi + \frac{1}{r^2}R\Phi'' + K^2 R\Phi = 0$, when making a division between $R=\Phi$ we have $\frac{R''}{R} + \frac{1}{r}\frac{R'}{R} + \frac{1}{r}\frac{\Phi''}{\Phi} + K^2 = 0$, that is, $\frac{rR''+R'}{rR} + \frac{\Phi''}{r^2\Phi} = -k^2$

separating variables $\frac{(rR')'}{rR}+k^2=-\frac{\Phi''}{r^2\Phi}$, that is, $\frac{r(rR')'}{R}+r^2k^2=-\frac{\Phi''}{\Phi}=m$.

For Φ, $\Phi''+m\Phi=0$ it is the equation of a simple harmonic motion, being m a parameter of separation of variables, which is taken as positive, keep in mind that the value of the parameter w will be defined when using the initial conditions. For the radial part, we have $R''+\frac{1}{r}R'+\left(k^2-\frac{m^2}{r^2}\right)$ $R=0$, the change of variable $\delta=kr$, allows us to do $R_{(r)}=Y_{(\delta)}$, it is like $R_r=Y_y=y_\delta y_r=\frac{\partial y}{\partial \delta}\frac{\partial \delta}{\partial r}=ky''$. As for the second derivative, $Y''+\left(\frac{1}{\delta}\right)Y'+\left(1-\frac{1}{\delta^2}m\right)Y=0$.

To which appropriate conditions must be added to obtain a contour problem of the form, $Y''+\left(\frac{1}{\delta}\right)Y'+\left(1-\frac{m^2}{\delta^2}\right)Y=0,\ Y(ka)=0,\ |Y(0)|<N$, which represents a Sturm problem Liouville, with Bessel differential equation is, then we see that the problem of the circular membrane and in general in most of the problems that we use the Laplace operator in cylindrical or polar coordinates usually leads us to a Bessel differential equation, it is a boundary problem, we have a Bessel differential equation under boundary conditions or the Sturm problem Liouville. Regarding the criterion for the change of variable $\delta=kr$, it has to do with circular symmetry, a dependence on the radius, a frequent change when there are radial dependencies and it is the way to reach the Bessel equation (Simmons, 1993).

PROBLEM EIGENVALUE AND EIGENFUNCTION

We have the Bessel equation

$$x^2y''+xy'+(x^2-p^2)y=0 \qquad (8)$$

with the conditions on the interval $\langle 0,L\rangle$ with $y(x)$ finite $x\to0^+$, $y(L)$, is the boundary problem to be solved. If, is a fixed number, then it is an eigenvalue problem, the problem satisfies $p\geq0$ Sturm 's conditions Liouville; therefore, the eigenvalues are greater than zero and ordered in decreasing order, $0\leq\lambda_1<\lambda_2<\lambda_3<\cdots<\lambda_k<\cdots$, to infinity λ, our boundary problem involves first finding which values of λ that are solutions to the problem, are the solutions corresponding to those λ. We analyze cases, for $\lambda=0$, is the equation $x^2y''-xy'-p^2y=0$, we are going to solve this equation in two cases, if $p=0$, then, $xy''-y'=0$, with general solution $y(x)=c_1+c_2lnx$.

When the conditions are taken $y(l)=0$, we have $0=c_1+c_2lnL$, which requires $c_1=0$, $c_2=0$, implies that there is only one trivial solution $y(x)=0$, in this case there is no solution, since non-trivial solutions are required.

If in (8) $p>0$, the characteristic equation is constructed, then it is written with two linearly independent solutions $y(x)=c_1x^p+c_2x^{-p}$.

In case $x\to0^+$, for it to be finite again we have a trivial solution, so there is no solution. Therefore, we discard the fact that $\lambda=0$, so all the eigenvalues are positive. Doing $\lambda=\varepsilon^2$, where $\varepsilon>0$, then $x^2y''+xy'+(\varepsilon^2x^2-p^2)y=0$, we must take into account that $x^2y''+xy'+(\lambda x^2-p^2)y=0$ is not the Bessel equation, since it does not have the factor λx.

The equation in its self-adjoint form $(xy')'-\frac{p^2}{x}y+(\varepsilon^2x)y=0$, we take to another conventional form by making the change of variable, $\varepsilon x=t$ we have, $t^2y_{tt}+ty_t+(t^2-p^2)y=0$, which is a Bessel equation of order p, whose general solution is, $y=c_1J_p(t)+c_2N_p(t)$, that is, taking into account the change is written, $y=c_1J_p(\varepsilon x)+c_2N_p(\varepsilon x)$ from $y(L)=0$, we obtain $c_1J_p(\varepsilon L)=0$, since c_1 it must be different from zero, then $J_p(\varepsilon L)=0$, but, being εL roots of the Bessel function, then $\varepsilon L=u_k$, thus $\varepsilon=\frac{u_k}{L}$, from which we have the eigenvalues $\lambda=\frac{u_k^2}{L^2}$. Therefore, the eigenvalues are $\lambda_1=\frac{u_1^2}{L^2},\lambda_2=\frac{u_2^2}{L^2},\lambda_3=\frac{u_3^2}{L^2},\ldots,\lambda_k=\frac{u_k^2}{L^2}$, in this way the importance of the roots of the Bessel equation and the eigenfunctions are established, $y_1=J_p\left(\frac{u_1}{L}x\right)$, $y_2=J_p\left(\frac{u_2}{L}x\right)$, $y_3=J_p\left(\frac{u_3}{L}x\right),\ldots,y_k=J_p\left(\frac{u_k}{L}x\right)$, the eigenvalues associated with the eigenfunctions have been found.

DISCUSSION

In the circular membrane problem, $u(r,\varphi,t)=U(r,\varphi,)e^{-iwt}$ by separation of variables, a function was obtained for Φ and R, where solutions are obtained for $\Phi(\varphi)=e^{im\varphi}$, $u(r,\varphi,t)=R(r).\Phi(\varphi)e^{-iwt}$, while is a $R(r)$ Bessel function, and are radial solutions, $R(r) = J_m(u_k^m r)$. The general solution of the problem is $u(r,\varphi,t) = \sum_{m,k} A_{mk} e^{-iw_{mk}t} e^{im\varphi} J_m(u_k^m r)$.

From now on, everything else depends on the specific problem, keeping in mind that you are working on a membrane which is a flat surface, we consider it harmonic, that is why w it A_{mk} is the greatest amplitude, after that everything decays.

In the case of using the initial conditions, with respect to the case of a membrane that is displaced to a $u=f(r)$, regardless of the variable φ and that at the moment $t=0$ is released starting from rest; that is, with the initial condition $u(r,\varphi,0)=f(r)$ and. The shape of $\left.\frac{du}{dt}\right|_{t=0} = 0$ at any time after, $t>0$ must be determined $u(r,\varphi,t)$ (Davis,1992).

According to the condition, the initial form is independent of φ; then $\Phi(\varphi)$ it has to be constant and p it has to be zero. Therefore, denoting the positive zeros of $J_0(x)$ by $u_1, u_2, u_3, \cdots, u_k, \cdots$, then $J_0(u_k r)[c_1 cos(u_k at) + c_2 sen(u_k at)], k = 1,2,3, \ldots,$ by by $\left.\frac{du}{dt}\right|_{t=0} = 0$, we have that and then, the particular solutions are the constant multiples of the functions $J_0(u_k r)[c_1 cos(u_k at)]$, $k=1,2,3,\cdots$, the sums of solutions also form the space of solutions, then the general solution is the infinite series, $u = \sum_{k=1}^{\infty} a_k J_o(u_k r) cos(u_k at)$.

To satisfy the initial condition $u(r,\varphi,0)=f(r)$, by making $t=0$, we set the results to $f(r)$. Theorem 9, ensures the stable behavior of $f(r)$, so the series converges to $f(r)$. The coefficients a_k can be defined by the expression

$$a_k = \frac{2}{[J_1(u_k)]^2} \int_0^1 r f(r) J_0(u_k r) dr \qquad (9)$$

Thus, (9) is a formal solution to the membrane problem, which satisfies the boundary condition and the initial conditions stated.

CONCLUSIONS

Bessel function is a mathematical solution for solving problems in cylindrical coordinates. It is used in fields such as engineering and originated in astronomy. Although it can be complicated to understand, its correct interpretation is very useful for finding solutions in different situations.

Vibration causes different types of waves to form, when you push something it moves it from its place and when we study noise, vibration and severity in a system we do so with some specific characteristics. The ways to control vibrations are different from the ways to control noise, although vibrations and noise often have the same causes and can cause noise, but they propagate differently.

In the end, calculations are made to describe how a circular membrane vibrates. Formulas involving time and position on the membrane are used, and these comply with specific rules at the edges of the membrane. The solution can be written as a sum of functions where the details correspond to given point conditions; thus, the Bessel functions act to organize the motion caused by the recovery of force caused by changes in deformation.

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