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USING GOOGLE EARTH TO EXPLORE CONCEPTS OF SPHERICAL GEOMETRY IN MATHEMATICS EDUCATION[1]

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1. This work was presented in the Teaching Section of the XL CNMAC – Congresso Nacional de Matemática Aplicada e Computacional, an online event that took place in September 2021.

Abstract: In this paper we present some concepts of spherical geometry, such as spherical surface, spherical line, spherical line segments, spherical coordinates, spherical triangles and spherical trigonometry, and propose two activities that contextualize these concepts. In the activities, designed for the Mathematics Degree Course, we covered cartography concepts using *Google Earth*, *a* computer program that renders *a* 3D representation of the Earth based on satellite images. The activities were organized with the aim of equipping mathematics undergraduate students to teach non-Euclidean geometry, specifically spherical geometry, in accordance with the Paraná State Department of Education's Curriculum Framework for High School, on the teaching of non-Euclidean geometry in high school. We conclude that *Google Earth* is *a* fantastic tool for teaching spherical geometry, enabling interdisciplinary interactions with geography, specifically cartography.
Keywords: Non-Euclidean geometries. Spherical geometry. Cartography. Teaching mathematics. Curriculum Framework for Secondary Education in Paraná.

INTRODUCTION

We live on *a* quasi-spherical surface. Euclidean geometry (Euclid, 2009) is not enough to explain our world, let alone the universe that surrounds it (Wolfson, 2005).

> The world around us has provided reasons for much of the development of mathematics. The fact that the Earth is itself *a* sphere, and the sky has the appearance of an inverted shell above us, has put curves, circles and spheres at the heart of geometry since the earliest times. These features of the world have given rise to challenging problems in explaining, representing and modeling the universe as we know it. How can we represent the three-dimensional environment we see in *a* flat drawing? How can we map the spherical Earth on *a* two-dimensional map? Struggling with these problems brought up other questions about dimensions and geometry. Sometimes the world doesn't seem to match the geometry established by Euclid, which has been accepted for 2000 years. New models for dealing with these situations have opened up exciting new avenues for mathematicians (Rooney, 2012, p. 97).

In this way, the study of non-Euclidean geometries, such as spherical geometry, for example, is *a* pertinent topic for teacher training and needs to be addressed in mathematics degree courses. In addition, the Curriculum Framework for Secondary Education in Paraná (Paraná, 2021) emphasizes the teaching of non-Euclidean geometries in secondary education.

> Non-Euclidean geometries emerged between the end of the century and the end of the 20th century.
> They gained importance at the beginning of the 20th century with the Theory of General Relativity and later with the development of Fractal Theory. The study of fractals at this stage of education allows students to develop their creativity, intuition and imagination, understanding the processes of regularity and interaction of these geometric entities. Their emergence has shown that, in order to understand various problems of reality and the scientific world, in addition to the mathematical relationships with Euclidean geometry itself, it is necessary to incorporate the study of non-Euclidean geometries into basic education (Paraná, 2021, p. 541).

Unfortunately, the National Common Core Curriculum - BNCC (Brazil, 2018) does not establish parameters for teaching non-Euclidean geometries. However, the BNCC for Mathematics and its Technologies proposes the use of technological tools and computer programs.

> It is also worth noting that the use of technologies provides students with alternatives for varied experiences that facilitate learning and reinforce their ability to reason logically, formulate and test conjectures, assess the validity of reasoning and construct arguments (Brasil, 2018, p. 536).

Thus, in this work we propose two activities using Google Earth (Nós; Almeida, 2023a, 2023b) that contextualize some characteristics and properties of spherical geometry, based on theorems demonstrated in Brannan *et al.* (2012), Doria (2019) and Motta (2018), and illustrated in three-dimensional figures built with CorelDRAW (Corel, 2024), which can be replaced by GeoGebra 3D (GeoGebra3D, 2024; Nós; Silva, 2018, 2020). These activities were planned for the Mathematics Degree Course (Nós; Motta, 2021; Motta; Nós, 2022), and in the research work we also propose spherical geometry activities for primary and secondary schools (Motta, 2018).

SPHERICAL GEOMETRY

Spherical geometry is *a* specialty of elliptical geometry, both developed by the German mathematician Georg Friedrich Bernhard Riemann (1826-1866). Spherical geometry satisfies Postulate 1, called the elliptic postulate of parallels (Brannan *et al.*, 2012), which characterizes it as *a* non-Euclidean geometry (Burton, 2011; Eves, 2004; Rooney, 2012).

Postulate 1: Given *a* line *r* and *a* point *P* not belonging to *r,* every line that passes through *P* intersects *r.*

Spherical geometry is geometry defined on the surface of *a* S^2_ρ of *a* sphere E^2_ρ with center *0=(x0, y0, z0)* and radius *p* illustrated in Figure 1. Thus, *a* point *P = (x,y,z)* of abscissa x, ordinate y and coordinate z belongs to S^2_ρ if, and only if,

$$(x-x_0)^2 + (y-y_0)^2 = (z-z_0)^2 = p^2. \quad (1)$$

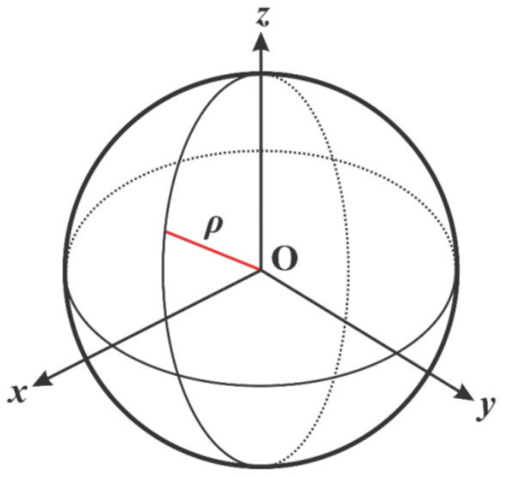

Figure 1: Spherical surface S^2_ρ with center *0* and radius *p*

Source: The authors.

The spherical line is *a* maximum circumference of S^2_ρ (Brannan et al., 2012), that is, *a* circle whose center is the center of the spherical surface. Circumferences of S^2_ρ whose centers do not coincide with the center of the spherical surface are called minimal, as illustrated in Figure 2(a). Unlike the Euclidean line, which is infinite, the spherical line has *a* finite length. Two spherical lines, i.e. two maximum circumferences of S^2_ρ always intersect at two diametrically opposite points - Figure 2(b), called antipodes.

The coordinates of *a* point $P \in S^2_\rho$ called the spherical coordinates of *P* are of great importance for proving results in spherical geometry. For every point *P= (x,y,z)* $\in S^2_\rho$ there are two angles ψ and φ bounded by the orthogonal axes *x0y0z*. These two angles are sufficient to define the spherical coordinates of P.

Let's consider the equatorial plane of S^2_ρ in which the points P_1 e *A=(p,0,0)* belong to the equator, which is the maximum circle passing through the points *A = (p,0,0)* e *B = (0,p,0).* We denote the angle AÔP1=φ which rotates around the axis z axis, i.e, $0 \leq \varphi < 2\pi$, and the angle *NÔP1=*ψ, where P $\in S^2_\rho$ e ψ does not exceed an angular measure of π radians, i.e, $0 \leq \psi \leq \pi$. The positive direction for φ e ψ is counterclo-

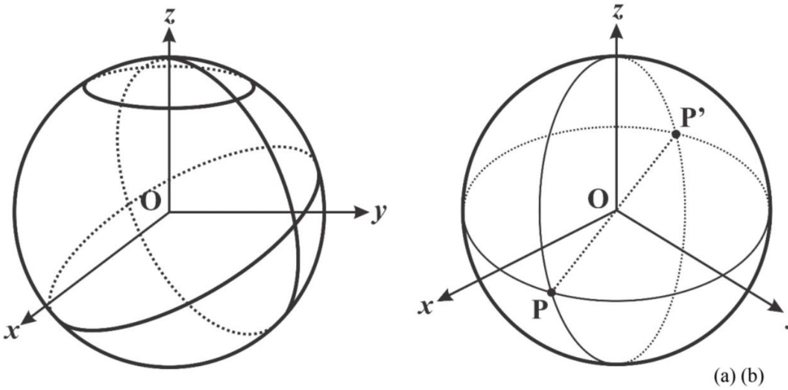

Figure 2: Circumferences of S_ρ^2 (a) minimum; (b) maximum

Source: The authors.

ckwise, defined by rotations in S_ρ^2 (Brannan et al., 2012; Doria, 2019). To define the spherical coordinates of *P* as *a* function of the angles φ e ψ given by Theorem 1, we must construct *a* maximum circle *C* that passes through *P1, P* and the pole *N* as shown in Figure 3. The poles *N* e *S* are defined by the intersection of the spherical surface with its axis of revolution.

Theorem 1: If P=(x,y,z) is *a* point belonging to S_ρ^2, with $0\leq \varphi \leq 2\pi$ e $0\leq \psi \leq \pi$ then the spherical coordinates of *P* are given by

$$P = p(cos\varphi\ sen\psi,\ sen\varphi\ sen\psi,\ cos\psi). \qquad (2)$$

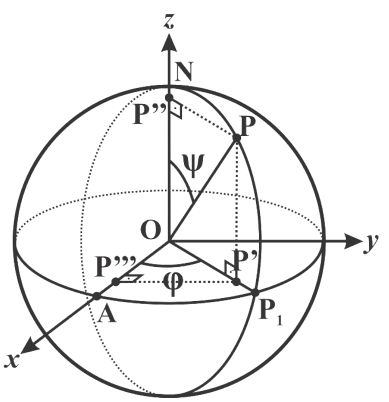

Figure 3: Coordinates of the point $P \in S_\rho^2$ as *a* function of the angles φ e ψ

Source: The authors.

Spherical line segments are called geodesics. *A* geodesic of S_ρ^2 is an arc of maximum circumference or, more specifically, it is every arc of maximum circumference that minimizes the distance between two points belonging to the spherical surface (Doria, 2019) - Theorem 2.

Theorem 2: If *P, Q* $\in S_\rho^2$ are two non-diametrically opposite points, then there is *a* single geodesic at S_ρ^2 connecting *P a Q*.

Given two points *P, Q* $\in S_\rho^2$, the arc *PQ* is defined by the angle $P\hat{O}Q = \alpha$ where 0 is the center of *P, Q* $\in S_\rho^2$ e $0 < \alpha \leq \pi$. Thus, the length of the geodesic with ends *P* e *Q* is the distance between the points *P* e *Q* on the spherical surface - Theorem 3, as illustrated in Figure 4.

Theorem 3: If $P=(p_1, p_2, p_3)$ and $Q=(q_1, q_2, q_3)$ are two points belonging to S_ρ^2 e α is the central angle corresponding to the arc of maximum circumference with ends *P* e *Q* then the distance $d_{S_\rho^2}(P,Q)$ from *P a Q* is equal to

$$d_{S_\rho^2}(P,Q) = \rho\alpha = \rho\ arccos\left(\frac{p_1q_1+p_2q_2+p_3q_3}{\rho^2}\right). \qquad (3)$$

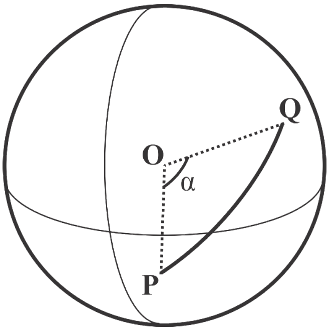

Figure 4: *A* geodesic of S_ρ^2: the arc *PQ* or *d(P,Q)*

Source: The authors.

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In Euclidean geometry, three distinct non--collinear points define *a* single triangle; in spherical geometry, three distinct points, not simultaneously belonging to the same geodesic and not diametrically opposed two by two, define eight spherical triangles, as illustrated in Figure 5(b). In *a* spherical triangle *ABC* - Figure 5(a): the points *A*, *B* e *C* are the vertices; the geodesics *AB*, *AC* e *BC* are the sides; α, β e γ are the internal angles.

Considering the points *A*=(*p*,0,0), *B*=(0,*p*,0) e *C*=(0,0,*p*) the spherical triangle *ABC* has an area, defined by Theorem 4, equivalent to one eighth of the spherical surface - Figure 5(c). In this case, the internal angles of *ABC* are all straight, i.e. equal to $\frac{\pi}{2}$ or 90° because the axes O_x, O_y e O_z are orthogonal to each other and each of the geodesics *AB*, *AC* e *BC* is contained in *a* single plane that is orthogonal to the planes containing the other two. *A* spherical triangle with these characteristics is called *a* trirectangle, and the sum of the internal angles of this triangle is equal to 3π or 270°. This is an important and interesting result since, unlike Euclidean geometry, the spherical triangle *ABC* has internal angles whose sum is greater than π or 180° - Theorem 5.

Theorem 4: If *ABC* is *a* triangle in S_ρ^2 whose interior angles measure α, β e γ then the area $\mathcal{A}$ of *ABC* is equal to

$$\mathcal{A}(\Delta ABC) = \rho^2[(\alpha + \beta + \gamma) - \pi]. \tag{4}$$

Measure (4) is positive and can be rewritten as

$$\mathcal{A}(\Delta ABC) = \rho^2 E, \tag{5}$$

where $E=\alpha+\beta+\gamma$ is the deficiency (excess) of the spherical triangle *ABC* i.e. how much the sum of the internal angles of *ABC* exceeds π. Relation (5) makes it possible to demonstrate the sum of the internal angles of *a* spherical triangle - Theorem 5.

Theorem 5: If *ABC* is *a* triangle in S_ρ^2 with interior angles of measures α, β e γ then the sum of *ABC* the internal angles of is given by

$$\pi < \alpha + \beta + \gamma < 3\pi. \tag{6}$$

The lower and upper bounds in inequality (6) can be proven through practical activities, both in secondary school and in mathematics degrees (Motta, 2018).

To calculate the area of *a* spherical triangle using relation (4), we need to know the measures of the internal angles. Once we know the measures of the sides of the spherical triangle - Theorem 3, the law of spherical sines and the law of spherical cosines - Theorems 6 and 7, respectively, make it possible to calculate the measure of the internal angles.

Theorem 6: If *ABC* is *a* triangle in S_ρ^2 whose interior angles are α, β e γ and whose sides opposite these angles measure respectively, *a*, *b* e *c* then

$$\frac{sen\alpha}{sen\left(\frac{a}{\rho}\right)} = \frac{sen\beta}{sen\left(\frac{b}{\rho}\right)} = \frac{sen\gamma}{sen\left(\frac{c}{\rho}\right)}. \tag{7}$$

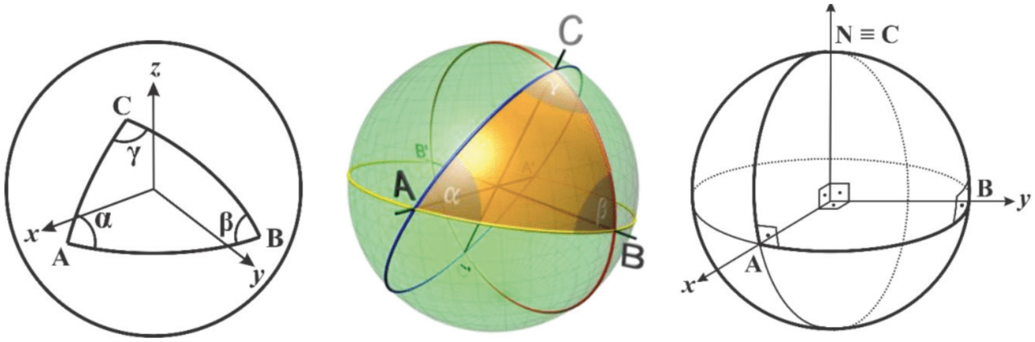
Theorem 7: If *ABC* is *a* triangle in S_ρ^2 whose interior angles are α, β e γ and whose sides opposite these angles measure, respectively, *a*, *b* e *c* then:

$$cos\alpha = \frac{cos\left(\frac{a}{\rho}\right) - cos\left(\frac{b}{\rho}\right)cos\left(\frac{c}{\rho}\right)}{sen\left(\frac{b}{\rho}\right)sen\left(\frac{c}{\rho}\right)}; \tag{8}$$

$$cos\beta = \frac{cos\left(\frac{b}{\rho}\right) - cos\left(\frac{a}{\rho}\right)cos\left(\frac{c}{\rho}\right)}{sen\left(\frac{a}{\rho}\right)sen\left(\frac{c}{\rho}\right)}; \tag{9}$$

$$cos\gamma = \frac{cos\left(\frac{c}{\rho}\right) - cos\left(\frac{a}{\rho}\right)cos\left(\frac{b}{\rho}\right)}{sen\left(\frac{a}{\rho}\right)sen\left(\frac{b}{\rho}\right)}. \tag{10}$$

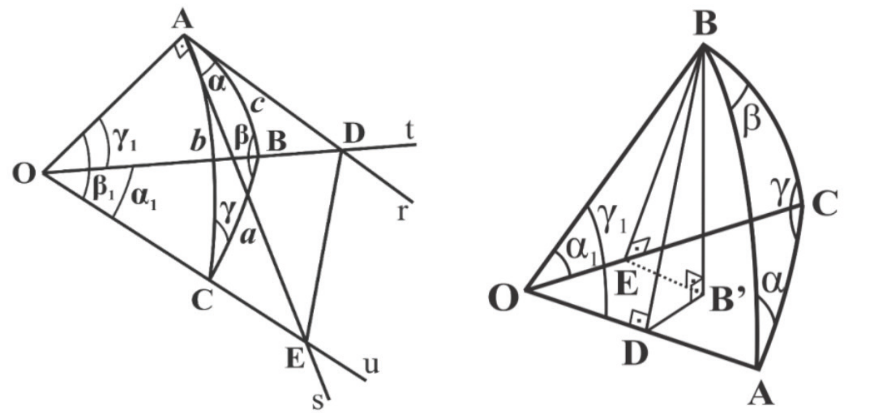
The demonstration of Theorems 6 and 7 depends on Theorem 3, Euclidean geometry and trigonometry relations (Brannan *et al.*, 2012; Motta, 2018). Figures 6(a) and 6(b) illustrate, respectively, the geometric approach required to demonstrate the law of cosines and spherical sines.



(a) (b) (c)

Figure 5: Spherical triangles in S_ρ^2 (a) internal angles of the triangle ABC (b) spherical triangles defined by three distinct points; (c) spherical triangle trirectangle

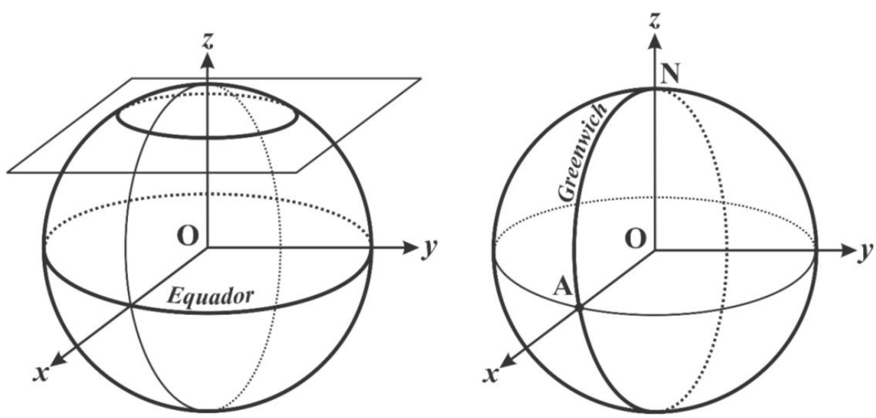
Source: (a) The authors; (b) Wikimedia Commons (2024); (c) the authors.



(a) (b)

Figure 6: Spherical trigonometry: (a) law of cosines; (b) law of sines

Source: The authors.



(a) (b)

Figure 7: Maximum circumferences at S_ρ^2 (a) equator; (b) Greenwich meridian

Source: The authors.

ACTIVITIES IN *GOOGLE EARTH*

The texts by Abrantes (2018), Alves (2009), Filho *et al.* (2018) and Jahn and Bongiovanni (2016) are recommended reading for the stage preceding the application of the activities. These texts explore the relationship between spherical geometry and cartography.[2]

ACTIVITY 1: THE CITY FURTHEST FROM CURITIBA

In this activity, we calculate the distance between the cities of Curitiba in Brazil and Uruma in Japan. To do this, students must be able to transform geographical coordinates into spherical coordinates and calculate the length of *a* geodesic.

According to FurthestCity (2024), the city closest to the antipode (diametrically opposite) of Curitiba on the Earth's surface is Uruma, Japan. According to Wikipedia (2024), the geographical coordinates of Uruma are:

$$26°22'45''N; \quad (11)$$

$$127°51'27''E. \quad (12)$$

The geographical coordinate (11) is the latitude of Uruma, which is north of the equator. In cartography, the equator divides the earth's surface into north (*N*) and south (*S*). Latitude is the spherical coordinate of *a point* $P \in S^2_\rho$ given by the distance between that point and the equator. The intersection between S^2_ρ and *a* plane secant to S^2_ρ parallel to the equator is *a* curve called the circumference of latitude. In particular, the equator is *a* circumference of latitude, illustrated in Figure 7(a).

The geographical coordinate (12) is the longitude of Uruma, which is east of the meridian[3] of Greenwich. In cartography, the meridian passing through the English city of Greenwich is considered to be ground zero for determining the longitude of *a* point, dividing the spherical surface into west (*W*) and east (*E*). Longitude is the spherical coordinate of *a* point $P \in S^2_\rho$ given by the distance between that point and the Greenwich meridian, illustrated in Figure 7(b).

The latitude (*N* or *S*) and longitude (*E* or IW) of *a* point *P* are given, respectively, by the angles

$$\psi' = \frac{\pi}{2} - \psi$$

e φ where ψ is the colatitude (complementary angle of latitude) of *P*. Figure 8 illustrates the angles ψ e φ which, together with the radius of the spherical surface *p* of the spherical surface, the spherical coordinates of the point *P* - Theorem 1.

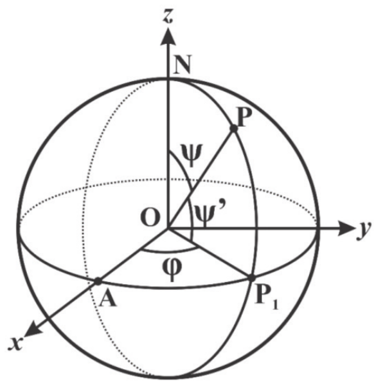


Figure 8: Colatitude ψ and longitude φ of the point *P* at S^2_ρ

Source: The authors.

Thus, using Theorem 1 and *p*=6371 *km* as *a* measure of the Earth's mean radius, the students transform the geographical coordinates of Curitiba and Uruma in Table 1 into spherical coordinates[4]. Then, using Theorem 3, they calculate the distance, in between the cities of Curitiba and Uruma. Finally, the students use *Google Earth* (Google, 2024) to locate Curitiba and Uruma on the earth's surface, and compare the distance between the two cities provided by *Google Earth* with the distance previously calculated.

We describe the steps of the *Google Earth* activity below.

2. The science of drawing geographical or topographical maps.
3. Every maximum circumference of that passes through the N pole.
4. Students use *a* calculator and convert degrees into radians.

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City	Latitude	Colatitude	Colatitude (rad)	Longitude	Longitude (rad)
Curitiba	25°25′47″ *S*	64°34′13″ *S*	1,117	49°16′19″ *W*	0,8552
Uruma	26°22′45″ *N*	63°37′15′ *N*	1,0996	127°51′27″ *E*	0,4712

Table 1: Geographical coordinates of the cities of Curitiba and Uruma
Source: Google (2024) and Wikipedia (2024).

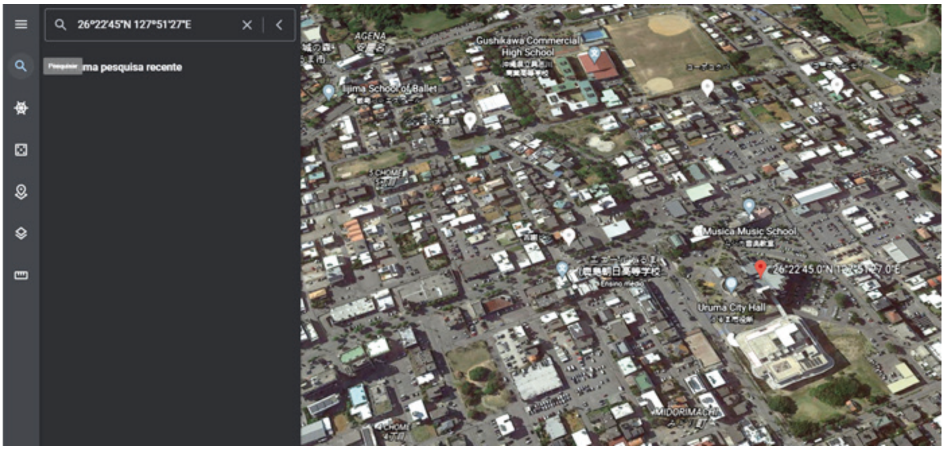


Figure 10: Locating Uruma in Japan
Source: Google (2024).

STAGES

1. First, create *a* project in *Google Earth* via the left-hand sidebar under the "Projects" icon, shown in Figure 9. Next, click on "Create" in the interaction menu and select "Create project in *Google Drive*" or "Create KML file", so that the project can be saved in the cloud or on *a* device, respectively.

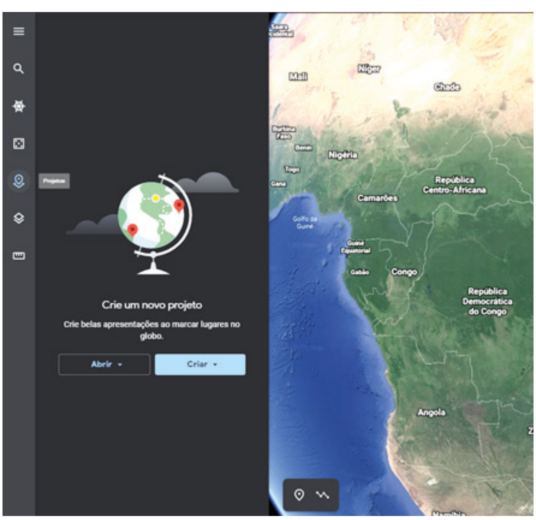


Figure 9: Creating *a* project in *Google Earth*
Source: Google (2024).

2. To locate Uruma in Japan, click on "Search" and type in the geographical coordinates of the city, as shown in Figure 10.

3. Once the location has been determined, click on "Add to project" - Figure 11, and store the project created for the activity.

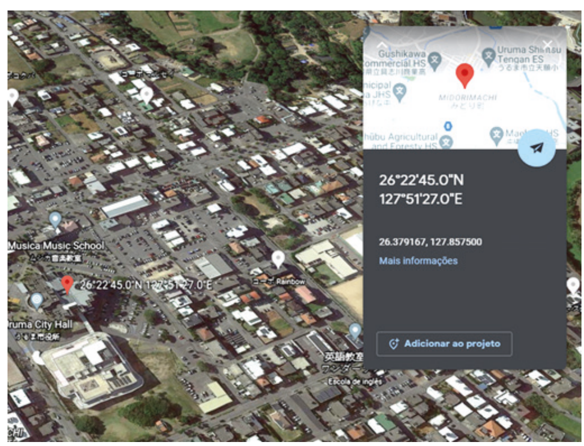


Figure 11: Adding the project
Source: Google (2024).

4. We repeat steps 2 and 3 to locate the city of Curitiba - Figure 12.

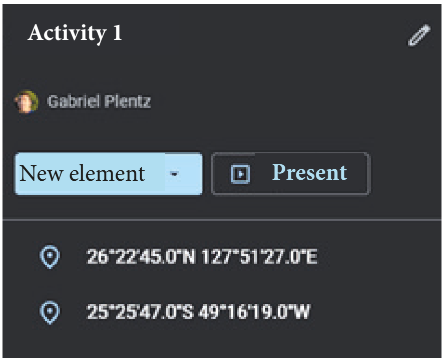


Figure 12: Locating Curitiba

Source: Google (2024).

5. Using the "Measure distance and area" tool in the left-hand side menu, click on the points determined in steps 2, 3 and 4 and then click on "Done" in the information box in the right-hand corner. Before that, we increased the scale to get more detail (just move the mouse wheel up or click on the "+" button in the bottom right corner). *Google Earth now* gives us the distance between Curitiba and Uruma - Figure 13.

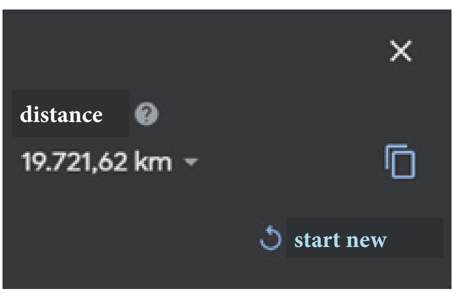


Figure 13: Checking the distance between Curitiba and Uruma

Source: Google (2024).

The distance between the two cities is represented by *a* line - Figure 14, which will remain visible until we click on "Start new" or close the frame in the right-hand corner.

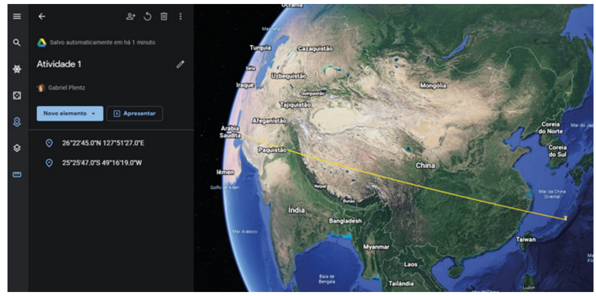


Figure 14: (Geodesic) line representing the distance between Curitiba and Uruma

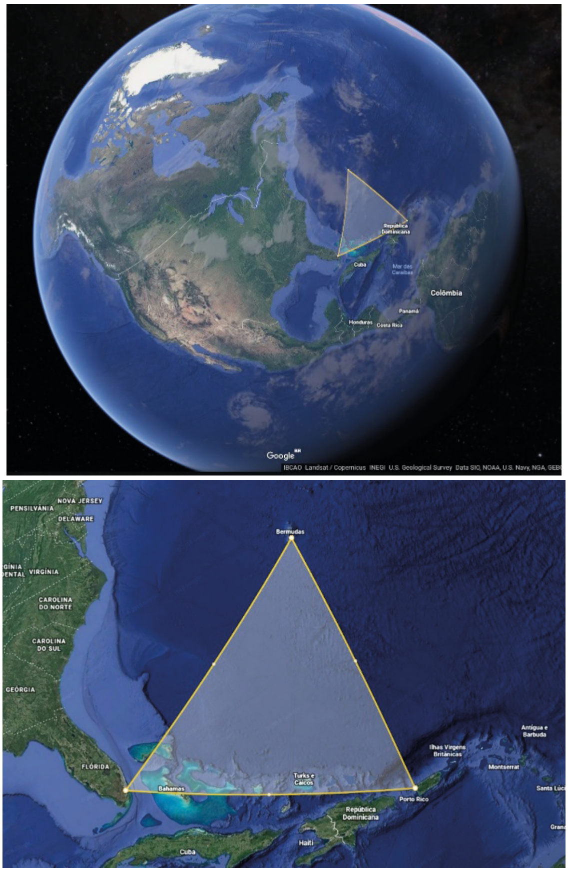
Source: Google (2024).

ACTIVITY 2: THE AREA OF THE BERMUDA TRIANGLE

In this activity, we use spherical coordinates to calculate the distance between the three cities that define the Bermuda triangle and the area of the spherical triangle whose vertices are these three cities. To do this, students must be able to calculate the length of *a* geodesic and the area of *a* spherical triangle.

The Bermuda Triangle, illustrated in Figure 15(a), is perhaps one of the most iconic subjects in terms of disastrous events for aviation and navigation. This is because several planes, boats and ships have mysteriously disappeared while traveling through its area, determined by Fort Lauderdale (USA), San Juan (Puerto Rico) and Hamilton (Bermuda), as illustrated in Figure 15(b).

There are scientific explanations for disappearances, such as those based on the region's magnetic field, climatic events such as earthquakes and whirlpools, sea currents, etc. But there are also sensationalist explanations, based on conspiracies, and even supernatural ones. The mysteries of the Bermuda Triangle became popularly known through the book *Invisible horizons: true mysteries of the sea*, by American sensationalist writer Vincent Gaddis (1913-1997).



(a) (b)

Figure 15: Bermuda Triangle: (a) surface; (b) vertices

Source: Google (2024).

Thus, using Theorem 1, p=6371 *km* as the measure of the Earth's average radius and the data in Table 2, the students transform the geographical coordinates of the cities of Fort Lauderdale, San Juan and Hamilton into spherical coordinates; then, using Theorem 3, they calculate the distance, in *km* Then, using Theorems 6 and 7, they determine the measures of the internal angles of the spherical triangle whose vertices are the three cities. Finally, using Theorem 4, students calculate the area, in km^2 of the Bermuda triangle and compare this measurement with the one provided by *Google Earth*.

The following are the stages of the *Google Earth* activity.

Stages

1. We created *a* new project in *Google Earth* and located the three vertices (points) of the Bermuda triangle (steps 1 to 3 of Activity 1) - Figure 16.

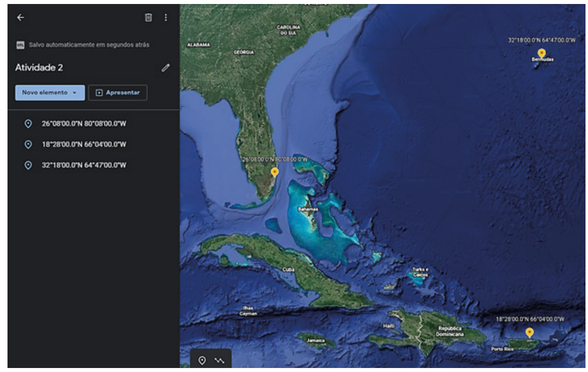


Figure 16: Locating the Bermuda triangle

Source: Google (2024).

2. Using the "Measure distance and area" tool, we join the three points to form the triangle. *Google Earth* provides the measurements of the perimeter and area of the triangle formed - Figure 17.

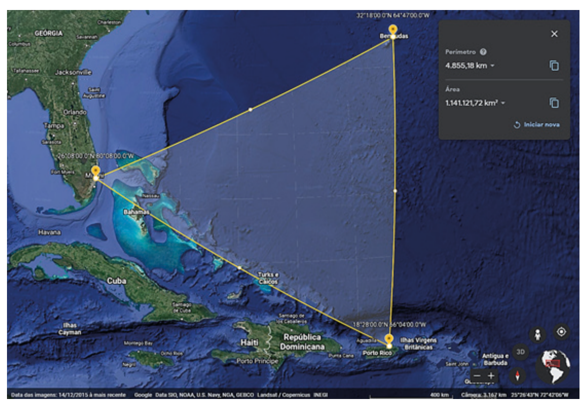


Figure 17: Calculating the area of the Bermuda triangle

Source: Google (2024).

Location	Latitude	Colatitude ψ	Longitude φ
Fort Lauderdale	$26°08'00''\,N$	$63°52'00''\,N$	$80°08'00''\,W$
San Juan	$18°28'00''\,N$	$71°32'00''\,N$	$66°04'00''\,W$
Hamilton	$32°18'00''\,N$	$57°42'00''\,N$	$64°47'00''\,W$

Table 2: Geographical coordinates of the Bermuda triangle

Source: Google (2024).

FINAL CONSIDERATIONS

In this paper, we discuss spherical geometry, presenting important geometric relationships such as the length of *a* geodesic, the sum of internal angles and the area of *a* spherical triangle. We also propose two activities to explore concepts and properties of spherical geometry using *Google Earth* in *a* Mathematics degree course.

The main difficulties faced in preparing this work were the scant bibliography in Portuguese on spherical geometry and the definition of the software to be used to build the three-dimensional images of the sphere, its elements and sections. As for the former, our basic reference was *Geometry* (Brannan *et al.*, 2012). However, the authors of this work demonstrate spherical geometry results on the unit sphere, i.e. the sphere of radius R=1. Thus, demonstrating these results on a sphere of radius R=p with p>0 proved to be an interesting challenge. As for the second, we opted for CorelDRAW (Corel, 2024), which can be replaced by *a* free dynamic geometry application such as GeoGebra 3D (GeoGebra3D, 2024).

We hope that this work will be useful to students on mathematics degree courses, particularly at UTFPR, Curitiba Campus, and also to mathematics teachers in basic education. As for the latter, we hope that the work will inspire the use of *Google Earth* in the planning/development of introductory activities to the study of spherical geometry, as well as in interdisciplinary activities involving mathematics and geography.

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