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BORIS GRIGORYEVICH GALIORKIN'S METHOD APPLIED TO BEAM DEFLECTION

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Abstract: The article focuses on the application of the Galerkin method to approximate the deflection of a beam. This method, which allows solving differential equations in an approximate way, stands out for its versatility and ease of application compared to methods such as Laplace and Fourier. The strategy consists of integrating the differential equation and proposing a test function that meets the boundary conditions, thus obtaining an approximate analytical solution.
Keywords: Deflection, Laplace, Fourier, Fourier, Galerkin, beam, beam

INTRODUCTION

In 1915 Galerkin, proposed a method of approximation of solution of boundary value problems that does not require the variational formulation of the problem, in the year 2016 was the 100th anniversary of this method, the article contains a brief description and origin of the method as well as its development [1], therefore provides a clearer and more general approximation. This method can be applied to the solution of partial differential equations of elliptic, hyperbolic and parabolic type, as well as linear and nonlinear problems. When the variational form of a problem with boundary conditions exists, it can be shown that the Ritz and Galerkin methods are equivalent and produce identical results. Therefore, instead of trying to develop the equivalent variational form for a given boundary-valued problem and applying Ritz's method, one can apply Galerkin's method directly to the boundary-valued problem. Galerkin's method is the means by which an ordinary or partial differential equation can be converted to an integral problem in order to transform it to a system of linear algebraic equations. Where the coefficients obtained are substituted into the test function.

METHODOLOGY OR DEVELOPMENT

In this study, the equation of a uniformly loaded embedded beam was solved. Galerkin's method was used to find an approximate solution and compared with the exact solution through graphs.

The fundamental idea of Galerkin's method can be exemplified by the boundary value problem described by equations (1).

$$L[u(r)] = 0 \text{ in the region } R \quad (1)$$
$$B[u(r_r)] = f(r_r) \text{ on the border } S \quad (2)$$

Where L is a linear differential operator for example:

$$L[u] = \nabla^2 u + Au + \left(\frac{1}{k}\right) g \quad (3)$$

Where B is a linear boundary condition operator

$$B[u] = k\left(\frac{\partial u}{\partial n}\right) + hu \quad (4)$$

where $\frac{\partial T}{\partial n}$ denotes the derivative along the outward normal to the boundary at the surface in steady state heat transfer application.

$$\nabla^2 T(r) + AT(r) + \frac{1}{k} g(r) = 0 \ en \ R \quad (5)$$
$$k\frac{\partial T}{\partial n} + hT = f(r_s) \quad (6)$$

equations (5- 6) is of the elliptic type which models steady state phenomena. These equations arise in areas such as fluid dynamics, heat transfer, electromagnetism, geophysics, biology, among others. The best known of these equations are the Laplace and Poisson equations.

The following procedure is used to solve these equations.

It is proposed ϕ_j (r) = 1,2,3 ..., n is a set of basis functions. The term $\tilde{T}_n$ (r) The one known as the test function is as follows:

$$\tilde{T}_n(r). = \Psi_0(r) + \sum_{j=1}^{n}$$
$$C_j \, \phi_j(r) \text{ in the region } (7)$$

Where the function $\psi_0(r)$ satisfies the non-homogeneous part of the boundary conditions of equation (2) and the functions ϕ_j (r)=1,2,...,n is a set of orthogonal functions, satisfy the homogeneous part, i.e.

$$B[\Psi_0(r)] = f(r_S) \qquad (8)$$

$$B\left[\phi_j(r)\right] = 0 \; j = 1,2 \ldots, n \quad (9)$$

An approximate solution is proposed that satisfies the boundary conditions but does not solve the differential equation exactly, generating an error. To minimize this error, the coefficients of the test solution are adjusted so that the error is orthogonal to a set of basis functions. This procedure is known as the Galerkin method.

$$\tilde{T}_n(r) = \Psi_0(r) + \sum_{j=1}^{n} C_j \, \phi_j(r) \quad (10)$$

To find an approximate solution, the same basis functions of equation (7) are used. If the non-homogeneous part is zero, the problem is simplified. When applying the Galerkin method, the error generated by the approximate solution is minimized by imposing conditions of orthogonality of the residual with respect to the basis functions.

$$\nabla^2 \tilde{T}_n(r) + A\tilde{T}_n(r) + \frac{1}{k} g(r) =$$

$$R(C_1, C_2, \ldots, C_n; r) \neq 0 \quad (11)$$

This method allows to calculate the unknown coefficients C_2, C_2, ..,C_n unknown coefficients by means of

$$\int_R \left(\nabla^2 \tilde{T}_n(r) + A\tilde{T}_n(r) + \frac{1}{k} g(r)\right)$$

$$\phi_j(r) dv = 0 \qquad (12)$$

Equation (12) can be expressed in a more compact form, as shown in eq. (13).

$$\int_R \phi_j(r) \, R(C_1, C_2, \ldots, C_n; r) dv = 0$$

$$j = 1,2, \ldots, n \quad (13)$$

The objective of establishing this relationship is to obtain a system of equations that allows the unknown coefficients to be calculated. Equation (13) ensures that the error is minimal in a specific sense. By restricting the solution to a finite space, the Galerkin method provides an approximate solution. This method takes advantage of the orthogonality principle to solve differential equations efficiently.

The construction of the test functions is described, following the recommendations of [2]. These functions must be smooth and form a complete functional space. For boundary conditions of the first kind, we look for functions that cancel on the boundary and are sufficiently regular in the interior of the domain. A common approach is to construct these functions from products of basis functions and powers of the independent variable, as shown in equation (14)

$$\phi_1 = w \,, \phi_2 = wx, \, \phi_3 = wy \,,$$

$$\phi_4 = wx^2 \;\;, \phi_5 = wxy \quad (14)$$

The constructed functions satisfy the boundary conditions, are sufficiently smooth and form a complete system. The problem is reduced to finding the coefficients of these functions. These coefficients are obtained by applying the boundary conditions

1. For domains with simple and smooth boundaries, such as the circle, orthonormal sets of basis functions are available.

$$F(x, y) = 0 \quad (15)$$

The function $F(x,y)$ is continuous and has continuous partial derivatives with respect to x e y. The function $w(x,y)$ can be selected as :

$$w(x, y) = \pm F(x, y) \quad (16)$$

For a circular region of radius R centered at the origin, the equation of the boundary satisfies equation

$$F(x, y) = R^2 - x^2 - y^2 = 0 \quad (17)$$

The weight function $w(x,y)$ is taken as

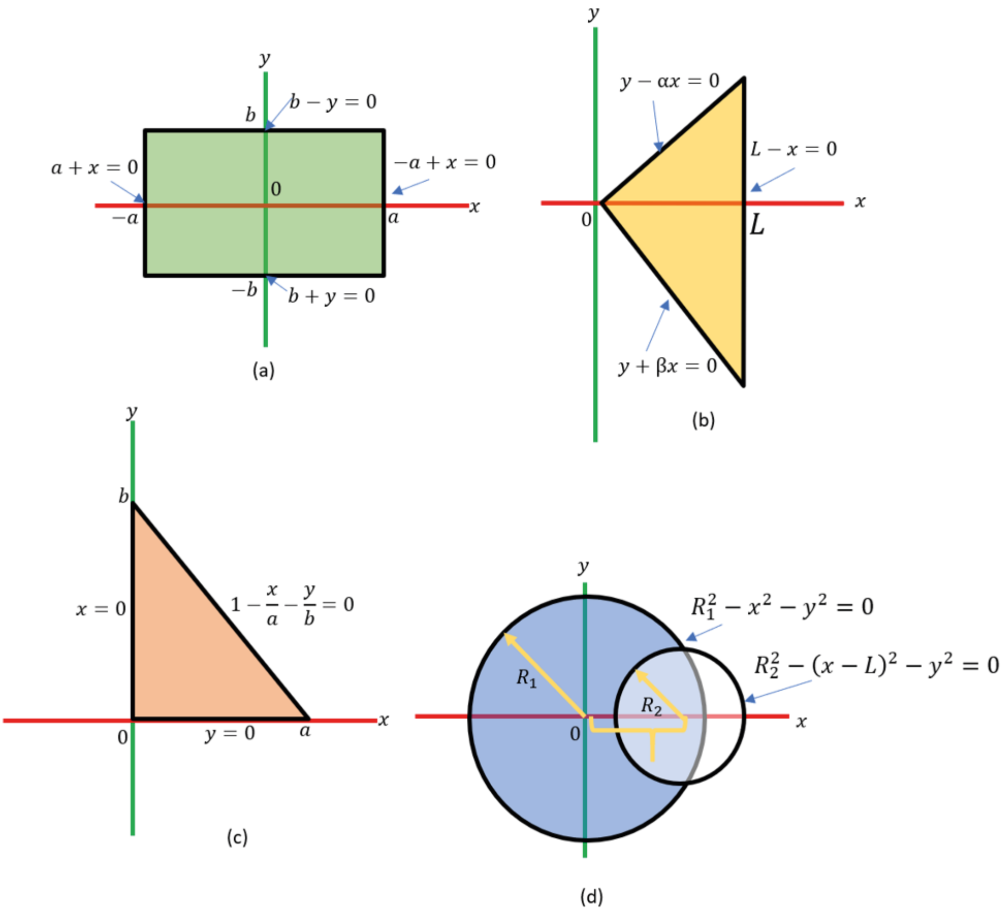

Figure (1) contour region obtained from [2].

$$w(x, y) = R^2 - x^2 - y^2 = 0 \qquad (18)$$

2. In regions bounded by convex polynomials, the equations of their sides are expressed as:

$$F_1(x, y) = a_1 x + b_1 y + d_1 = 0 \quad , \quad b_n y + d_n = 0 \ (19)$$

$$F_2(x, y) = a_2 x + b_2 y + d_2 = 0, .., $$

$$F_n(x, y) = a_n x + b_n y + d_n = 0 \ (19)$$

the weight function $w(x,y)$ is chosen as follows:

$$w(x, y) = \pm F_1(x, y), w(x, y) = \pm F_2(x, y) \ , \ ..., w(x, y) = \pm F_n(x, y) \quad (20)$$

A function is sought that cancels over the entire boundary of the domain and satisfies the homogeneous part of the first type boundary conditions in the region of interest. For each of the four geometries illustrated in Figure 1, multiple solutions will be obtained for such a function as previously defined. The specific boundary conditions for each of these geometries (1a, 1b, 1c and 1d) are detailed below.

$$a - x = 0 \ , a + x = 0 \ , b - y = 0 \quad b - y = 0 \qquad (21)$$

$$y - \alpha x = 0 \quad , y + \beta x = 0 \ , L - x = 0 \qquad (22)$$

$$x = 0 \ , y = 0 \ , 1 - \frac{x}{a} - \frac{y}{b} = 0 \qquad (23)$$

$$R_1^2 - x^2 - y^2 = 0. \ R_2^2 - (x - L)^2 - y^2 = 0 \qquad (24)$$

The weight functions $w(x,y)$ for each geometry are presented below:

$$w(x, y) = (a^2 - x^2)(b^2 - y^2) \qquad (25)$$

$$w(x, y) = (y - \alpha x)(y + \beta x)(L - x) \quad (26)$$

$$w(x,y) = xy(1 - \frac{x}{a} - \frac{y}{b}) \quad (27)$$

$$w(x,y) = (R_1^2 - x^2 - y^2)(R_2^2 - (x-L)^2 - y^2) \quad (28)$$

For each set of boundary conditions, a specific weight function is required ϕ_j function is required for each set of boundary conditions. The test solution is constructed as a linear combination of these functions.

Application

An application consider a beam as shown in Figure. 2.

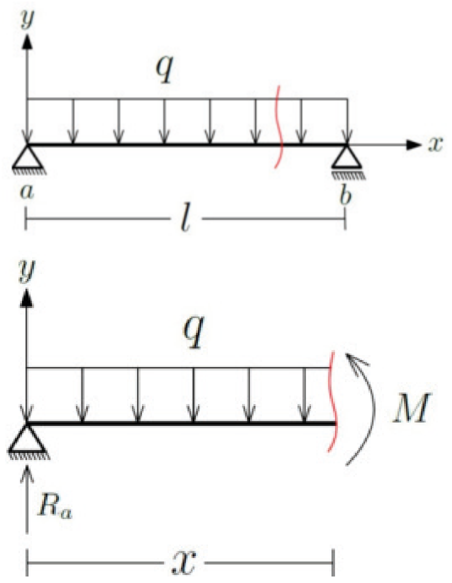

Figure 2. Double-supported beam with distance cut x where $0 \le x \le l$. [3]

As a case study, the application of Galerkin's method to a simply supported beam subjected to bending moments at its ends is considered (Figure 2). where l is the length of the horizontal distance between the points of attachment of the conductor at two consecutive supports, q is the intensity of the static load uniformly distributed over the entire surface and Ra is the reaction at the support, the bending moment equation for the beam in question must be obtained.

The governing equation in general:

$$EI\frac{d^2u(x)}{dx^2} - M(x) = 0 \quad (29)$$

boundary conditions:

$$\begin{matrix} u(0) = 0 \\ u(l) = 0 \end{matrix} \quad (30)$$

For a particular case [3].

$$\frac{d^2u}{dx^2} = \frac{qx}{EI}(l - x) \quad (31)$$

Where: $M(x) = \frac{qx}{EI}(l - x)$

boundary conditions:

$$\begin{matrix} u(0) = 0 \\ u(l) = 0 \end{matrix} \quad (32)$$

The represents deflection of the beam as a function of position x is denoted by $M(x)$. The parameters M(x) represent the bending moment, I y E the moment of inertia and Young's modulus, respectively. The length of the beam is l. Substituting these values in equation (29), the analytical solution is obtained [3]. It is given by equation (33).

$$u(x) = \frac{q}{EI}\left(\frac{lx^3}{16} - \frac{x^4}{24} - \frac{xl^3}{48}\right) \quad (33)$$

Next, the Galerkin method will be applied to the problem.

In general, a linear combination of orthogonal polynomials of the form given by equation (31) is selected as the test function.

$$\tilde{y} = C_1\phi_1 + C_2\phi_2 +, \ldots, C_n\phi_n \quad (31)$$

where the ϕ_j satisfy the boundary conditions, the following is proposed for this particular case

$$\tilde{y} = c_1\phi_1 \text{where } \phi_1 = sen\left(\frac{\pi x}{l}\right) \quad (32)$$

$$\tilde{y} = C_1\phi_1 \quad \text{where } \phi_1 = seno\left(\frac{\pi x}{l}\right) \quad (33)$$

Where:

$$\tilde{y} = C_1 seno\left(\frac{\pi x}{L}\right) \quad (34)$$

The equation satisfies the boundary conditions as shown below:

$$\tilde{y}(0) = C_1 seno\left(\frac{\pi(0)}{l}\right) = 0 \quad (35)$$

$$\tilde{y}(l) = C_1 seno\left(\frac{\pi l}{l}\right) = C_1 seno(\pi) = 0\ para\ \pi = 0, 2\pi, \ldots, n\pi, n\ \text{múltilpo de dos} \quad (36)$$

Substituting (20) in (7) we obtain

$$\int_o^L \left(EI\frac{d^2\tilde{y}}{dx^2} - M(x)\right)\phi_1\, dx = 0 \quad (37)$$

$$\int_o^l \left(\left[\left(C_1 seno\left(\frac{\pi x}{l}\right) - \frac{qx}{EI}(l-x)\right)\right]\right) seno\left(\frac{\pi x}{l}\right) dx = 0 \quad (38)$$

where

$$\tilde{y} = \left(C_1 seno\left(\frac{\pi x}{l}\right)\right) \quad (39)$$

$$\tilde{y}' = C_1\left(\frac{\pi}{l} coseno\left(\frac{\pi x}{l}\right)\right) \quad (40)$$

$$\tilde{y}'' = -C_1\left(\left(\frac{\pi}{l}\right)^2 seno\left(\frac{\pi x}{l}\right)\right) \quad (41)$$

$$\int_o^l \left(-C_1\left(\left(\frac{\pi}{L}\right)^2 seno\left(\frac{\pi x}{l}\right)\right) - \frac{qx}{EI}(l-x)\right) seno\left(\frac{\pi x}{l}\right) dx = 0 \quad (42)$$

$$-C_1\left(\frac{\pi}{l}\right)^2 \int_0^l seno^2\left(\frac{\pi x}{l}\right) dx - \int_0^l \frac{qx}{EI}(l-x) seno\left(\frac{\pi x}{l}\right) dx = 0 \quad (43)$$

They integrate using identity:

$$seno^2(\theta) = \frac{1-coseno(2\theta)}{2} \quad (43)$$

$$-C_1\left(\frac{\pi}{L}\right)^2 \int_0^l seno^2\left(\frac{\pi x}{l}\right) dx = -C_1\left(\frac{\pi}{l}\right)^2\left(\frac{l}{2}\right) = -C_1\left(\frac{\pi^2}{2l}\right) \quad (44)$$

$$\int_0^l x(l-x) seno\left(\frac{\pi x}{l}\right) dx = -4\frac{q}{EI}\left(\frac{l}{\pi}\right)^3 \quad (45)$$

Substituting in equation (38), the equations (44-45)

$$-C_1\left(\frac{\pi^2}{2l}\right) - 4\frac{q}{EI}\left(\frac{l}{\pi}\right)^3 = 0 \quad (46)$$

$$C_1 = -\frac{4\frac{q}{EI}\left(\frac{l}{\pi}\right)^3}{\left(\frac{\pi^2}{2l}\right)} = -\frac{q}{EI}\left(\frac{8l^3}{\pi^5}\right) \quad (47)$$

Substituting equation (47) into equation (33) gives the approximate solution.

$$\tilde{y} = -\frac{q}{EI}\left(\frac{8l^3}{\pi^5}\right) seno\left(\frac{\pi x}{L}\right) \quad (48)$$

Taking *Values* $q=1$ $E\,I=1$ $l=1$ we will plot equation (33) and equation (48)

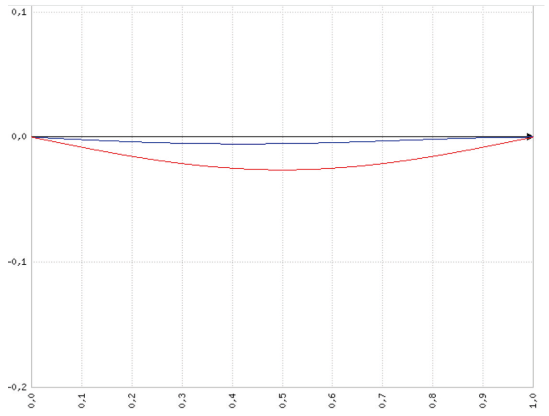

Figure. Graph of the analytical (red) and approximate (blue) solution.

RESULTS AND ANALYSIS

With the advancement of technology, today's software makes it possible to quickly and accurately calculate symbolic derivatives and integrals. This tool is invaluable for the application of Galerkin's method. By increasing the number of terms in the test function, a closer and closer approximation to the exact analytical solution is obtained. While in some cases it is possible to find this analytical solution, in most situations, especially when dealing with nonlinear equations, it is very difficult or even impossible to obtain.

Figure 3 illustrates how, when using a single term in the test function, the difference between the numerical solution and the analytical solution (if any) can be significant. However, as more terms are incorporated, this difference is reduced considerably.

CONCLUSIONS

Galerkin's method, when applied to partial or ordinary differential equations, allows transforming the problem into an integral form. This transformation facilitates obtaining an approximate analytical solution, which can be compared with the exact solution to evaluate the error. This method constitutes a viable alternative to solve problems of simply supported beams subject to moments concentrated at the ends.

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