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APPLICATION TO MICROECONOMIC OF A PROXIMAL METHOD FOR A CLASS OF NONCONVEX MULTI-OBJECTIVE MINIMIZATION

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Abstract: In this article we show numerical examples and an application to microeconomics of a proximal point method, introduced by Papa Quiroz, Bermeo and Ichpas (2023), for a class of non-convex multiobjective minimization problems. In the work the authors demonstrated the theoretical convergence of the algorithm to a Pareto-Clarke critical point and to a Pareto solution when the functions are convex. The present work extends the numerical experimentation of the algorithm by applying it to a specific problem with the intention of showing the practicability of the proposed algorithm. The algorithm was implemented in MATLAB and the results show that the algorithm promises to solve deterministic microeconomic problems.
Keywords: Proximal point method, Multiobjective minimization, Clarke subdifferential, Pareto critical point Microeconomics.

INTRODUCTION

The problems of the real context imply the search to simultaneously optimize several objectives that are generally in conflict with each other, that is, if one improvement leads to the worsening of the other, for example, to buy a car you have to look at the cost, the material of construction, the brand among other factors that imply the purchase decision, this type of problem motivated the development of research works in optimization. Edgeworth [21] and Pareto [19] were the first researchers to introduce multi-objective optimization in the field of economics, then Stadler [23] and Steuer [24] developed multi-objective algorithms in Applied Mathematics and Engineering. Currently several researchers have developed methods to solve this class of multiobjective optimization problems.

In 2019, Papa Quiroz and Cruzado [17] introduced the inexact proximal method to solve unrestricted multi-objective quasi-convex minimization problems in two versions. Where they showed the convergence of the sequence generated by the algorithm, under certain conditions of the objective function, they demonstrated that the sequence converges towards a Pareto-Clarke critical point, they performed the numerical experimentation of the proposed method.

In 2020, Papa Quiroz, Borda and Collantes [1] presented the Proximal method for quasiconvex multi-objective minimization in the non-negative orthant and its application to the theory of demand in microeconomics. Where they showed the convergence of the sequence generated by the algorithm, under certain conditions of the objective function, they proved that the sequence converges towards a Pareto-Clarke critical point, they carried out the numerical experimentation of the proposed method by varying the parameter $\{\lambda_k\}$ of a specific example.

In 2022, Papa Quiroz, Bermeo and Ichpas [15] introduced an inexact proximal method to solve multiobjective quasi-convex minimization problems with constraints. Where they presented the convergence of the sequence generated by the algorithm, under certain conditions of the objective function, where they proved that the sequence converges towards a Pareto-Clarke critical point, they carried out the numerical experimentation of the proposed method for biobjective problems.

In this work we are interested in the application of the proximal point method in multiobjective optimization see [5,8,10,14,17]. That is, we will apply the method to the multiobjective quasi-convex minimization problem with constraints defined by:

$$min\{G(x): x \in C\}$$

where $G: R^n \to R^m$ is a quasi-convex vector function, with each of the functions $G_i: R^n \to R, i = 1,2 \dots . . m$ being quasi-convex defined on the convex set *C*. The aforementioned model is motivated by applications in microeconomic theory where quasi-convex objective functions are strongly related to convex preferences in the diversification of consumption choice, in demand and production theory, see Mas-Colell et al. [4] and Madden Paul [11].

In this article we will carry out more numerical experiments and the application, of the proposed method, to a specific example applied to economics.

BASIC TOOLS

PROXIMAL DISTANCES

We start with the definition of proximal distance on $\boldsymbol{C} \subset \mathbb{R}^n$, which, for general convex sets, has been introduced in [9].

Definition 2.1. A function $L_d: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_+ \cup \{+\infty\}$ is called a proximal distance in *C* if for each $y \in C$, it satisfies the following properties:

1) $L_d(.,y)$ is proper, lower semicontinuous, strictly convex and continuously differentiable on *C*

2) $dom(L_d(.,y)) \subset \overline{C}$ *and* $dom\ \partial_1 L_d(.,y) = C$ where $\partial_1 L_d(.,y)$ denote the classical subdifferential map of the function $L_d(.,y)$ with respect ti the first variable

3) $L_d(.,y)$ is coercive on $\mathbb{R}^n$

4) $L_d(y,y) = 0$

We denote by *D(C)* the family of functions satisfying this definition.

Definition 2.2 Given $L_d \in D(C)$, a function $L_H: \mathbb{R}^n x \mathbb{R}^n \to \mathbb{R}_+ \cup \{+\infty\}$ is called the induced proximal distance to *d* if *H* is a finite valued function on *CxC* and for each $a, b \in C$, satisfies

Ii. $L_H(a,a) = 0$

Iii. $\langle c - b, \nabla_1 L_d(b, \mathrm{a})\rangle \leq L_H(c,a) - L_H(c,b) - \gamma L_H(b.a)$, for all $c \in C$, where $\gamma \in (0,1]$.

We denote by $(L_d, L_H) \in \mathcal{F}(C)$ to the proximal and induced proximal distance that satisfies the conditions of the Definition 2.2. We also denote by $(L_d, L_H) \in \mathcal{F}(\overline{C})$ if there exists *H* such that:

Iiii. L_H is finite-valued on $\overline{C}xC$, satisfying Ii and Iii for each $c \in \overline{C}$.

Iiv. for each $c \in \overline{C}$, $L_H(c,.)$ has level bounded sets on *C*.

Finally, we write $(L_d, L_H) \in \mathcal{F}_+(\overline{C})$ if

Iv. $(L_d, L_H) \in \mathcal{F}(\overline{C})$

Ivi. For all $y \in \overline{C}$ and for all $\{y^k\} \subset C$ bounded with $\lim_{k\to\infty} L_H(y, y^k) = 0$, then $\lim_{k\to\infty} y^k = y$.

Ivii. For all $y \in \overline{C}$ and all $\{y^k\} \subset \mathbb{R}^n_{++}$ such that $\lim_{k\to\infty} y^k = y$, then $\lim_{k\to\infty} L_H(y, y^k) = 0$.

CLARKE SUBDIFFERENTIAL

Definition 2.3 Let $f: \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a locally lipschitzian function at the point $x \in \mathbb{R}^n$. The generalized directional derivative of *f* at *x*, in the direction of $v \in \mathbb{R}^n$, is defined by:

$$f^0(x,v) = \lim_{\substack{y\to x \\ t\downarrow 0}} \sup \frac{f(y+tbv) - f(y)}{t}$$

We should note that f^o exists thanks to the locally function lipschitzian condition of *f*.

Definition 2.4 Let $f: \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a locally lipschitzian function at the point $x \in \mathbb{R}^n$, then the sub-differential, in sense of Clarke, of *f* at *x* is the set

$$\hat{\partial} f(x) = \{\xi \in \mathbb{R}^n : f^0(x,v) \geq \langle \xi; v\rangle, \forall v \in \mathbb{R}^n\}.$$

Each element of $\xi \in \hat{\partial} f(x)$ is called subgradient of *f* at *x*, in the sense of Clarke.

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LOCALLY LIPSCHITZ, CONVEX AND QUASICONVEX FUNCTIONS

Definition 2.5 A function $f: \mathbb{R}^n \to \mathbb{R}$ is locally lipschitzian with positive constant k in $x \in R^n$ if there is $\varepsilon > 0$ such that; $|f(y) - f(z)| \leq k\|y - z\|$ for all $z, y \in B(x, \varepsilon)$.

Definiton 2.6 A function $f: \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is convex in $\mathbb{R}^n$ if for each $x, y \in \mathbb{R}^n$ and each $\lambda \in [0,1]$ it holds:

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y)$$

Definiton 2.8 A function $f: \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is strictly quasi-convex in $\mathbb{R}^n$ if for each $x, y \in \mathbb{R}^n, x \neq y$, and for each $\lambda \in \langle 0,1 \rangle$ It is true that:

$$f(\lambda x + (1-\lambda)y) < \max\{f(x), f(y)\}.$$

Definiton 2.9 A function $f: \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is semi-strict quasi-convex in $\mathbb{R}^n$ if for each $x, y \in \mathbb{R}^n$, and for each $\lambda \in \langle 0,1 \rangle$ It is true that:

$$f(x) < f(y) \Rightarrow f(\lambda x + (1-\lambda)y) < f(y).$$

MULTIOBJECTIVE OPTIMIZATION

In this section we consider some basic definitions and properties of multiobjective optimization.

Let $\mathbb{R}^m$ be the m-dimensional Euclidean space with the partial order $\preccurlyeq$ induced by the Paretian cone $\mathbb{R}^m_+$ given by: For each $x, y \in \mathbb{R}^m$, $x \preccurlyeq y$ if and only if $y - x \in \mathbb{R}^m_+$, this means that $x_j \leq y_j, \forall j \in \{1, \dots, m\}$, and the strict partial order $\prec$ induced by the cone $\mathbb{R}^m_{++}$, where For each $x, y \in \mathbb{R}^m$, $x \prec y$ if only if $y - x \in \mathbb{R}^m_{++}$, this means that $x_j < y_j, \forall j \in \{1, \dots, m\}$. These partial orders establish a class of problems known in the multiobjective optimization literature.

Consider the mathematical model defined by

$$\text{Optimize } F(x) = Opt\{(F_1(x), \dots \dots F_m(x)) : x \in C \subset \mathbb{R}^n\}$$

where $F: C \subset \mathbb{R}^n \to \mathbb{R}^m$, with C being a nonempty convex set, the point $\hat{x}$ is a weak Pareto solution (or Pareto optimal) of the vector function F restricted to the set C if there is no other $\bar{x} \in C$ such that $F(\bar{x}) \prec F(\hat{x})$..

A function $F: C \subset \mathbb{R}^n \to \mathbb{R}^m$ is called locally Lipschitz on $\mathbb{R}^{\boldsymbol{n}}$ is each component $F_i: C \subset \mathbb{R}^n \to \mathbb{R}$ is locally Lipschitz on C where $F(x) = (F_1(x), \dots, F_m(x))$.

A function $F: C \subset \mathbb{R}^n \to \mathbb{R}^m$ is called convex (quasiconvex) on C is each component $F_i: C \subset \mathbb{R}^n \to \mathbb{R}$ is convex (quasi-convex).

OPTIMALITY CONDITIONS AND PARETO-CLARKE CRITICAL POINT

Definition 2.8 Let $F: C \subset \mathbb{R}^n \to \mathbb{R}^{\boldsymbol{m}}$ with $F(x) = (F_1(x), \dots, F_m(x))$ be a locally Lipschitz function on C. We say that $\hat{x} \in C$ is a Pareto-Clarke critical point of F yes for any $y \in C$ exists $i_0 \in \{1,2, \dots, m\}$ such that $F^0_{i_0}(\hat{x}, y - \hat{x}) \geq 0$.

SEQUENCE PROPERTIES

Lemma 2.9 Let $\{v_k\}$, $\{\tau_k\}$, and $\{\beta_k\}$, and three nonnegative sequences satisfying

$$v_{k+1} \leq (1 + \tau_k)v_k + \beta_k,$$

for all k, where it holds

$$\sum_{k=0}^{+\infty} \tau_k < +\infty, \sum_{k=0}^{+\infty} \beta_k < +\infty,$$

Then the sequence $\{v_k\}$ converges.

INEXACT ALGORITHM WITH PROXIMAL GENERALIZED DISTANCE

We are interested in solving the multiobjective optimization problem (POM) with constraints:

$$min\,\{F(x): x \in C\} \qquad (1)$$

where $F: \boldsymbol{C} \subset \mathbb{R}^{\boldsymbol{n}} \to \mathbb{R}^{\boldsymbol{m}}$ satisfies the following assumptions:

(H1) $F_i(x) \geq 0$, for all $i = 1,2, \dots, m$ for all and for all $x \in \mathbb{R}^n$;

(H2) F is locally Lipschitz on $\mathbb{R}^{\boldsymbol{n}}$;

(H3) F is strict or semi-strict quasi-convex in $\mathbb{R}^n$.

In order to solve the problem (1), we propose an Inexact Scalarization Proximal Point Method with proximal distance regularization, which we call ISPPMR.

ISPPMR algorithm

Initialization: choice an arbitrary initial point

$$x^0 \in C \qquad (2)$$

Iterative step: For $k = 0,1,2,\ldots$ and given x^k, find $x^{k+1} \in \Omega_k$ and $e^{k+1} \in \mathbb{R}^n$ such that:

$$e^{k+1} \in \partial^0(\langle F(.), z_k\rangle + \lambda_k L_d(., x^k))(x^{k+1}) + \mathrm{N}_{\Omega_k}(x^{k+1}), \quad (3)$$

where ∂^0 is the Clarke subdifferential, $\lambda_k > 0$, $\{z_k\} \subset \mathbb{R}^n_+$, $\|z_k\| = 1$, $L_d(.,.)$ is a proximal distance.

Stop criterion: If $x^{k+1} = x^k$ or x^{k+1} or is a Pareto-Clarke critical point, then stop.

Otherwise, make $k \leftarrow k+1$ and return to Iterative Step.

Remark: In practice to obtain a point x^{k+1} satisfying the equation of the algorithm, we should find only an approximate critical point (local minimum, local maximum or saddle point) of the following optimization problem:

$$\boldsymbol{min}\left\{\sum_{i=1}^{m} z_i^k F_i(.) + \lambda_k L_d(., x^k): F_i(x) \le F_i(x^k), i = 1, \ldots, m\right\}.$$

Theorem 3.1 If $F: C \subset \mathbb{R}^n \to \mathbb{R}^m$ is a function that satisfies the hypotheses (H1),(H2),(H3) and $(d, H) \in \mathcal{F}(\bar{C})$ then the sequence $\{x^k\}$ generated by the ISPPMR algorithm is well defined

Proof. Simmilar to Theorem 5.1 (see [15]).

Next, we present another assumption about the objective function F and the initial point x^0, which is mentioned in several works related to proximal method algorithms, review, for example, [12], [13] and [14]. Therefore, The following additional statement is then considered:

$(\mathbf{H}_4)$ $(F(x^0) - \mathbb{R}^m_+) \cap F(\mathbb{R}^n)$ is $\mathbb{R}^m_+$ - complete.

This assumption means that for all sequence $\{l^k\} \subset \mathbb{R}^n$, with with $l^0 = x^0$,, such that $F(l^{k+1}) \preceq F(l^k)$, for all $k \in \mathbb{N}$, there exists $a \in \mathbb{R}^n$ such that $F(l) \preceq F(l^k)$, for all .

Denote the set:

$$E := \{x \in \mathbb{R}^n : F(x) \preceq F(x^k),\ \forall k \in \mathbb{N}\}$$

CONVERGENCE RESULTS

Lemma 3.2 Let $F: \boldsymbol{C} \subset \mathbb{R}^n \to \mathbb{R}^m$ be a function satisfying (H1), (H2), (H3), (H4), $(L_d, L_H) \in \mathcal{F}_+(C)$ and $\{x^k\}$ the sequence generated by the ISPPMR algorithm. If $\hat{x}$ is an accumulation point of , then $\hat{x} \in E \cap C$.

Proof. See Lemma 3.3.1 [16]

Proposition 3.3 Let $\{\boldsymbol{x}^k\}$ and $\{\boldsymbol{e}^k\}$ be the sequences generated by the ISPPMR algorithm. If the assumptions (H1), (H4) are satisfied and $(L_d, L_H) \in \mathcal{F}(\bar{C})$, then for all $\bar{x} \in E$ and for all $k \in \mathbb{N}$, we have

$$L_H(\bar{x}, x^{k+1}) \le L_H(\bar{x}, x^k) - \gamma L_H(x^{k+1}, x^k) - (1/\lambda_k)\langle e^{k+1}, \bar{x} - x^{k+1}\rangle.$$

Proof. See Proposition 3.3.2 [16]

Proposition 3.4 Let $\{\boldsymbol{x}^k\}$ and $\{\boldsymbol{e}^k\}$ be the sequences generated by the ISPPMR algorithm. Suppose that the assumptions (H1), (H3), (H4) are satisfied, $(L_d, L_H) \in \mathcal{F}(\bar{C})$. If the following conditions hold:

i. $\frac{\|e^{k+1}\|}{\lambda_k} \le \eta_k \sqrt{L_H(x^k, x^{k+1})}$

ii. $\sum_{k=0}^{+\infty} \eta_k < +\infty$

iii. There exists $\theta > 0$, such that

$$\|x - y\|^2 \le \theta L_H(x, y),$$

for all $x \in \bar{C}$ and for all $y \in C$ and for all then we have

a. There exists a natural number k_0 such that for all $k \ge k_0$ and for all $\bar{x} \in E$ we have $L_H(\bar{x}, x^{k+1}) \le \left(1 + \frac{\theta\eta_k}{1-\theta\eta_k}\right) L_H(\bar{x}, x^k) + \left(\frac{\eta_k}{4} - \gamma\right) L_H(x^{k+1}, x^k);$

b. $\{L_H(\bar{x}, x^{k+1})\}$ converges for all $\bar{x} \in E$;

c. $\{x^k\}$ is bounded;

d. $\lim_{k\to\infty} L_H(x^{k+1}, x^k) = 0$.

e. If $(L_d, L_H) \in \mathcal{F}_+(\bar{C})$, then $\{x^k\}$ converges in *E*.

Proof. See Proposition 3.3.3 [16]

Theorem 3.5 If the function $F: R^n \to R^m$, such that $F_i: \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$, satisfies the hypotheses (H1), (H2), (H3) and (H4), with $0 < \lambda_k \leq \bar{\lambda}$ then the sequence generated by ISPPMR converges towards some Pareto Clarke critical point of the multi-objective optimization problem.

Proof. See Theorem 3.3.4 [16]

NUMERICAL EXPERIMENT AND APPLICATION TO THE THEORY OF MICROECONOMICS

In this section we give a numerical example that shows the functionality of the proposed method and then we apply the algorithm to solve a specific optimization problem in microeconomy.

For that we use an Intel Core i7 2.30 GHz computer, 5GB of RAM, Windows 10 as operating system with SP1 64 bits and we implement our code using MATLAB R2022 a.

NUMERICAL EXPERIMENT

Let the multiobjective minimization problem be defined by

$$min\{(F_1(x), F_2(x)) : x \in C \subset \mathbb{R}^2\}$$

where $F_1(x_1, x_2) = -x_1^{1/3} x_2^{2/3}$, $F_2(x_1, x_2) = -x_1^{0.25} x_2^{0.75}$, and $C = \{(x_1, x_2) \in \mathbb{R}^2 : -x_1 + x_2 \leq 2; 3x_1 + x_2 \leq 18 : 3x_1 - x_2 \leq 12;\ x_1 \geq 0; x_2 \geq 0\}$

The functions F_1, F_2 are quasi-convex and differentiable functions, therefore $(F_1(.), F_2(.))^T$ is a quasi-convex vector function of class $C^1(\mathbb{R}^2_+)$ this problem satisfies the hypotheses **(H1), (H2)** and **(H3)**). Observe that this problem only has a unique Pareto solution given by (4, 6) which is a feasible point in the set *C*, as shown in figure 1.

We take x^0 as initial point and given $x^k \in C$, the main step of the proposed algorithm is to find a critical of the following problem

$$\begin{cases} ming(x_1, x_2) = (-x_1^{1/3} x_2^{2/3}) z_1^{\,k} + (-x_1^{0.25} x_2^{0.75}) z_2^{\,k} + \\ \qquad \lambda_k L_d\left(\left((x_1, x_2), (x_1^{\,k}, x_2^{\,k})\right)\right) \\ Sa: \\ \quad -x_1^{1/3} x_2^{2/3} + (x_1^{\,k})^{1/3} (x_2^{\,k})^{2/3} \leq 0 \\ \quad -x_1^{0.25} x_2^{0.75} + (x_1^{\,k})^{0.25} (x_2^{\,k})^{0.75} \leq 0 \end{cases}$$

where we consider as the proximal distance $L_d(.,)$, for the numerical experimentation of this example, the Kullback -Leibler Bregman distance defined by

$$L_d(x, y) := \sum_{i=1}^{2} x_i ln\left(\frac{x_i}{y_i}\right) + y_i - x_i$$

Next, we present the numerical experiments, taking the proximal distance mentioned above, and making a variation of the fixed parameters.

1.-Let $z_k = (z^k{}_1, z^k{}_2) = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$, be $\lambda_k = 0.2$; be the point $x^0 = (1, 1)$ is the starting point of the algorithm and the point $p^0 = (2, 3)$ is used to solve all the **subproblems** using the MATLAB functionality with the SQP algorithm and we consider the stopping criterion $\|x^{k+1} - x^k\| < 0.0000001$ to finish the ISPPMR algorithm.

k	N[x^k]	$x^k = (x_1^k, x_2^k)$	$\sum F_i(x^k) z^k{}_i$
1	9	(2.85303 4.58408)	-5.64657
2	11	(3.75158 5.73600)	-7.16821
3	14	(3.95622 5.95520)	-7.47601
4	14	(3.99266 5.99258)	-7.52931
9	14	(3.99978 5.99978)	-7.53963
10	15	(3.99978 5.99978)	-7.53963

Table 1. Results of iterations of the ISPPMR algorithm using the Kullback-Leibler Bregman distance with $\lambda_k = 0,2$

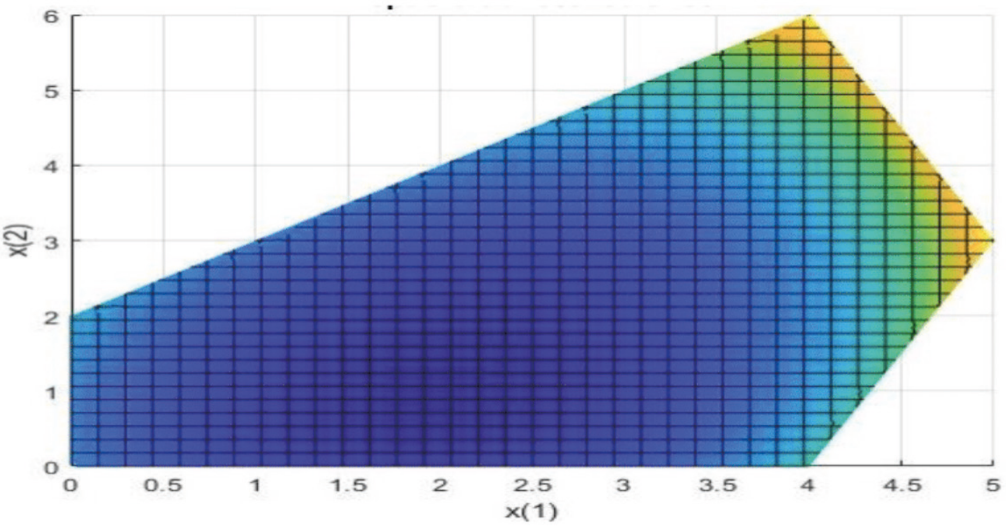

Fig 1. Constraint Set

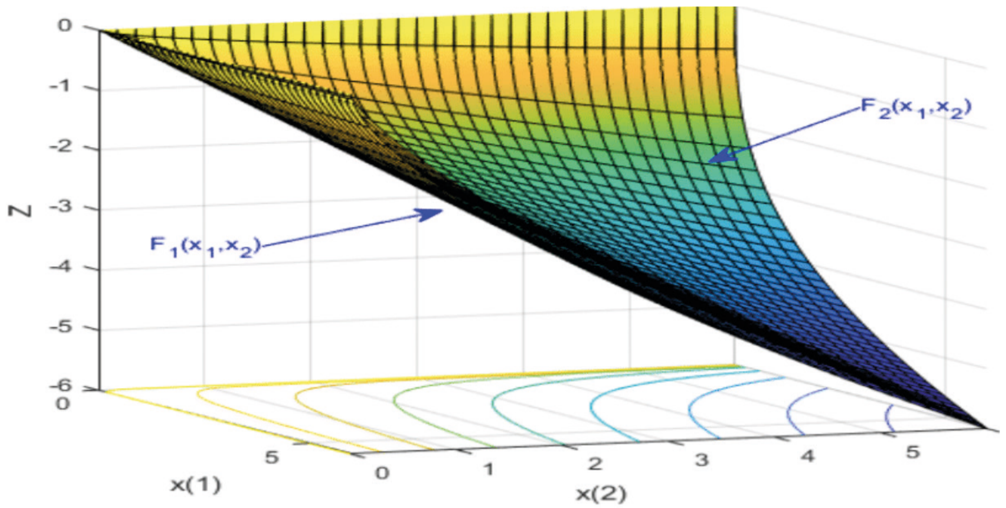

Fig 2. Objective Component Functions

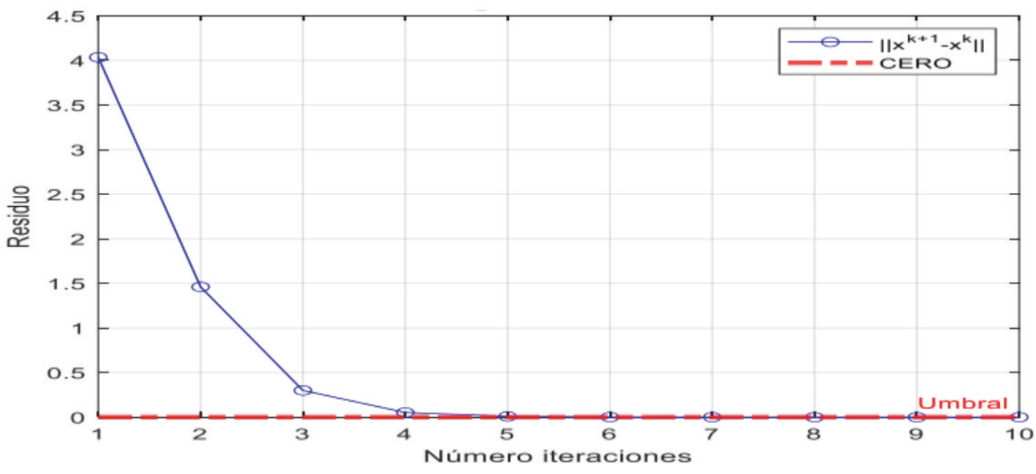

Fig 3. Distances of consecutive points with $\lambda_k = 0{,}2$.

Table 1. shows that with k=10, iterations approach the solution of the problem. Likewise, we can observe in the table that the sequence x^k converges to the point (4,6), which, as we mentioned, is the unique Pareto solution, implying that the problem has a unique solution on the Pareto frontier, determined by the proposed algorithm. Figure 3 shows the difference of two consecutive iterations that are converging to zero.

2.-Let $z_k = ({z^k}_1, {z^k}_2) = \left(\frac{1}{\sqrt{3}}, \sqrt{\frac{2}{3}}\right)$, be $\lambda_k = 0.4$, the point is the initial point $x^0 = (1, 2)$ of the algorithm and we choose $p^0 = (2, 3)$ as starting point of each subproblem. To solve all the subproblems we use the MATLAB functionality with the SQP algorithm and we consider the stopping criterion $\|x^{k+1} - x^k\| < 0.000001$ to finish the ISPPMR algorithm.

k	N[x^k]	$x^k = (x_1^k, x_2^k)$	$\sum F_i(x^k){z^k}_i$
1	6	(2.08783 3.76654)	-4.43995
2	9	(2.98041 4.90343)	-5.93316
3	13	(3.51488 5.49713)	-6.74782
4	12	(3.78491 5.78064)	-7.14378
15	11	(3.99999 5.99999)	-7.45288
16	16	(3.99999 5.99998)	-7.45288

Table 2. Results of iterations of the ISPPMR algorithm using the Kullback-Leibler Bregman distance with $\lambda_k = 0,4$

Table 2. shows that with k=16 iterations it approaches the solution of the problem, having a difference with respect to the first numerical experimentation in which needed only k=10 iterations. Likewise, we can indicate that the convergence of the sequence and the solution on the Pareto frontier is analogous to the first numerical experimentation, except for the behavior of the difference of two consecutive steps that converges to zero, has more iterations, as can be seen in the Figure 4.

APPLICATION TO THE THEORY OF MICROECONOMICS

In economic theory, it is always needed optimize resources, for example, in the field of production it is necessary to use mathematical models to optimize production (see [3], [11]). One of the functions that is widely used in this area is the Cobb-Douglas function initially proposed by Knut Wicksell (1851-1926), then Charles Coob and Paul Douglas in 1928 evidenced concrete statistical studies, they considered that the production function described by $Q = AL^{\alpha}K^{1-\alpha}$, was linked to the labor factor denoted by L to the capital factor denoted by K, and to the total factor of production denoted by A, where $\alpha \in \langle 0,1 \rangle$ and A>0 Next we present an example using the Cobb-Douglas function.

The company RENAP produces a good. The production function of a good Q is given by the function, $Q = f(K, L) = 80K^{1/4}L^{1/2}$, and its sale price is P=40 dollars. The prices of the capital and labor factors are 80 and 40 respectively. Calculate the levels of capital and work, with which they maximize the utility and minimize the costs of the company, considering that the company has an investment amount of 10240 dollars.

To formulate the mathematical model, the utility and cost functions of the company must be defined, in fact, the income function $I(K, L) = PQ = (40)80K^{1/4}L^{1/2}$, the cost function $C(K, L) = 80K + 40L$ and the utility function $U(K, L) = (40)80K^{1/4}L^{1/2} - 80K - 40L$ and also considering the restrictions $80K + 40L \leq 10240, K > 0, L > 0$.

Therefore, the problem to be optimized is given by:

$$\min\{(-U(K, L), C(K, L))\}$$
$$\text{s.a} \quad 80K + 40L \leq 10240$$
$$K > 0, L > 0$$

where

$$U(K, L) = (40)80K^{1/4}L^{1/2} - 80K - 40L, \; C(K, L) = 80K + 40L$$
$$C = \{(K, L) \in \mathbb{R}^2 : 80K + 40L \leq 10240; K > 0, L > 0\}$$

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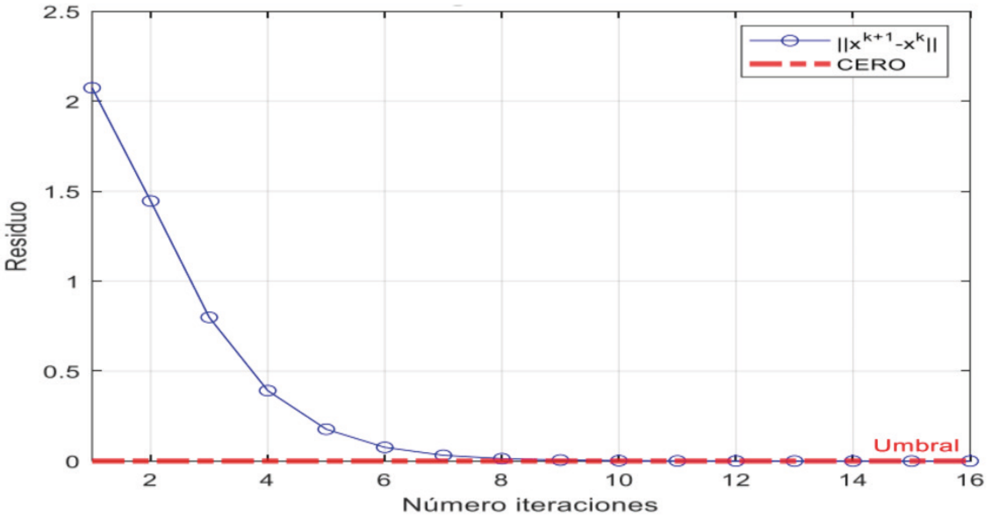

Fig 4. Distances of consecutive points with $\lambda_k = 0,4$

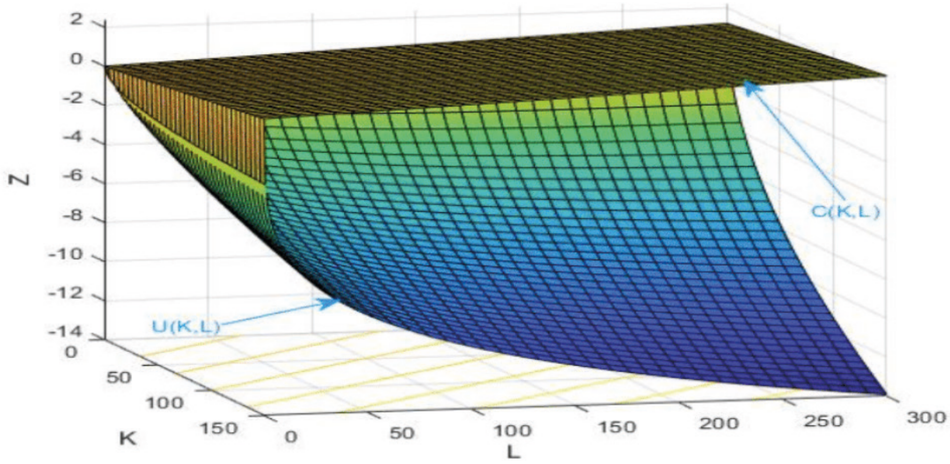

Fig 5. Component functions objective utility and cost

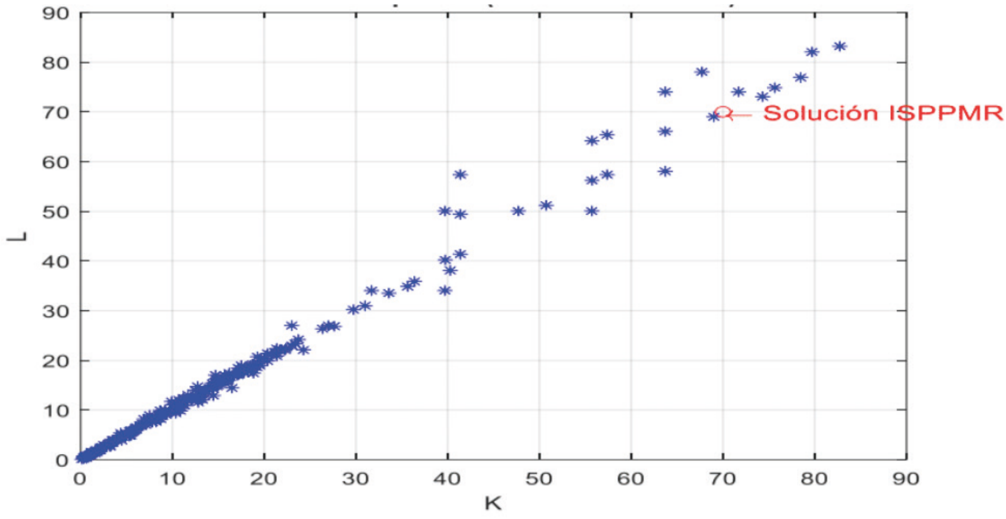

Fig 6. Pareto solutions in set C

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With the cost and utility functions defined above, the vector function $(U(.), C(.))^T$ is quasi-convex on the convex set C. This problem satisfies hypotheses (H1), (H2) and (H3), just like the examples in the previous chapter. Furthermore, it can be seen in Figure 6 that the point (70,70) is a Pareto solution.

Now taking as starting point $x^0 = (x_1{}^0, x_2{}^0)$ and given $x^k \in C$, the main step of the algorithm proposal is to find a critical point (local minimum) of the following problem.

$$\begin{cases} min g(K,L) = (3200K^{1/4}L^{1/2} - 80K - 40L)z_1{}^k + \\ (80K + 40L)z_2{}^k + \lambda_k L_d\left(\left((K,L),(K^k,L^k)\right)\right) \\ \\ Sa: \\ 3200K^{1/4}L^{1/2} - 80K - 40L \leq 3200(K^k)^{1/4}(L^k)^{1/2} - \\ 80K^k - 40L^k \\ 80K + 40L \leq 80K^k + 40L^k \end{cases}$$

Considering the fixed parameters $z_k = (z^k{}_1, z^k{}_2)$, λ_k for each *k*. In addition, by doing $(K,L) = (x_1, x_2)$, the problem described as:

$$\begin{cases} min g(x_1,x_2) = (3200x_1{}^{1/4}x_2{}^{1/2} - 80x_1 - 40x_2)z_1{}^k + \\ (80x_1 + 40x_2)z_2{}^k + \lambda_k L_d\left(\left((x_1,x_2),(x_1{}^k,x_2{}^k)\right)\right) \\ \\ Sa: \\ 3200x_1{}^{1/4}x_2{}^{1/2} - 80x_1 - 40x_2 - 3200(x_1{}^k)^{1/4}(x_2{}^k)^{1/2} + \\ +80x_1{}^k + 40x_2{}^k \leq 0 \\ 80x_1 + 40x_2 - 80x_1{}^k - 40x_2{}^k \leq 0 \end{cases}$$

where we consider, for the numerical experimentation of this example, the following proximal distances.

i.- Kullback -Leibler Distance Bregman,

$$L_d(x,y) := \sum_{i=1}^{2} x_i ln\left(\frac{x_i}{y_i}\right) + y_i - x_i$$

ii. - Itakura-Saito distance

$$L_d(x,y) := \sum_{i=1}^{2} \left[\frac{x_i}{y_i} - log\left(\frac{x_i}{y_i}\right) - 1\right]$$

iii. -Second order Homogeneous distance

$$L_d(x,y) := \sum_{i=1}^{2} \frac{\nu}{2}(x_i - y_i)^2 + \sigma\left(y_i{}^2 log\frac{y_i}{x_i} + x_i y_i - y_i{}^2\right).$$

With $\sigma = 0001$and $\nu = 0.01.$ and .

Next, we present the numerical experiments, taking the proximal distances mentioned above and the fixed parameters z_k, λ_k.

1.- Kullback -Leibler Distance Bregman, defined by

$$L_d(x,y) := \sum_{i=1}^{2} x_i ln\left(\frac{x_i}{y_i}\right) + y_i - x_i$$

and the fixed parameters given by $z_k = (z^k{}_1, z^k{}_2) = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$, $\lambda_k = 0.1$, also the point x^0=(100, 10) where iteration 0 is obtained and the initial point p^0=(100, 10) to solve all the subproblems using the MATLAB functionality with the SQP algorithm and we consider the stopping criterion $\|x^{k+1} - x^k\| < 0.00001$ to finish the ISPPMR algorithm.

k	N[x^k]	$x^k = (x_1^k, x_2^k)$	$\sum F_i(x^k)z^k{}_i$
1	6	(70.02529,69.94940)	-42879.97922
2	6	(69.99998,70.00002)	-42879.98458
3	8	(70.00001, 69.99996)	-42879.98458
4	6	(69.99997,70.00005)	-42879.98458
11	6	(69.99997,70.00005)	-42879.98458
12	6	(69.99997,70.00005)	-42879.98458

Table 3. Results of iterations of the ISPPMR algorithm using the Kullback-Leibler Bregman distance with $\lambda_k = 0.1$

Table 3. shows that with *k*=12 iterations we approach of the solution of the problem. Likewise, we can observe in the table that the sequence $\{x^k\}$ converges to the point (70, 70), which is a Pareto solution as indicated in Figure 6. On the other hand, Figure 7 shows the solutions that form the Pareto frontier, in particular the solution of the problem determined by the proposed algorithm.

In addition, also we observe that Figure 8 shows us the difference of two consecutive steps that are converging to zero, according to the iterations.

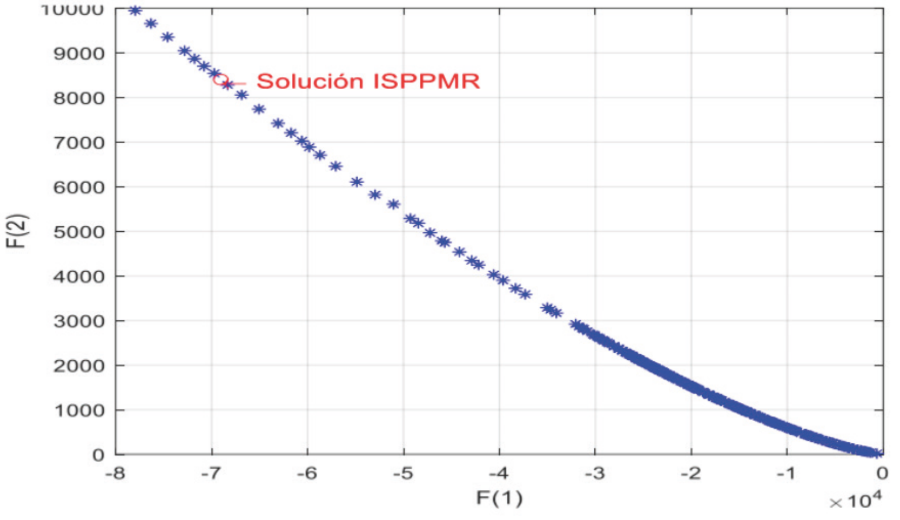

Fig 7. Pareto frontier utility and cost function

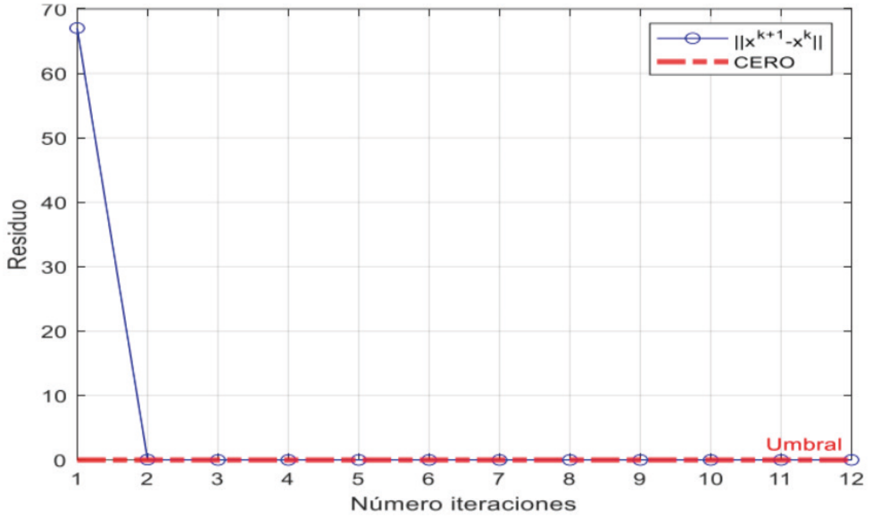

Fig 8. Distances of consecutive points

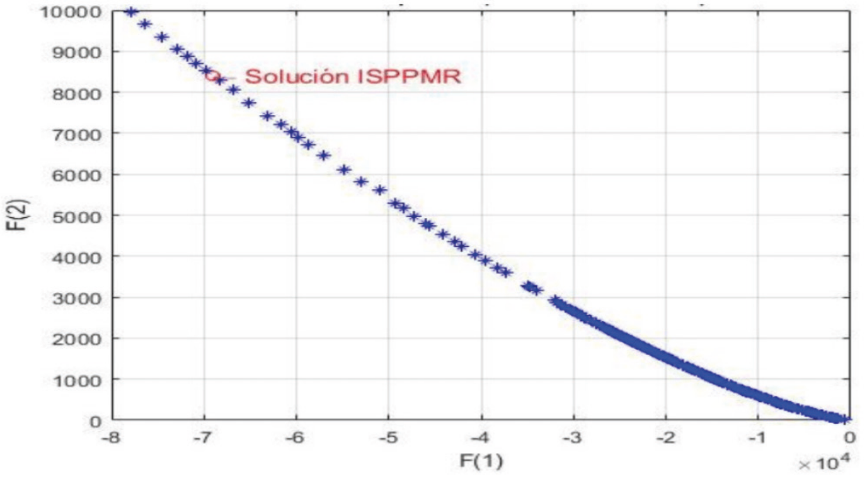

Figure 9. Pareto frontier utility and cost function

2. Itakura-Saito distance defined by

$$L_d(x,y) := \sum_{i=1}^{2}\left[\frac{x_i}{y_i} - log\left(\frac{x_i}{y_i}\right) - 1\right]$$

and the fixed parameters $z_k = (z^k{}_1, z^k{}_2) = \left(\frac{1}{\sqrt{3}}, \sqrt{\frac{2}{3}}\right)$, $\lambda_k = 0.01$, also the point x^0=(100, 10) where iteration 0 is obtained and the initial point p^0=(100, 10) to solve all the subproblems using the MATLAB functionality with the SQP algorithm and we consider the stopping criterion $\|x^{k+1} - x^k\| < 0.00001$ to finish the ISPPMR algorithm.

k	N[x^k]	$x^k = (x_1^k, x_2^k)$	$\sum F_i\,(x^k) z^k{}_i$
1	6	(70.00256, 69.99487)	-33002.53174
2	6	(69.99999,70.00001)	-33002.53178
3	6	(69.99999,70.00001)	-33002.53178

Table 4. *Results of numerical experimentation with Itakura distance Saito*

Table 4. shows that with k=3, internal iterations approach the solution of the problem. Likewise, we can see in the table that the sequence x^k converges towards the point (70, 70), which is a Pareto solution like the previous case that was considered the Kullback -Leibler proximal distance. Bregman. On the other hand, Figure 9 shows the solutions that form the Pareto frontier, in particular the solution to the problem determined by the proposed algorithm.

Furthermore, it should also be noted that Figure 10 shows us the difference of two consecutive steps that converge to zero, according to the iterations.

3.- Second order Homogeneous Distance defined by

$$L_d(x,y) := \sum_{i=1}^{2}\frac{\nu}{2}(x_i - y_i)^2 + \sigma\left(y_i{}^2 log\frac{y_i}{x_i} + x_i y_i - y_i{}^2\right).$$

With $\sigma = 0001$ and $\nu = 0.01$, and the fixed parameters $z_k = (z^k{}_1, z^k{}_2) = \left(\frac{1}{2}, \sqrt{\frac{3}{4}}\right)$, $\lambda_k = 0.01$, also the point x^0=(100, 10) where iteration 0 is obtained and the initial point p^0=(10, 10) to solve all the subproblems using the MATLAB functionality with the SQP algorithm and we consider the stopping criterion $\|x^{k+1} - x^k\| < 0.00001$ to finish the ISPPMR algorithm.

k	N[x^k]	$x^k = (x_1^k, x_2^k)$	$\sum F_i\,(x^k) z^k{}_i$
1	6	(70.01312, 69.97374)	-27246.11346
2	6	(69.99998, 70.00003)	-27246.11448
3	6	(69.99997, 70.00004)	-27246.11448
4	7	(70.00000, 69.99999)	-27246.11448
5	6	(69.99997, 70.00004)	-27246.11448
6	6	(69.99997 70.00004)	-27246.11448

Table 5. *Results of numerical experimentation with second-order Homogeneous distance*

Table 5. shows that with k=6, internal iterations approach the solution of the problem. Likewise, we can see in the table that the sequence x^k converges towards the point (70, 70), which is a Pareto solution like the two previous cases where the Kullback -Leibler proximal distances were used. Bregman and Itakura -Saito. On the other hand, Figure 11 shows the solutions that form the Pareto frontier, in particular the solution to the problem determined by the proposed algorithm.

Furthermore, it should also be noted that Figure 12 shows us the difference of two consecutive steps that converge to zero, according to the iterations.

In the numerical experiments carried out, it has been observed that with the three proximal distances used, a Pareto optimal solution has been reached, with the difference that with the Kullback -Leibler distance Bregman, and the parameters $(z^k{}_1, z^k{}_2) = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$, $\lambda_k = 0.1$, 12 iterations were obtained, with the Itakura-Saito distance and the parameters $(z^k{}_1, z^k{}_2) = \left(\frac{1}{\sqrt{3}}, \sqrt{\frac{2}{3}}\right)$, $\lambda_k = 0.01$. 3 iterations were obtained and with the

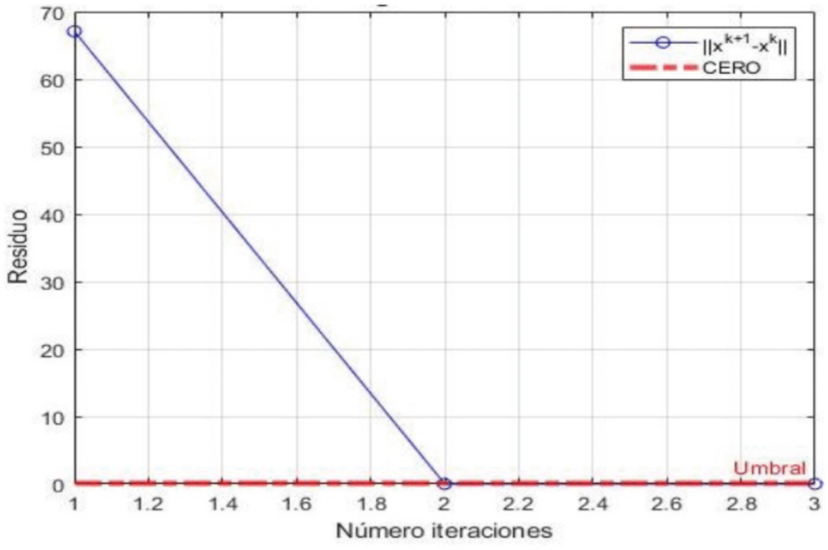

Figure 10. Distances of consecutive points

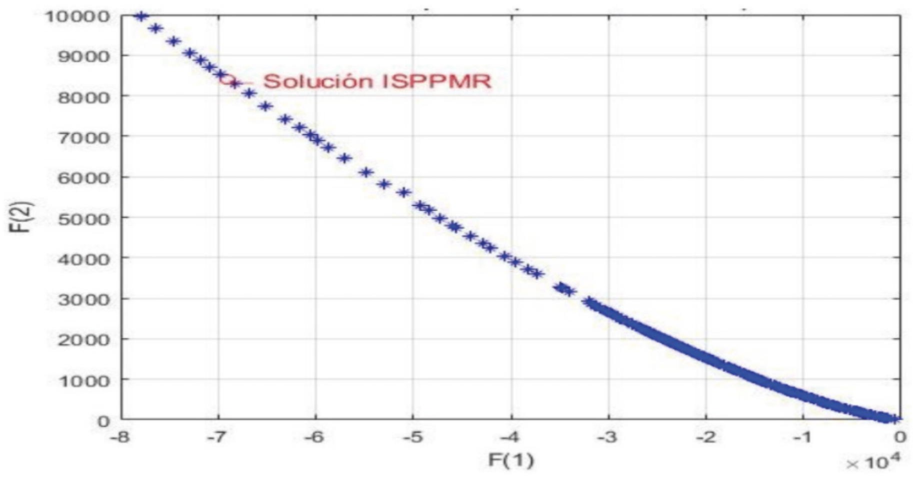

Figure 11. Pareto frontier utility and cost function

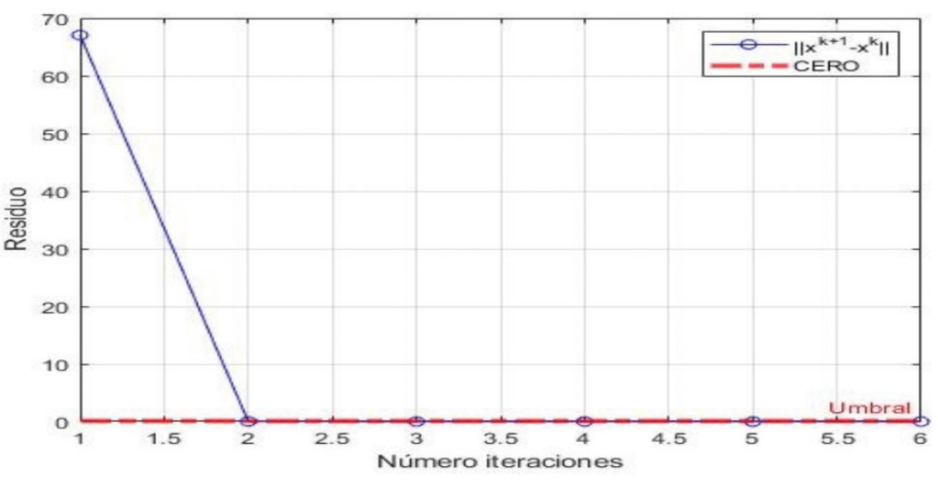

Figure 12. Distances of consecutive points

second-order Homogeneous distance and the parameters $(z^k{}_1, z^k{}_2) = \left(\frac{1}{2}, \sqrt{\frac{3}{4}}\right)$, $\lambda_k = 0.01$, 6 iterations were obtained.

Finally, we can conclude that the company RENAP, when producing the good, will have an approximate maximum utility of 69,041 dollars and an approximate minimum cost of 8,400 dollars, when the prices of the capital and labor factors are 69.99 and 70 respectively.

METHODOLOGY

In the present investigation we use the Optimization methodology mathematics, in which it consists of giving a good approach to the multi-objective optimization problem to then apply the algorithm of the proximal method, in such a way that the sequence of points generated by the algorithm exists and also converges to a solution of the problem. On the other hand, it is of a non-experimental type because no manipulation of the independent variables is done, where its design is correlational descriptive since its purpose is to investigate the convergence of values that present the properties of the proximal optimization method.

RESULTS AND DISCUSSIONS

The examples presented in this document extend those presented by Papa Quiroz et al. [15] to solve constrained quasi-convex multiobjective optimization problems.

The numerical experimentation shows in detail the convergence of the proximal algorithm to a Pareto critical point using a Bregman proximal distance and the variation of the $\{\lambda_k\}$ parameter. With respect to the application example, the proximal distances, Kullback--Leibler Bregman, Itakura-Saito, and second-order homogeneous distances were used, where we show the Pareto solutions and the Pareto frontier solutions, evidencing the convergence of the proximal algorithm to a critical Pareto point.

CONCLUSIONS AND FUTURE WORK

In conclusion, the present work consists in the elaboration of a desktop application, which seeks to generate This article presents a numerical experimentation and application of the proximal algorithm for multiobjective quasi--convex problems with restrictions. The algorithm can be used to solve models in the theory of consumption, demand, and production. Future research may be the computational implementation of the studied method for some real context problems in decision theory.

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