v. 1, n. 3, 2024

International Journal of
Exact Sciences

Acceptance date: 23/10/2024

A NEW METHOD FOR FINDING A SIMPLE ROOT

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Abstract: A new variant of the cubic order iterative method of Weerakoon Sunethra and Fernando is presented. The main modification lies in the replacement of the integral approximation by the arithmetic mean. This adaptation simplifies the method without compromising its order of convergence. Numerical experiments will be carried out to compare the performance of both methods.
Keywords: method, arithmetic mean, order of convergence, Weerakoon Sunethra, Fernando.

INTRODUCTION

In this paper, a new numerical algorithm for solving nonlinear equations is proposed. This method, inspired by the work of Sunethra Weerakoon et al (2000), uses the arithmetic mean and the trapezoid rule to obtain approximate solutions. The numerical results obtained demonstrate the high efficiency of this new approach be it f in an open interval $I_{(a,b)}$ which satisfies that $f(a) = 0$ (*It is the root of the function*) y $f'(a) \neq 0$. Newton approximates a single root of a nonlinear equation $f(x) = 0$ where by evaluating $a \in \mathbb{R}$. It mathematically satisfies that $f(a) = 0$. According to Gutierrez (2019), the method exhibits a quadratic convergence rate.

SOME DEFINITIONS

Babajee (2006) defines the convergence of a sequence as , $x_n \in \mathbb{R}$, $n = 0,1,2,3...$ to a root as follows: the sequence converges to to if the limit of the difference between the terms of the sequence is equal to zero. $\lim_{n\to\infty} |x_n - \alpha| = 0$ is equal to zero.

According to Babajee (2006), given a sequence and a root, a succession $\{x_n\}$ and a root, it is said that the sequence converges to a with order of convergence q if there exists a positive constant $c \geq 0$ and a **positive** integer $n_0 \geq 0$ and $q \geq 1$ such that for all , Babajee (2006) indicates that, when the **error in** the nth iteration, one has the following relation $e_n = x_n - a$ in the nth iteration, we have the following **relation** $e_{n+1} = ce_n^q + O\left(e_n^{q+1}\right) = O\left(e_n^{q+1}\right)$ in a successive iteration is less than or equal to a constant c multiplied by the error in the previous iteration raised to power q being q equal to 2 or 3, the convergence is said to be quadratic or cubic, respectively. This $|x_{n+1} - a| \leq c|x_n - a|^q$ is known as the error equation and the value of q represents the order of convergence of the method. Likewise, Babajee (2006) defines the efficiency index of an iterative method through equation (1).

$$EEF = q^{\frac{1}{d}} \qquad (1)$$

Babajee (2006) indicates that, in this equation, q represents the order of convergence of the method, while d corresponds to the total number of function evaluations and their derivatives required in each iteration.

METHODOLOGY OR DEVELOPMENT

Weerakoon et al (2000) proposed a third order method of convergence, based on Newton's theorem, for the numerical approximation of indefinite integrals.

$$f(x) = f(x_n) + \int_{x_n}^{x} f(\lambda)d\lambda \qquad (2)$$

The indefinite integral of Equation (2) is approximated numerically using the method of trapezoids.

$$\int_{x_n}^{x} f(\lambda)d\lambda \approx \frac{1}{2}(x - x_n)(f'(x_n) + f'(x)) \qquad (3)$$

Substituting Equation (3) into Equation (2) allows us to derive Equation (4).

$$x_{n+1} = x_n - \frac{2f(x_n)}{f'(x_n) + f'(y_n)} \qquad (4)$$

Where

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)} \qquad (5)$$

The two-stage method proposed by Weerakoon et al in (2000) exhibits a cubic order of convergence, as indicated by Equation (6).

$$e_{n+1} = \left(C_2^2 + \frac{1}{2}C_3\right)e_n^3 + O(e_n^4) \qquad (6)$$

Its efficiency index is given by:

$$EEF = q^{\frac{1}{d}} = 5^{\frac{1}{3}} = 1.709975945759 \quad (7)$$

The proposed method requires the fulfillment of certain relationships based on the mean theorem, developed by the author in collaboration with Ilhan (2013). Although a general relationship between means is established, the present study will focus specifically on the arithmetic mean, defined in Equation (8).

$$\bar{x} = \frac{\sum_{i=1}^{n} x_i}{n} \quad (8)$$

$$f'\left(\frac{x+x_n}{2}\right) = \frac{f'(x)+f'(x_n)}{2} \quad (9)$$

By substituting the integral of Equation (9) in Equation (4), an equivalent equation is obtained which allows the development of a new algorithm, a variant of the method proposed by Weerakoon et al (2000).

$$x_{n+1} = x_n - \frac{f(x_n)}{f'\left(\frac{x_n+x_{n+1}^*}{2}\right)} \quad (10)$$

$$x_{n+1}^* = x_n - \frac{f(x_n)}{f'(x_n)} \quad (11)$$

Unlike the method proposed by Weerakoon et al (2000), this new variant exhibits a convergence of cubic order. Its convergence rate, which quantifies the speed of approaching the solution is:

$$EEF = q^{\frac{1}{d}} = 4^{\frac{1}{3}} = 1.5874010446328 \quad (12)$$

From an initial value, the iterative algorithm based on Equations (4) and (5) of Weerakoon et al. (2000) produces a succession of values that converge to the root of the equation.

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)} \quad (13)$$

$$x_{n+1} = x_n - \frac{2f(x_n)}{f'(x_n)+f'(y_n)} \quad (14)$$

The stopping criterion defined by equations (15-16) can be used.

$$|x_{n+1} - x_n| \leq \varepsilon \quad (15)$$

$$|f(x_{n+1})| \leq \varepsilon \quad (16)$$

Where ε the maximum tolerance error in the approximation.

The proposed algorithm demonstrates a significant improvement in computational efficiency compared to the methods developed by Weerakoon et al. (2000).

the following iterative process allows us to approximate the root x_{n+1} from a given initial value x_0.

$$y_n = \frac{x_n+z_n}{2} \quad (11)$$

$$z_n = x_n - \frac{f(x_n)}{f'(x_n)} \quad (12)$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'\left(\frac{x_n+z_n}{2}\right)} \quad (13)$$

The same stopping criterion is used in this new method as in the previous one.

The same procedure is used to establish the order of convergence.

Application 1. Taking figure (1) as a reference, the problem of determining the simple root is posed by the following values x_0=2.5, 3.5, $\varepsilon=10^{-5}$ a=2.

Figure 1. shows where the exact root is found in the value of x=2.

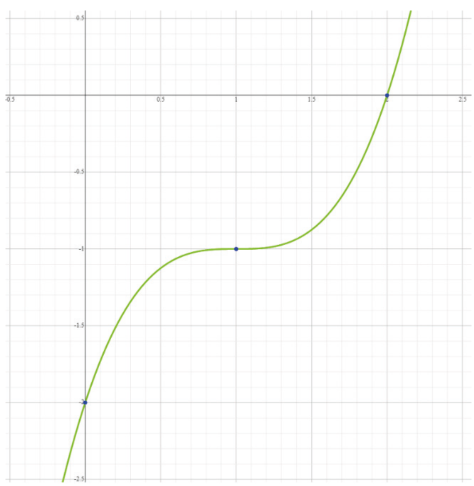

Figure 1. Graphical representation of the Function $f(x) = (x=1)^3 - 1$

itera-tion	Weerakoon and Fer-nando	New Method	Error a for Weerakoon and Fer-nando	Error a for New Method
1	2.0562711	2.0484376	0.056711	0.0484376
2	2.0001843	2.0000951	0.0001849	0.0000951
3	2	2	0	0

Table 1. Comparison of the two iterative methods the function $f(x) = (x-1)^3 - 1$, x_0=2.5, $\varepsilon = 10^{-5}$ a =2

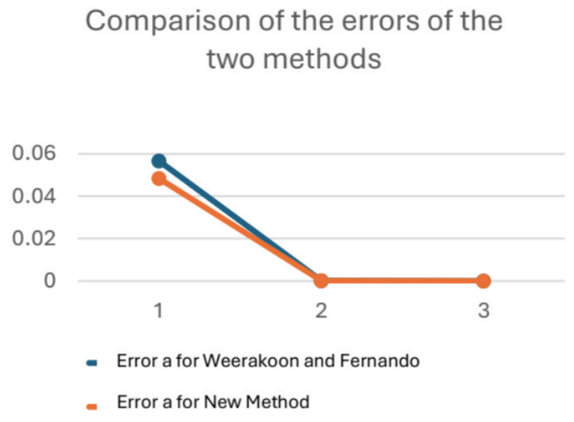

Figure (2). Comparison of the errors of the two methods

Application 2. carry out the same procedure for the function $f(x) = e^{x^2+7x-30} - 1$, $x_0 = 3.25$, 3.5, $\varepsilon=10^{-5}$ $\alpha=3$ where, as shown in Figure (2), the exact solution is observed.

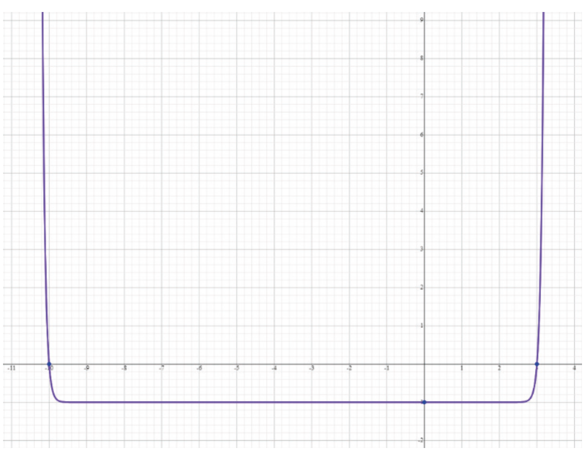

Figure 3. Graphical representation of $f(x)=e^{x^2+7x-30} - 1$

itera-tion	Wee-rakoon and Fer-nando	New Method	Error a for Weerakoon and Fer-nando	Error a for New Method
1	3.1465125	3.1339786	0. 1465125	0. 1339786
2	3.0562022	3.0392299	0. 0562022	0.0392299
3	3.0063872	3.0017476	0. 0063872	0. 0017476
4	3.0000143	3.0000002	0. 0000143	0.000002
5	3	3	0	0

Table 2. Comparison of the two iterative methods the function $f(x)=e^{x^2+7x-30}-1$, $x_0=3.25$, $\varepsilon=10^{-5}$ $\alpha=3$

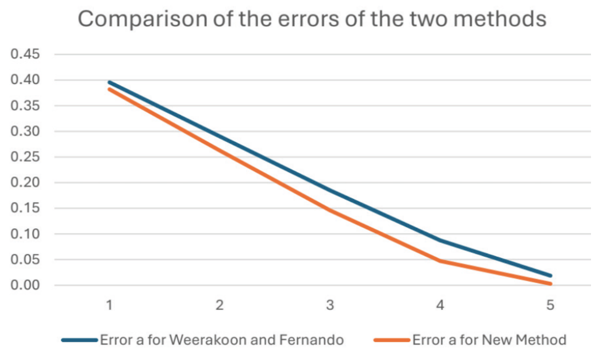

Figure (4). It shows the errors made in the comparison of the two methods.

itera-tion	Wee-rakoon and Fer-nando	New method	Error for the method of Weerakoon and Fernando	Error for the new method
1	3.3955736	3.381989	0.3955736	0.3819890
2	3.290076	3.2628131	0.290076	0.2628131
3	3.1851311	3.1459777	0.1851311	0.1459777
4	3.0875299	3.0472534	0.0875299	0.0472534
5	3.0186534	3.0029125	0.0186534	0.0029125
6	3.0003214	3.000001	0. 0003214	0. 000001
7	3	3	0	0

Table 3. Comparison of the two iterative methods the function $f(x)=e^{x^2+7x-30}-1$, $x_0=3.25$, $\varepsilon=10^{-5}$ $\alpha=3$

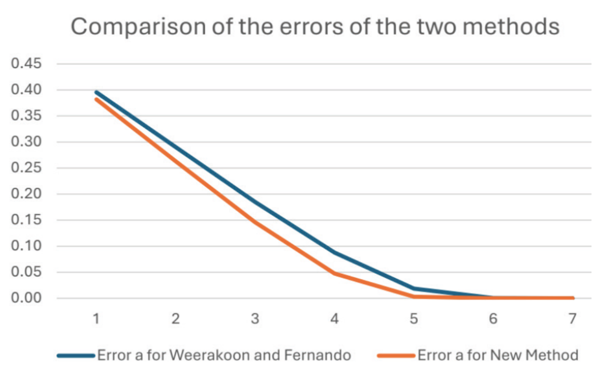

Figure (5). Shows the iteration and error committed by each of the methods.

Application 4. Let the function of Fig. 4. Given by $f(x)=x^5+17x$ We want to calculate the simple root.

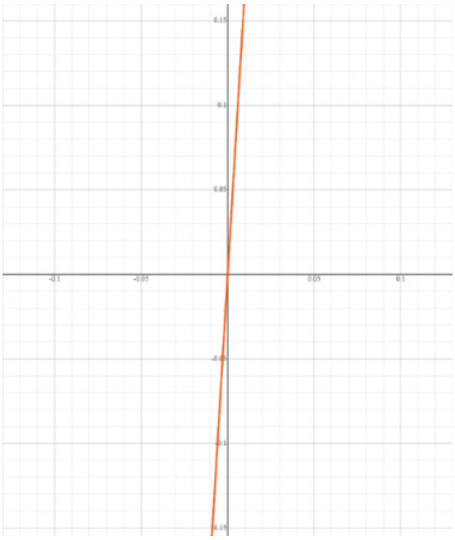

Figure 6. Graphical representation of the function $f(x)=x^5+17x$

itera-tion	Wee-rakoon and Fer-nando	New method	Error for Weerakoon and Fer-nando	Error for New method
1	0.0770505	-0.022169	0.0770505	0.0221691
2	0.0000002	0	0.0000002	0
3	0	0	0	0

Table 4. Comparison of the two iterative methods the function $f(x)=x^5+17x$, $X_0=1$, $\varepsilon=10^{-5}$ $\alpha=0$

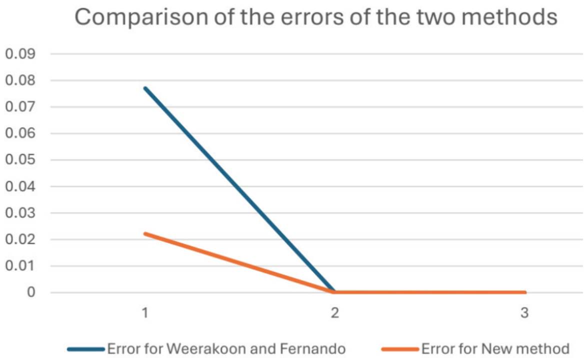

Figure (7). Shows the iteration and error committed by each of the methods.

itera-tion	Weerakoon and Fer-nando	New method	Error for We-erakoon and Fernando	Error for New method
1	0.0770505	-0.022169	0.0770505	0.0221691
2	0.0000002	0	0.0000002	0
3	0	0	0	0

Table 5. Iteration for solving the function $f(x)=x^5+17x$, $X_0=1$, $\varepsilon=10^{-5}$ $\alpha=0$

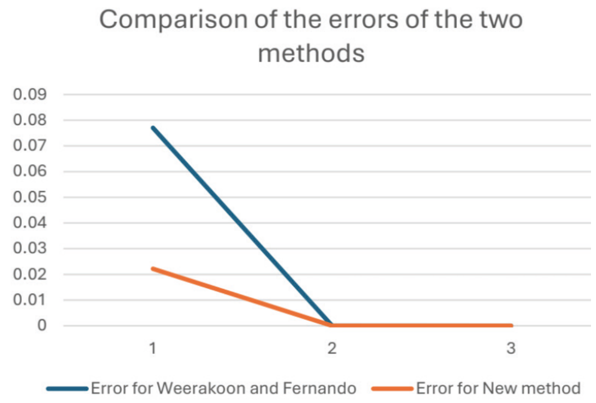

Figure 8. Shows the iteration and error committed by each of the methods.

Itera-tion	Weerakoon and Fer-nando	New method	Error for Weerakoon and Fer-nando	Error for New method
1	0.0027322	-0.012284	0.0027322	0.012284
2	0	0	0	0
3	0	0	0	0

Table 6. Comparison of the two iterative methods the function $f(x)=x^5+17x$, $X_0=1$, $\varepsilon=10^{-5}$ $\alpha=0$

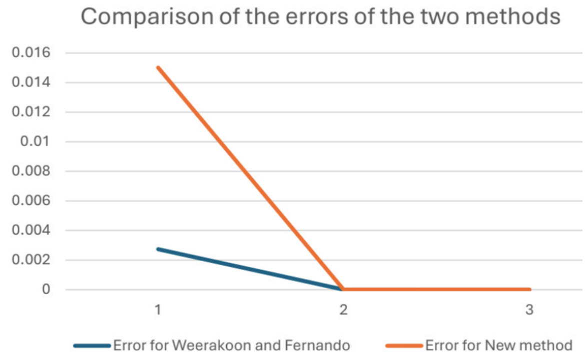

Figure (8). Shows the iteration and error committed by each of the methods.

RESULTS AND ANALYSIS

The numerical results obtained indicate that the new proposed method exhibits a faster convergence to the desired approximation error compared to the method of Sunethra Weerakoon and T.G.I. Fernando. This statement is supported by multiple simulations implemented in C language, which demonstrate a higher efficiency of the new method, as evidenced by the indices calculated according to equations (7) and (12).

CONCLUSIONS

The results obtained in this research show that the proposed new iterative method offers a more efficient alternative for finding a simple root of a nonlinear equation compared to the methods of Sunethra Weerakoon and T.G.I. Fernando. Specifically, the new method demands a smaller number of functional evaluations and exhibits a higher efficiency index, which translates into a faster convergence and, therefore, a reduction in the number of iterations required to reach the desired solution.

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