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DESIGN, CONSTRUCTION, PAINTING, AND PRICING OF A QUADRANGULAR MOSAIC WITH APPLICATION OF DIFFERENTIAL AND INTEGRAL CALCULUS USING GEOGEBRA AND MATLAB

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Abstract: The objective of the article is that a student, when faced with a mathematical problem related to the design, construction and painting of an abstract art mosaic, can understand the problem posed. Under a constructivist approach, the student must understand the procedure to follow and the tools available to solve it. In this process, the understanding of it is expanded in the algebraic field by applying knowledge of Euclidean geometry and differential and integral calculus. Geometric definitions, such as points, line segments, contours, arc lengths, and areas, are essential. In addition, calculus concepts are explored, such as functions, critical points, local minimums and maximums, optimization and numerical methods to find roots of equations and areas between functions. In a complementary way, the use of scientific software such as GeoGebra allows the student to expand their graphic capacity in the comprehensive design and construction of a work. Regarding the economic valuation of a work of abstract art, the article also addresses how to evaluate a mosaic and apply mathematical concepts related to costing and pricing.

Keywords: Mosaic, Euclidean geometry, differential and integral calculus, segment, function, area between curves, optimization, pricing.

INTRODUCTION

The art of mosaic has a rich history dating back to the earliest civilizations. The Sumerians, Egyptians and inhabitants of the Minoan-Mycenaean region used this technique to decorate their buildings, floors and funerary places. However, it was in ancient Rome where the mosaic reached its maximum splendor. The Romans considered it a luxury object and incorporated it into baths and public spaces. Mural mosaics also emerged around this time, with notable examples in the Grottoes of the

Muses, Pompeii and Herculaneum. [2]

As we progress through history, the mosaics experienced ups and downs. During the medieval, Renaissance and Baroque periods, they were set aside in favor of cheaper frescoes and glazed tiles. However, during the Renaissance, they reemerged with force. Mannerism inspired a technique based on imperial Rome, where pebbles and natural elements were used along with paintings and sculptures. Despite this, the art of mosaic remained subordinate to architecture and painting. [1]

In the 19th century, mosaics were adapted to faster and more cost-effective techniques. But it was in the following century when eclectic and visionary artists, such as Antoni Gaudí, Gustav Klimt and Gino Severini, revolutionized the art of mosaic. Their unique and personal interpretations took this form of expression to new horizons, fusing tradition and avant-garde. [3]

Today, contemporary mosaic art, similar to that of the last century, is characterized by very individual approaches to its technique; and has been shaped by the work of artists such as Oskar Kokoschka, Enzo Cucchi, Aldo Mondino, Ezio Frigerio, Niki de Saint Phalle among others. [4]

Mosaics, which are defined as designs and images created from partitions or pieces (tiles) of decorative materials of various colors, are used to cover the plane using tessellations. Currently, these mosaics are widely used to decorate floors, walls, ceilings, paintings and objects. Its main purpose is to provide added value and turn any space into a work of art. Furthermore, in their evolution, mosaics have permeated the manifesto of the culture, myths and traditions of the civilization where they develop, therefore allowing them to capture their worldview. [5]

In the field of architecture and design, differential calculus plays a crucial role. It is

used to design, model, visualize and simulate both two-dimensional and three-dimensional shapes. This allows the individual to analyze the biunivocal and permanent relationship between space, forms and their dimensional structure. In addition, it facilitates the calculation of segments, curves, areas and volumes. [6]

However, one of the deficiencies of differential and integral calculus books, such as that of Bruce and Larson (2010), is that they do not address the subject of design, architecture and art with examples of practical application.

This limits the student's options to develop their knowledge in a less formal way, but maintaining the mathematical rigor that characterizes calculus.

Therefore, the main objective of this work is to fill this gap. It seeks for the student to discover the unknown properties of mosaics and space coverings. At the same time, the student is expected to learn to investigate, manipulating algebraic concepts and the construction of graphs as an exercise in mathematical application. This way, their ability to solve problems on their own will be strengthened, under the explanation of a logical framework and with the help of different scientific technical tools.

PROBLEM STATEMENT

The challenge lies in the design, construction, painting and pricing of a quadrangular mosaic. This process involves the use of the Cartesian plane, the location of points in the plane, the use of functions, the determination of line segments and the arc length of a function. The concepts of local maximum and minimum values, the calculation of areas between curves and the financial valuation of the base rate are also applied to create a painting. This painting, with a mosaic structural arrangement, is

presented in abstract art. The costing resource of a good is applied and the convolution of functions is used. All this is done under previously established mathematical and pictorial conditions.

MOSAIC STRUCTURE

- Shape: Quadrangular.
- Dimension of side: e^1+1 .
- Units of measurement: [square meters - m^2].

MOSAIC LOCATION FOR DESIGN AND CONSTRUCTION

- First quadrant of the Cartesian plane.

STRUCTURAL ASPECTS IN MOSAIC DESIGN

QUADRANGULAR EDGE

- POINTS
 - $O = (0,0)$ [Edge origin]
 - $A = (e^1+1, e^1+1)$.
 - $C = (0, e^1+1)$.
- OUTLINE
 - Line segments that generate the square: $OABC$.

INSCRIBED SQUARE

- POINTS
 - $D = (\frac{1}{2}, \frac{1}{2})$ [Origin of the inscribed square].
 - $E = (e^1 + \frac{1}{2}, \frac{1}{2})$.
 - $F = (e^1 + \frac{1}{2}, e^1 + \frac{1}{2})$.
 - $G = (\frac{1}{2}, e^1 + \frac{1}{2})$.

- OUTLINE
 - Line segments that generate the square: *DEFG*.
- ARC LENGTH
 - Union of points D and F through the arc length of the function: $y = x \cdot (\ln(x))^2$.
- AUXILIARY POINTS
 - Auxiliary points are generated as a result of determining critical points of the function: $y = x \cdot (\ln(x))^2$; in such a way that point H corresponds to the local maximum and point I to the local minimum.
- STRAIGHT SEGMENTS
 - Segment orthogonal to the x axis that begins and passes through point I, and continues until the intersection with segment FG and generates point J.
 - Segment parallel to the x axis that begins in the line segment DG and generates point K, passes through point H and extends to the intersection with the segment described in the previous paragraph, and generates point L.
 - Line segment that joins points E and J; and whose intersection with the function $y = x \cdot (\ln(x))^2$; it is generated the point .

STRUCTURE OF THE FRAME DESIGN BETWEEN THE EDGE OF THE OUTER SQUARES AND ITS INSCRIBED

- Generate straight line segments inside the frame between the edge of the external and inscribed squares, under the following algorithm:
 - Define point N at the center of gravity of the mosaic.
 - Define the auxiliary straight lines that correspond to the axes of the Cartesian plane with center point N.

- Generate auxiliary straight lines that result from the diagonals of the mosaic.
- Laying out straight line segments inside the edge between the external squares and their inscription, highlighting the auxiliary straight lines previously described.

PAINTING OF REGIONS THAT CORRESPOND TO THE INTERNAL AREAS OF THE MOSAIC ONCE ITS DESIGN IS DEFINED

SIGNING AND CALCULATION OF INTERNAL AREAS

Sign each area of the region with a code (A_i is suggested by varying i from 1 (one) to the total number of areas generated).

The area coding of the regions will be carried out, starting with the internal areas of the inscribed square.

The area coding of the mosaic regions continues with the frame areas between the border of the outer and inscribed squares.

Calculate each of the areas (the use of integral calculus is suggested; and at the same time, the use of code in Geogebra for programming and visualization).

DEFINITION OF COLORS

It is suggested that, to color each region within the mosaic, the attached color palette be used, therefore the Table containing the color, name and price per square unit is included.









Color palette with cost per square unit		
Color		Value (US\$)
	Purple	\$ 10,00
	Yellow	\$ 20,00
	Red	\$ 40,00
	Orange	\$ 80,00
	Green	\$ 100,00
	Pink	\$ 130,00
	Purple	\$ 170,00
	Sky blue	\$ 300,00

Table 01: Color palette with names and values per square unit.

MOSAIC PAINTING PROCESS

The mosaic painting process begins with the internal regions of the inscribed square. Each region is filled with a unique color, avoiding repetition of colors. Subsequently, we continue with the border regions between the external and inscribed square, maintaining the same filling principle with a single color. It is important to keep in mind that the color used in these areas must be different from the colors that were applied in the adjacent regions of the inscribed square.

MOSAIC RATE

PRICING METHODOLOGY RESEARCH

Investigate and define a basic methodology for pricing the price of a good related to art, which considers the rules of any market economy, which includes the concept of profitability, and mainly considers covering the basic production costs of the mosaic, based on concepts such as cost, profit margin and price (rate) calculation.

DETERMINATION OF THE COST OF THE MOSAIC BY THE AMOUNT OF PAINTED AREA, BASED ON THE LAYOUT AND COST OF THE COLOR

Define a calculation formula that allows determining the production cost of the mosaic, considering the cost arrangement by color type and the amount of painted area of the internal regions.

CALCULATION AND PAINTING OF MAXIMUM AND MINIMUM RATE MOSAICS

Define a procedure to capture the final art of the mosaics with a maximum and minimum rate, corresponding to the colors to be used.

ANALYSIS OF RATES FOR OTHER MOSAICS

Starting from any of the two mosaics, be it the maximum or minimum rate; Analyze the result on the variable rate of the mosaics when permuting the colors on different areas.

LOGIC FRAME

Use of definitions and theorems of Euclidean and analytical geometry, therefore such as differential and integral calculus, use of GeoGebra software and mathematics related to pricing of goods and products.

PROBLEM RESOLUTION

To solve the problem, it is recommended to use a basic theoretical framework, based on the definitions and theorems used in differential and integral calculus, according to Larson, R. Edwards, B. (2010). In addition, the use of GeoGebra software is suggested for graphing and to perform complementary calculations [7].

The resolution of the problem is addressed and presented as an algorithm to generate the design of a mosaic with a defined structure.

The process begins with loading the GeoGebra software, either in its web or desktop version. Next, a Cartesian plane is displayed and fixed points are located. The auxiliary points are determined based on what is established in the theoretical framework of differential calculus.

Once the preliminary design has been obtained, with the arc lengths and straight line segments that define its contours, therefore as the construction of internal lines within the square that the mosaic contemplates, we proceed to calculate the areas between curves. To do this, the definitions and theorems of the theoretical framework of integral calculus are used, relying on the real functions established in the mosaic design.

Regarding the financial pricing of the mosaic, the function that relates the total base production cost and the profit margin variable is used. To maximize or minimize the price of the mosaic, the amount of painted area must be considered in relation to the cost per color type and the painting procedure.

MOSAIC DESIGN

In the design of the mosaic, everything established in the structural aspects is used, such as fixed and auxiliary points, contours, lengths of arcs and straight line segments; therefore:

LOCATION OF FIXED POINTS – QUADRANGULAR EDGE

Point O = (0,0)[Edge Origin]

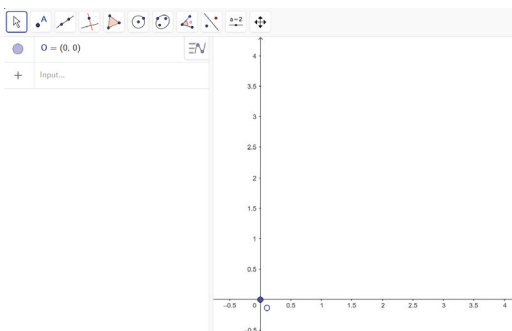


Figure 01: Tile Edge Origin Location.

Point A = ($e^1+1,0$)

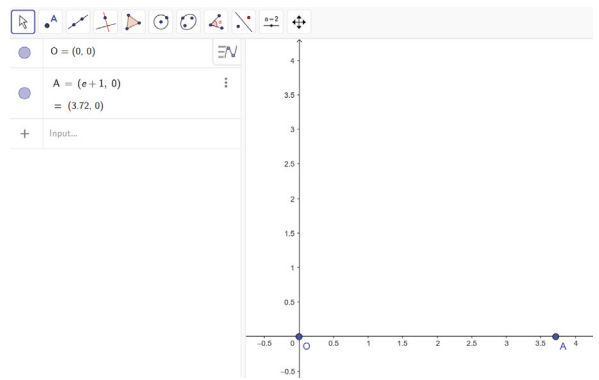


Figure 02: Location of point A.

Point B = (e^1+1, e^1+1).

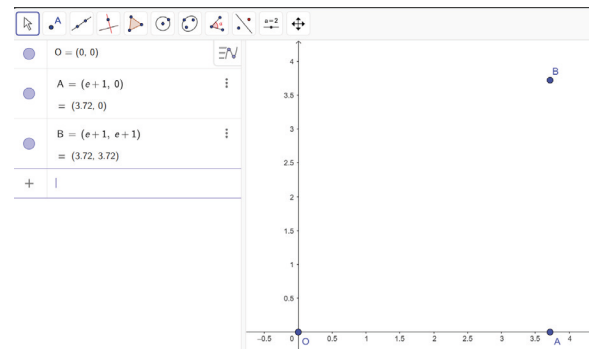


Figure 03: Location of point B.

Point C = (0, e^1+1)

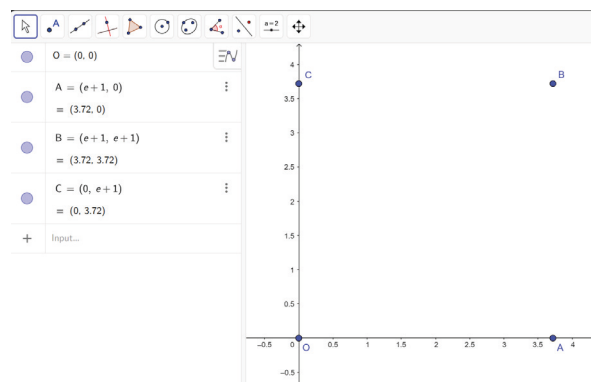


Figure 04: Location of point C.

DELIMITATION OF THE CONTOUR OF THE EXTERNAL SQUARE

Outline of $OABC$ generated from line segments of its previously located points.

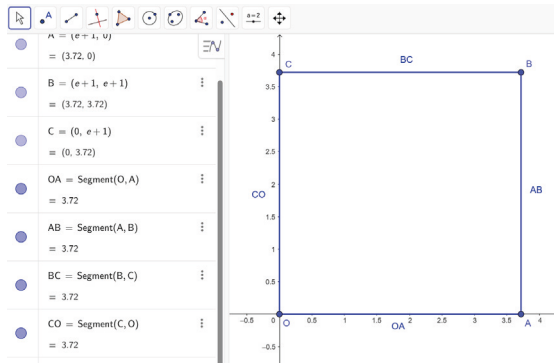


Figure 05: Delimitation of the contour of the external square.

LOCATION OF FIXED POINTS – INSCRIBED SQUARE

$$\text{Point D} = \left(\frac{1}{2}, \frac{1}{2}\right)$$

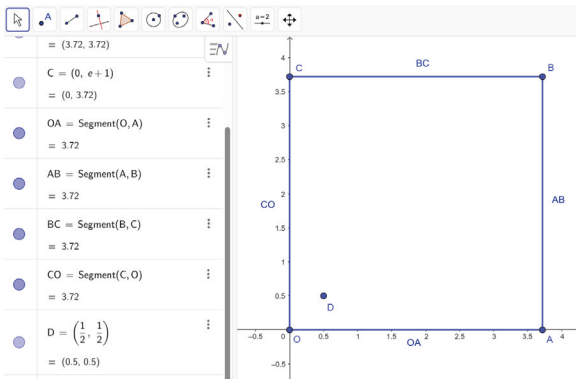


Figure 06: Location of point D.

$$\text{Point E} = \left(e^1 + \frac{1}{2}, \frac{1}{2}\right)$$

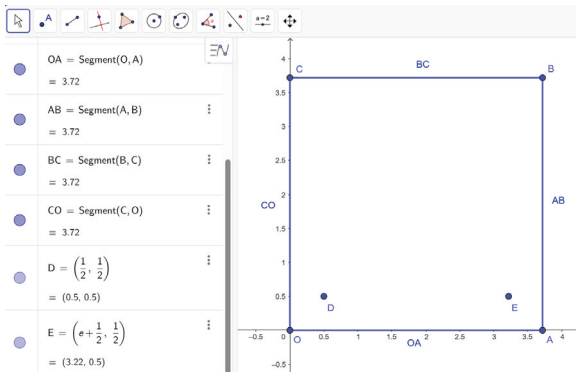


Figure 07: Location of point E.

$$\text{Point F} = \left(e^1 + \frac{1}{2}, e^1 + \frac{1}{2}\right).$$

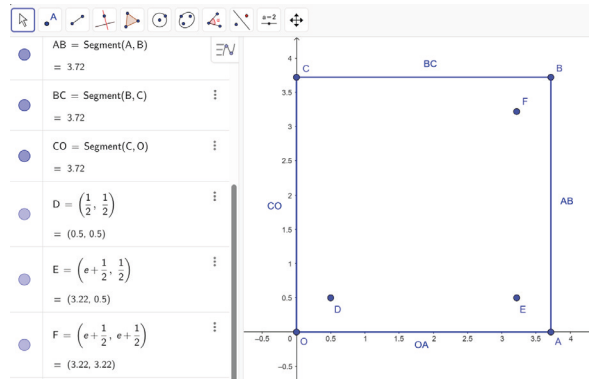


Figure 08: Location of point F.

$$\text{Point G} = \left(\frac{1}{2}, e^1 + \frac{1}{2}\right)$$

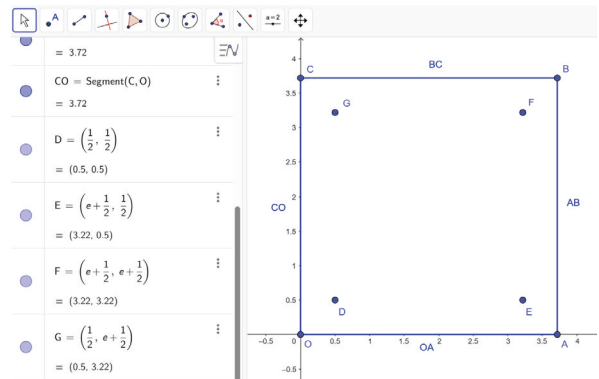


Figure 09: G-spot location.

DELIMITATION OF THE CONTOUR OF THE INSCRIBED SQUARE

Outline of $DEFG$ generated from the line segments of its previously located points.

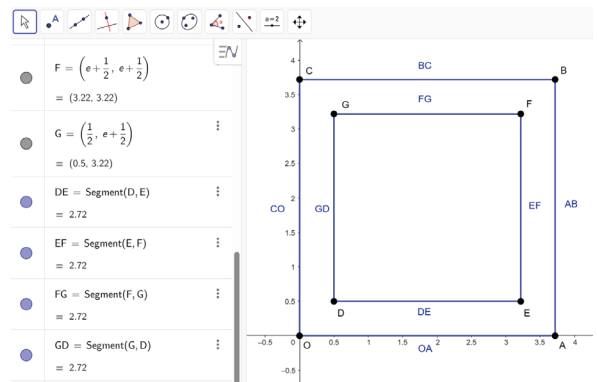


Figure 10: Delimitation of the contour of the inscribed square.

ARC LENGTH – INSCRIBED SQUARE

Arc that joins points D and F through the function $y = (x - \frac{1}{2}) \cdot (\ln(x - \frac{1}{2}))^2 + \frac{1}{2}$.

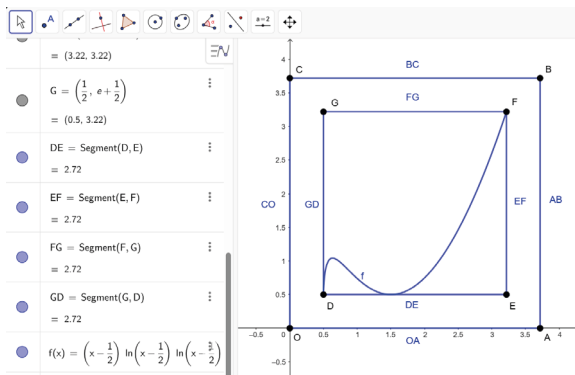


Figure 11: Delimitation of the arc length inside the inscribed square.

DETERMINATION AND LOCATION OF AUXILIARY POINTS

These points are determined based on the following theoretical framework, given the definitions and theorems of differential calculus; namely:

Definition 1 (Critical number or point); (Larson, 2010)

Let f be defined in c . If $f'(c) = 0$ or if f is not differentiable in c , therefore c is a critical point of f .

Theorem 1 (Relative extremes occur only at critical numbers or points); (Larson, 2010)

If f has a relative minimum or maximum at $x=c$, therefore c is a critical point of f .

Theorem 2 (Criteria for increasing and decreasing functions); (Larson, 2010)

Let f be a function that is continuous on the closed interval $[a, b]$ and differentiable on the open interval $]a, b[$.

1. If $f'(x) > 0$ for every x in $]a, b[$, therefore f is increasing in $[a, b]$.
2. If $f'(x) < 0$ for every x in $]a, b[$ therefore f is decreasing in $[a, b]$.
3. If $f'(x) = 0$ for every x in $]a, b[$ therefore f is constant in $[a, b]$.

Theorem 3 (First derivative criterion); (Larson, 2010)

Let c be a critical point of a function f that is continuous on an open interval I containing c . If f is differentiable on the interval, except possibly on c , then $f(c)$ can be classified as follows.

1. If $f'(x)$ changes from negative to positive at c , therefore f has a relative minimum at $(c, f(c))$.
2. If $f'(x)$ changes from positive to negative at c , therefore f has a relative maximum at $(c, f(c))$.
3. If $f'(x)$ is positive on both sides of c or negative on both sides of, therefore $f'(c)$ is neither a relative minimum nor a relative maximum.

Once the theoretical framework has been established, and considering what is established in the problem statement regarding the auxiliary points that are generated as a result of the location of critical points of the function: $y = (x - \frac{1}{2}) \cdot (\ln(x - \frac{1}{2}))^2 + \frac{1}{2}$; we have:

$$y' = f'(x) = \left(\ln\left(x - \frac{1}{2}\right)\right)^2 + 2 \cdot \left(x - \frac{1}{2}\right) \cdot \ln\left(x - \frac{1}{2}\right) \cdot \frac{1}{\left(x - \frac{1}{2}\right)}$$

$$y' = \ln\left(x - \frac{1}{2}\right) \left[\ln\left(x - \frac{1}{2}\right) + 2\right]$$

By setting y' equal to zero, we obtain:

$$\ln\left(x - \frac{1}{2}\right) \left[\ln\left(x - \frac{1}{2}\right) + 2\right] = 0$$

By solving the previous equation, we obtain the solutions:

$$\ln\left(x - \frac{1}{2}\right) = 0 \vee \left[\ln\left(x - \frac{1}{2}\right) + 2\right] = 0$$

$$x = \frac{3}{2} \vee x = e^{-2} + \frac{1}{2}$$

Since the function: $y = (x - \frac{1}{2}) \cdot (\ln(x - \frac{1}{2}))^2 + \frac{1}{2}$; is defined for all values of $x \in \left[\frac{1}{2}, e + \frac{1}{2}\right]$; we have: $x = e^{-2} + \frac{1}{2}$ and $x = \frac{3}{2}$ they are the only critical points.

Generating interval partitions $\left[\frac{1}{2}, e + \frac{1}{2}\right]$; with

its critical points $x = e^{-2} + \frac{1}{2}y x = \frac{3}{2}$, a Table is obtained of the results obtained by applying the theorems of the theoretical framework for each of the points, therefore:

Interval	$\frac{1}{2} < x < e^{-2} + \frac{1}{2}$	$e^{-2} + \frac{1}{2} < x < \frac{3}{2}$	$\frac{3}{2} < x < e^1 + \frac{1}{2}$
Test point	$x_1 = e^{-2} + \frac{1}{4}$	$x_2 = e^{-2} + 1$	$x_3 = e^1 - \frac{1}{2}$
Symbol of	+	-	+
Monotony	Growing	Decreasing	Growing
Conclusion regarding the critical point	Maximum relative	Minimum relative	

Table 02: Results of the behavior of the function, when applying theorems 1, 2 and 3 within the interval $[\frac{1}{2}; e + \frac{1}{2}]$.

Under Theorem 3 (First Derivative Criterion); (Larson, 2010); It is concluded that the function $y = (x - \frac{1}{2}) \cdot (\ln(x - \frac{1}{2}))^2 + \frac{1}{2}$ has a relative maximum at $x = e^{-2} + \frac{1}{2}$ and a relative minimum in: $x = \frac{3}{2}$. The auxiliary points being located as local maximum $H = (e^{-2} + \frac{1}{2}, 4(e^{-2} + \frac{1}{2}))$ and minimum local $I = (\frac{3}{2}, \frac{1}{2})$.

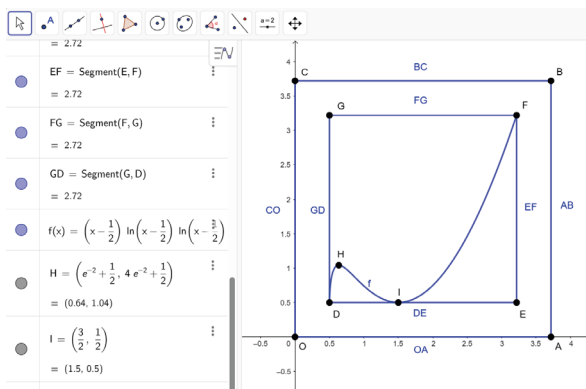


Figure 12: Location of points H and I.

STRAIGHT SEGMENTS – INSCRIBED SQUARE

Line segment generated orthogonal to the x axis that begins at point I, and extends until it intersects the segment:

$$\overline{FG} \text{ and generates the point } J = \left(\frac{3}{2}, e + \frac{1}{2}\right).$$

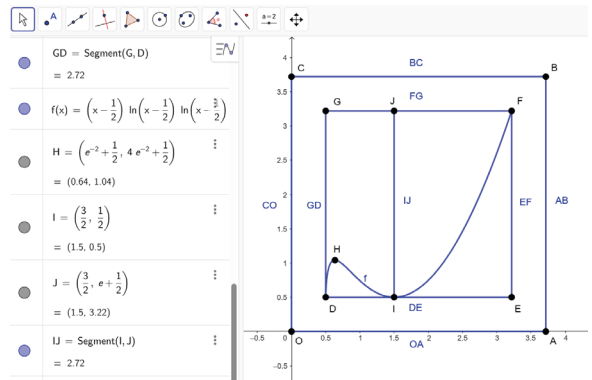


Figure 13: Location of point J and generation of the line segment \overline{IJ} .

Line segment that is generated parallel to the x axis and that passes through point H, and extends from the segment (GD) until it intersects the segment (IJ); same as, they generate the location of the points $K = (\frac{1}{2}, 4(e^{-2} + \frac{1}{2}))$ y $L = (\frac{3}{2}, 4(e^{-2} + \frac{1}{2}))$.

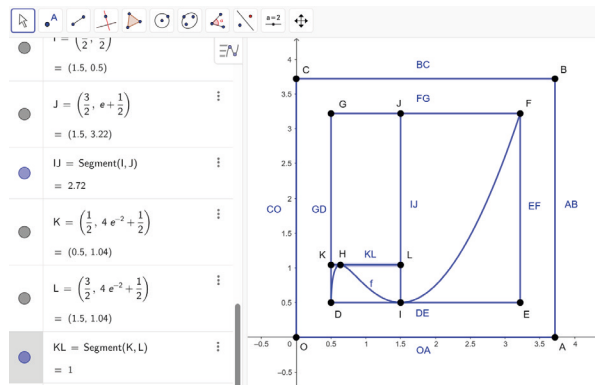


Figure 14: Location of points K and L; and, generation of the line segment: \overline{KL} .

Line segment: \overline{EJ} which is generated from the union of the points $E = (e + \frac{1}{2}, \frac{1}{2})$ y $J = (\frac{3}{2}, e + \frac{1}{2})$.

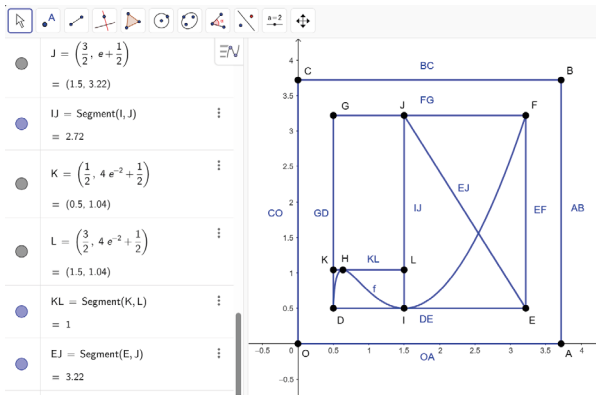


Figure 15: Generation of the line segment: \overline{EJ} .

The function associated with the line segment: \overline{EJ} is shown by:

$$\frac{y - \frac{1}{2}}{x - e - \frac{1}{2}} = \frac{e + \frac{1}{2} - \frac{1}{2}}{\frac{3}{2} - e - \frac{1}{2}}$$

Which, when solved, its explicit function is given by:

$$\begin{aligned} \frac{y - \frac{1}{2}}{x - e - \frac{1}{2}} &= \frac{e}{1 - e} \\ \left(y - \frac{1}{2}\right)(1 - e) &= e\left(x - e - \frac{1}{2}\right) \\ y(1 - e) - \frac{1}{2}(1 - e) &= ex - e\left(e + \frac{1}{2}\right) \\ y(1 - e) &= ex - e\left(e + \frac{1}{2}\right) + \frac{1}{2}(1 - e) \\ y(1 - e) &= ex - e^2 - \frac{e}{2} + \frac{1}{2} - \frac{e}{2} \\ y &= \frac{ex}{1 - e} - \frac{e^2 + e - \frac{1}{2}}{1 - e} \end{aligned}$$

The intersection between the function associated with the line segment: \overline{EJ} with arc length function: $y = \left(x - \frac{1}{2}\right) \cdot \left(\ln\left(x - \frac{1}{2}\right)\right)^2 + \frac{1}{2}$ generates the point M, which is determined by solving the following system of equations:

$$\begin{cases} y = \left(x - \frac{1}{2}\right) \cdot \left(\ln\left(x - \frac{1}{2}\right)\right)^2 + \frac{1}{2} \\ y = \frac{ex}{1 - e} - \frac{e^2 + e - \frac{1}{2}}{1 - e} \end{cases}$$

Transforming the previous system into an equivalent algebraic equation, we obtain:

$$\left(x - \frac{1}{2}\right) \cdot \left(\ln\left(x - \frac{1}{2}\right)\right)^2 - \frac{ex}{1 - e} + \frac{2e^2 + 3e - 2}{2(1 - e)} = 0$$

Given the complexity of the algebraic equation, for its resolution, a numerical method is chosen, which for ease can be the Newton Raphson method, considering the following theoretical framework given by its algorithm; namely:

Algorithm 1 (Newton's method to approximate the zeros of a function); (Larson, 2010)

Let $f(c)=0$, Where f is differentiable on an open interval containing c . Therefore, to approximate c , the following steps are followed.

1. An initial estimate is made: x_1 which is close to c .
2. A new approach is determined

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

3. If $|x_n - x_{n+1}|$ is within the desired precision, let x_{n+1} used as the final approach. Otherwise, go back to step two and calculate a new approximation.

Note: Each successive application of this procedure is called an iteration.

Based on the algorithm established in the theoretical framework, it is transferred to a script (Newton_Raphson.m) that is programmed and executed in the MATLAB package; therefore:

Obtaining the following estimates of the root, therefore:

Therefore, expressing the approximate result in decimal form; The solution of the algebraic equation allows us to locate the point M= (2.55,1.56).

```

Newton_Raphson.m x +
/MATLAB Drive/Newton_Raphson.m
1      clc
2      clear
3      fun = @(x) ((x-0.5).*(log(x-0.5))^2)-((exp(1).*x)/(1-exp(1)))+(2*exp(2)+3*exp(1)-2)/(2*(1-exp(1)));
4      raices = NR(fun, 2, 5)
5
6      function salida = NR(fun,a,n)
7      syms x;
8      z = fun(x);
9      derZ = diff(z);
10     out = zeros(1,n+1);
11     out(1) = a;
12
13     for i = 1 : n
14         numerador = subs(z,x,out(i));
15         denominador = subs(derZ,x,out(i));
16         out(i+1) = out(i) - double(numerador)/double(denominador);
17     end
18     salida = out;
19     end

```

Figure 16: Newton-Raphson algorithm script in Matlab.

```

Command Window
New to MATLAB? See resources for Getting Started.
raices =
    2.0000    2.6614    2.5558    2.5533    2.5533    2.5533

```

Figure 17: Root Value Results after Running Newton-Raphson Algorithm Script in Matlab.

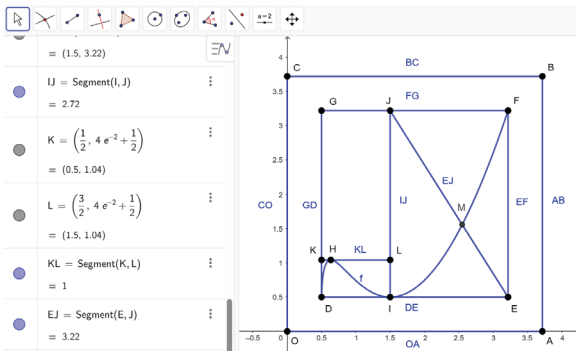


Figure 18: Location of point M.

STRUCTURE OF THE FRAME BETWEEN THE EDGE OF THE EXTERNAL SQUARE AND ITS INSCRIBED SQUARE

To generate straight line segments inside the frame between the edge of the external square and its inscribed square, first the central point corresponding to the barycenter of the mosaic is obtained, through the intersection of its diagonals, for this the two functions are established associated with line segments: \overline{OB} and \overline{AC} , that are respectively:

$$LR_{\overline{OB}}: y = x$$

$$LR_{\overline{AC}}: y = -x + e + 1$$

When solving by substitution, the intersection of the lines generates the point N $= \left(\frac{e+1}{2}, \frac{e+1}{2}\right)$.

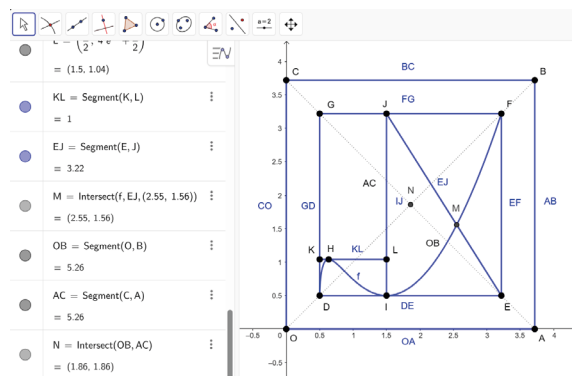


Figure 19: Location of point N.

Once point N is obtained $= \left(\frac{e+1}{2}, \frac{e+1}{2}\right)$, a Cartesian plane is drawn with origin at said point; and the points of intersection with the edge of the square are established as with the inscribed square; obtaining the

points: $P = (e + 1, \frac{e+1}{2}), Q = (0, \frac{e+1}{2}), R = (\frac{e+1}{2}, e + 1), y$
 $S = (\frac{e+1}{2}, 0)$; therefore also with its inscribed square, obtaining the points $T = (e + \frac{1}{2}, \frac{e+1}{2}),$
 $U = (\frac{1}{2}, \frac{e+1}{2}), V = (\frac{e+1}{2}, e + \frac{1}{2}), y W = (\frac{e+1}{2}, \frac{1}{2}).$

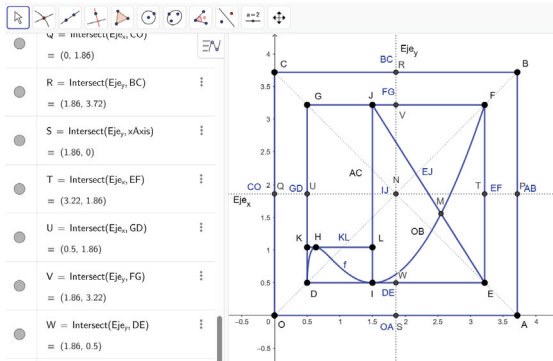


Figure 20: Layout of the auxiliary lines of the plane with center in N and the diagonals of the mosaic, locating the points P,Q,R and S on the contour of the external square and the points T,U,V and W on the contour of the inscribed square.

With the points obtained, we proceed to diagram the line segments established inside the space between the edge of the square and its inscribed square, correspondingly: $\overline{PT}, \overline{BF}, \overline{RV}, \overline{CG}, \overline{QU}, \overline{DO}, \overline{SW}$ y \overline{AE} ; therefore, we have:

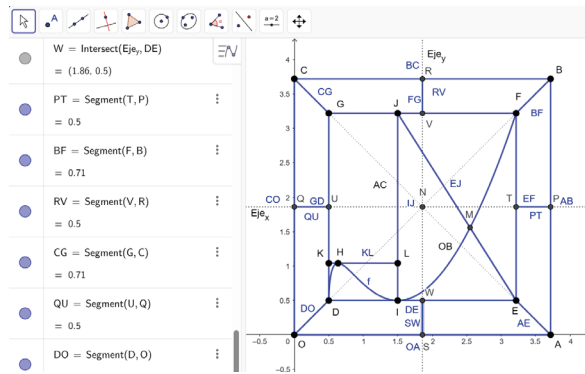


Figure 21: Generation of line segments; $\overline{PT}, \overline{BF}, \overline{RV}, \overline{CG}, \overline{QU}, \overline{DO}, \overline{SW}$ y \overline{AE} .

Once the mosaic design has been diagrammed, the auxiliary lines are hidden in order to visualize the actual mosaic design with its strokes; therefore:

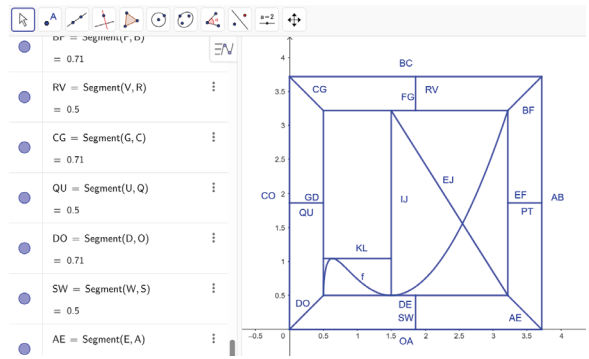


Figure 22: Real design of the mosaic are its strokes.

Based on the above and without definition of segments, the final design of the mosaic is established:

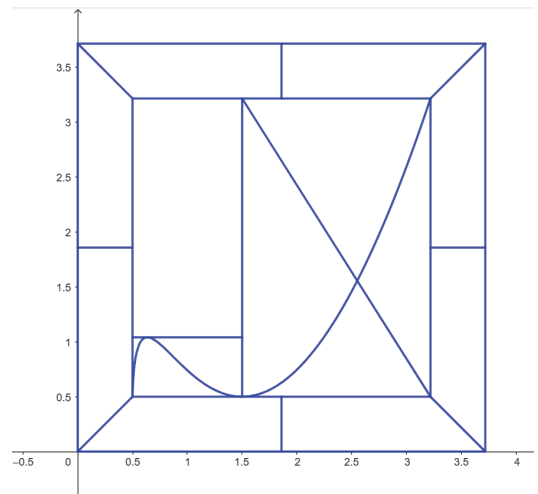


Figure 23: Final mosaic design.

MOSAIC PAINTING

Before filling the mosaic with color, the areas of its different elements must be calculated, in order to obtain the maximum value of the work or, in contrast, its minimum; considering what is established in the cost conditions of the colors, therefore as their distribution.

For the calculation of areas, the following theoretical framework given by definition and theorems related to integration and areas between curves will be considered; namely:

Definition 2 (Antiderivative or primitive); (Larson, 2010)

A function F is said to be an antiderivative or primitive of f , on an interval I if $F'(x) = f(x)$ for all x in I .

Definition 3 (Definite integral); (Larson, 2010)

If f is defined in the closed interval $[a, b]$ and the limit of the Riemann sums over the partitions: Δ .

$$\lim_{\|\Delta\| \rightarrow 0} \sum_{i=1}^n f(c_i) \Delta x_i$$

It exists, therefore f is integrable on $[a, b]$ and the limit is denoted by:

$$\lim_{\|\Delta\| \rightarrow 0} \sum_{i=1}^n f(c_i) \Delta x_i = \int_a^b f(x) dx$$

The limit is called the definite integral from f from a to b . The number a is the lower limit of integration, and the number b is the upper limit of integration.

Theorem 4 (The integral defined as the area of a region); (Larson, 2010)

If f is continuous and non-negative in the closed interval $[a, b]$, therefore the area of the region bounded by the graph of f , the x axis and the vertical lines: $x=a$ and $x=b$ is showed by:

$$Area = \int_a^b f(x) dx$$

Theorem 5 (Fundamental theorem of calculus); (Larson, 2010)

If a function f is continuous on the closed interval $[a, b]$, and F is an antiderivative of f , on the interval $[a, b]$, therefore:

$$\int_a^b f(x) dx = F(b) - F(a)$$

Theorem 6 (Area of a region between two curves); (Larson, 2010)

If f and g are continuous in $[a, b]$, and $g(x) \leq f(x)$ for all x in $[a, b]$, therefore the area of the region bounded by the graphs of f and g and the vertical lines: $x=a$ and $x=b$ is:

$$Area = \int_a^b [f(x) - g(x)] dx$$

Definition 4 (Improper integrals with infinite discontinuities); (Larson, 2010)

1. If f has an infinite discontinuity in b therefore

$$\int_a^b f(x) dx = \lim_{c \rightarrow b^-} \int_a^c f(x) dx$$

2. If f is continuous in the interval $]a, b[$ and has an infinite discontinuity in a therefore:

$$\int_a^b f(x) dx = \lim_{c \rightarrow a^+} \int_c^b f(x) dx$$

3. If f is continuous in the interval $[a, b]$, except for some: c in $]a, b[$ in which f has an infinite discontinuity, then:

$$\int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx$$

In the first two cases, the improper integral converges if the limit exists, otherwise the improper integral diverges. In the third case, the improper integral on the left diverges if any of the improper integrals on the right diverge.

DETERMINATION OF THE AREAS OF THE MOSAIC REGIONS

For ease of understanding and calculation, the following graph is included that contains the detail of the area and the respective location of the region within the mosaic, therefore:

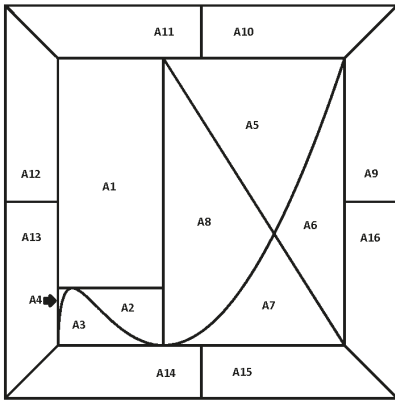


Figure 24: Arrangement of areas of the regions of the signed mosaic from the inscribed square towards its outer edge.

- Calculation of area 1:

This area is calculated with the distances of the segments: \overline{KL} and \overline{KG} that form a rectangular structure; therefore its geometric solution is:

$$\overline{KL} = 1 \wedge \overline{KG} = (e - 4e^{-2})$$

Therefore:

$$A_1 = (e - 4e^{-2}) \approx 2,176$$

Equivalently, the area between the curves can be calculated, $y = (e + \frac{1}{2})$; e , $y = (4e^{-2} + \frac{1}{2})$, defined between the integration limits $\frac{1}{2}$ to $\frac{3}{2}$; therefore:

$$A_1 = \int_{\frac{1}{2}}^{\frac{3}{2}} \left(e + \frac{1}{2} - 4e^{-2} - \frac{1}{2} \right) dx$$

$$A_1 = \int_{\frac{1}{2}}^{\frac{3}{2}} (e - 4e^{-2}) dx$$

$$A_1 = (e - 4e^{-2})x \Big|_{\frac{1}{2}}^{\frac{3}{2}}$$

$$A_1 = (e - 4e^{-2}) \approx 2,176$$

It can be evaluated numerically using Geogebra commands; therefore:

Geogebra code:

$$A_1 = \text{The whole part between} \left(e + \frac{1}{2}, 4e^{-2} + \frac{1}{2}, \frac{1}{2}, \frac{3}{2} \right)$$

GeoGebra Chart:

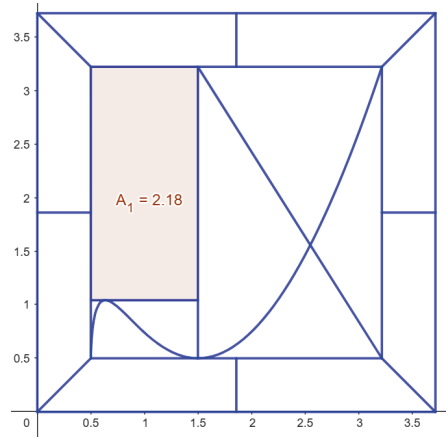


Figure 25: Determining the value and shading of area 1.

- Calculation of area 2:

This area is calculated as the integral between the curves $y = (4e^{-2} + \frac{1}{2})$ e $y = (x - \frac{1}{2}) \cdot (\ln(x - \frac{1}{2}))^2 + \frac{1}{2}$, defined between the integration limits y ; therefore:

$$A_2 = \int_{e^{-2} + \frac{1}{2}}^{\frac{3}{2}} \left(\left(4e^{-2} + \frac{1}{2} \right) - \left(x - \frac{1}{2} \right) \cdot \left(\ln \left(x - \frac{1}{2} \right) \right)^2 - \frac{1}{2} \right) dx$$

$$A_2 = \int_{e^{-2} + \frac{1}{2}}^{\frac{3}{2}} \left(4e^{-2} - \left(x - \frac{1}{2} \right) \cdot \left(\ln \left(x - \frac{1}{2} \right) \right)^2 \right) dx$$

$$A_2 = \int_{e^{-2}}^1 \left(4e^{-2} - z \cdot (\ln(z))^2 \right) dz$$

$$A_2 = 4e^{-2}z \Big|_{e^{-2}}^1 - \int_{e^{-2}}^1 \left(z \cdot (\ln(z))^2 \right) dz$$

$$A_2 = \left(4e^{-2}z - \frac{1}{2}z^2 \cdot (\ln(z))^2 + \frac{1}{2}z^2 \cdot \ln(z) - \frac{z^2}{4} \right) \Big|_{e^{-2}}^1$$

$$A_2 = \left(4e^{-2} - \frac{1}{4} \right) - \left(4e^{-4} - 2e^{-4} - 1e^{-4} - \frac{e^{-4}}{4} \right)$$

$$A_2 = \left(4e^{-2} - \frac{3e^{-4}}{4} - \frac{1}{4} \right) \approx 0,277$$

It can be evaluated numerically using Geogebra commands; therefore:

Geogebra code:

$$A_2 = \text{The whole part between} \left(4e^{-2} + \frac{1}{2}, \left(x - \frac{1}{2} \right) \cdot \left(\ln \left(x - \frac{1}{2} \right) \right)^2 + \frac{1}{2}, e^{-2} + \frac{1}{2}, \frac{3}{2} \right)$$

GeoGebra Chart:

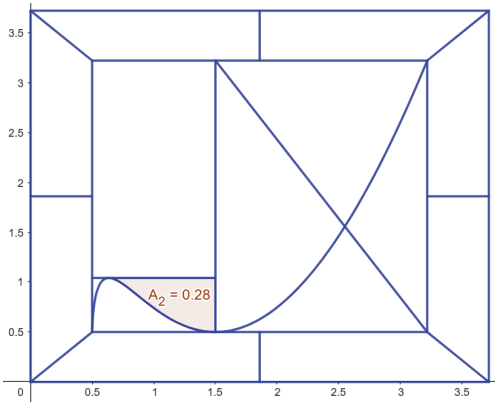


Figure 26: Determination of the value and shading of the area 2.

- Calculation of area 3:

This area is calculated as the integral between the curves $y = (x - \frac{1}{2}) \cdot (\ln(x - \frac{1}{2}))^2 + \frac{1}{2}$ e $y = \frac{1}{2}$ defined between the integration limits $\frac{1}{2}$ y $\frac{3}{2}$, therefore:

$$A_3 = \int_{\frac{1}{2}}^{\frac{3}{2}} \left((x - \frac{1}{2}) \cdot (\ln(x - \frac{1}{2}))^2 + \frac{1}{2} - \frac{1}{2} \right) dx$$

$$A_3 = \int_{\frac{1}{2}}^{\frac{3}{2}} \left((x - \frac{1}{2}) \cdot (\ln(x - \frac{1}{2}))^2 \right) dx$$

$$A_3 = \int_0^1 (z \cdot (\ln(z))^2) dz$$

$$A_3 = \lim_{t \rightarrow 0^+} \int_t^1 (z \cdot (\ln(z))^2) dz$$

$$A_3 = \lim_{t \rightarrow 0^+} \left(-\frac{1}{2} z^2 \cdot (\ln(z))^2 - \frac{1}{2} z^2 \cdot \ln(z) + \frac{z^2}{4} \right) \Big|_t^1$$

$$A_3 = \frac{1}{4} = 0,25$$

It can be evaluated numerically using Geogebra commands; therefore:

GeoGebra code:

$$A_3 = \text{The whole part between} \left((x - \frac{1}{2}) \cdot (\ln(x - \frac{1}{2}))^2 + \frac{1}{2}, \frac{1}{2}, \frac{3}{2} \right)$$

GeoGebra Chart:

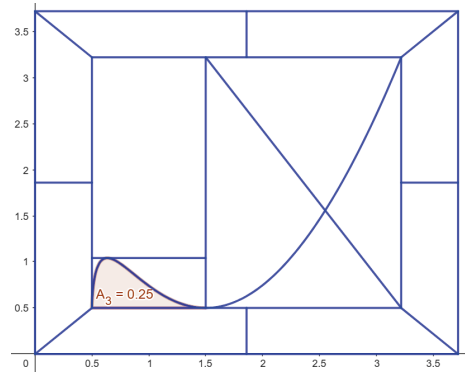


Figure 27: Determining the value and shading of area 3.

- Calculation of area 4:

This area is calculated as the integral between the curves: $y = (4e^{-2} + \frac{1}{2})$ e $y = (x - \frac{1}{2}) \cdot (\ln(x - \frac{1}{2}))^2 + \frac{1}{2}$, defined between the integration limits: $\frac{1}{2} y (e^{-2} + \frac{1}{2})$; therefore:

$$A_4 = \int_{\frac{1}{2}}^{e^{-2} + \frac{1}{2}} \left((4e^{-2} + \frac{1}{2}) - (x - \frac{1}{2}) \cdot (\ln(x - \frac{1}{2}))^2 - \frac{1}{2} \right) dx$$

$$A_4 = \int_{\frac{1}{2}}^{e^{-2} + \frac{1}{2}} \left(4e^{-2} - (x - \frac{1}{2}) \cdot (\ln(x - \frac{1}{2}))^2 \right) dx$$

$$A_4 = \int_0^{e^{-2}} (4e^{-2} - z \cdot (\ln(z))^2) dz$$

$$A_4 = 4e^{-2} z \Big|_0^{e^{-2}} - \lim_{t \rightarrow 0^+} \int_t^{e^{-2}} (z \cdot (\ln(z))^2) dz$$

$$A_4 = 4e^{-2} z \Big|_0^{e^{-2}} - \lim_{t \rightarrow 0^+} \left(-\frac{1}{2} z^2 \cdot (\ln(z))^2 - \frac{1}{2} z^2 \cdot \ln(z) + \frac{z^2}{4} \right) \Big|_t^{e^{-2}}$$

$$A_4 = 4e^{-4} - \frac{13e^{-4}}{4}$$

$$A_4 = \left(\frac{3e^{-4}}{4} \right) \approx 0,013$$

It can be evaluated numerically using Geogebra commands; therefore:

GeoGebra code:

$$A_3 = \text{The whole part between}$$

$$\left(4e^{-2} + \frac{1}{2}, (x - \frac{1}{2}) \cdot (\ln(x - \frac{1}{2}))^2 + \frac{1}{2}, \frac{1}{2}, e^{-2} + \frac{1}{2} \right)$$

GeoGebra Chart:

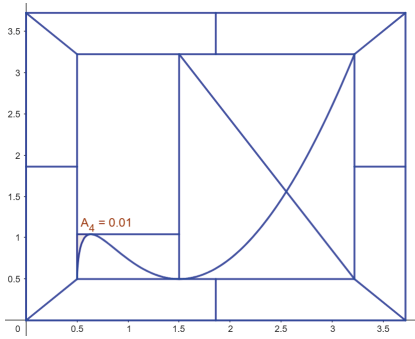


Figure 28: Determining the value and shading of area 4.

- Calculation of area 5:

This area is calculated as the sum of two integrals; The first corresponds to the area between the curves: $y = (e + \frac{1}{2})$; e , $y = \frac{ex}{1-e} - \frac{e^2+e-\frac{1}{2}}{1-e}$, defined between the integration limits $\frac{3}{2}$ y 2,55; therefore:

$$A_{51} = \int_{\frac{3}{2}}^{2,55} \left(\left(e + \frac{1}{2} \right) - \frac{ex}{1-e} + \frac{e^2+e-\frac{1}{2}}{1-e} \right) dx$$

$$A_{51} = \int_{\frac{3}{2}}^{2,55} \left(\frac{e + \frac{1}{2} - e^2 - \frac{e}{2} - ex + e^2 + e - \frac{1}{2}}{1-e} \right) dx$$

$$A_{51} = \int_{\frac{3}{2}}^{2,55} \left(\frac{e \left(x - \frac{3}{2} \right)}{e-1} \right) dx$$

$$A_{51} = \int_0^{1,05} \left(\frac{e(z)}{e-1} \right) dz$$

$$A_{51} = \left(\frac{ez^2}{2(e-1)} \right) \Big|_0^{1,05}$$

$$A_{51} = \left(\frac{1,49845}{e-1} \right) \approx 0,872$$

Meanwhile, the second corresponds to the area between the curves: $|y = (e + \frac{1}{2})$; e , $y = (x - \frac{1}{2}) \cdot \left(\ln(x - \frac{1}{2}) \right)^2 + \frac{1}{2}$, defined between the integration limits 2,55 y $(e + \frac{1}{2})$; therefore:

$$A_{52} = \int_{2,55}^{e+\frac{1}{2}} \left(\left(e + \frac{1}{2} \right) - \left(x - \frac{1}{2} \right) \cdot \left(\ln(x - \frac{1}{2}) \right)^2 - \frac{1}{2} \right) dx$$

$$A_{52} = \int_{2,55}^{e+\frac{1}{2}} \left(e - \left(x - \frac{1}{2} \right) \cdot \left(\ln(x - \frac{1}{2}) \right)^2 \right) dx$$

$$A_{52} = \int_{2,05}^e \left(e - (z) \cdot \left(\ln(z) \right)^2 \right) dz$$

$$A_{52} = \left(ez - \frac{1}{2} z^2 \cdot (\ln(z))^2 - \frac{1}{2} z^2 \cdot \ln(z) + \frac{z^2}{4} \right) \Big|_{2,05}^e$$

$$A_{52} = \left(e^2 - \frac{e^2}{2} - \frac{e^2}{2} + \frac{e^2}{4} - \left(2,05e - \frac{(2,05)^2 \cdot \ln^2(2,05)}{2} - \frac{(2,05)^2 \cdot \ln(2,05)}{2} + \frac{(2,05)^2}{4} \right) \right)$$

$$A_{52} \approx 0,594$$

Therefore, the entire area would be the sum of the two partial areas, therefore:

$$A_5 = A_{51} + A_{52} \approx 1,466$$

It can be evaluated numerically using GeoGebra commands; therefore:

GeoGebra code:

$$A_{51} = \text{The whole part between} \\ \left(4e^{-2} + \frac{1}{2} \cdot \frac{ex}{1-e} - \frac{e^2+e-\frac{1}{2}}{1-e}, \frac{3}{2}, 2,55 \right)$$

$$A_{52} = \text{The whole part between} \\ \left(4e^{-2} + \frac{1}{2} \cdot \left(x - \frac{1}{2} \right) \cdot \left(\ln(x - \frac{1}{2}) \right)^2 + \frac{1}{2}, 2,55, e + \frac{1}{2} \right)$$

GeoGebra Chart:

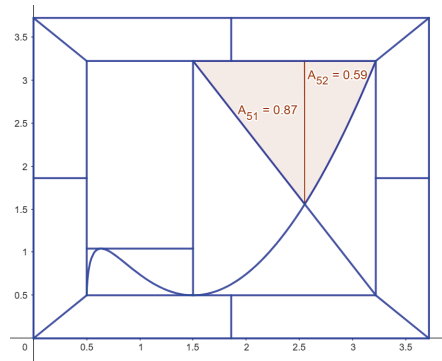


Figure 29: Determining the value and shading of area 5.

- Calculation of area 6:

This area is calculated as the integral between the curves: $y = (x - \frac{1}{2}) \cdot \left(\ln(x - \frac{1}{2}) \right)^2 + \frac{1}{2}$; e , $y = \frac{ex}{1-e} - \frac{e^2+e-\frac{1}{2}}{1-e}$, defined between the integration limits $2,55 y (e + \frac{1}{2})$ therefore:

$$A_6 = \int_{2,55}^{e+\frac{1}{2}} \left(\left(x - \frac{1}{2}\right) \cdot \left(\ln\left(x - \frac{1}{2}\right)\right)^2 + \frac{1}{2} - \frac{ex}{1-e} + \frac{e^2 + e - \frac{1}{2}}{1-e} \right) dx$$

$$A_6 = \int_{2,55}^{e+\frac{1}{2}} \left(\left(x - \frac{1}{2}\right) \cdot \left(\ln\left(x - \frac{1}{2}\right)\right)^2 - \frac{e\left(x - \frac{1}{2}\right) - e^2}{(1-e)} \right) dx$$

$$A_6 = \int_{2,05}^e \left(z \cdot (\ln(z))^2 - \frac{ez - e^2}{1-e} \right) dz$$

$$A_6 = \left(-\frac{1}{2}z^2 \cdot (\ln(z))^2 - \frac{1}{2}z^2 \cdot \ln(z) + \frac{z^2}{4} - \frac{(e^2 + e)z}{1-e} + \frac{ez^2}{2(1-e)} \right) \Big|_{2,05}^e$$

$$A_6 = \left(-\frac{e^2}{2} - \frac{e^2}{2} + \frac{e^2}{4} - \frac{(e^2 + e)e}{1-e} + \frac{e^3}{2(1-e)} \right) - \left(-\frac{2,05^2}{2} \cdot (\ln(2,05))^2 - \frac{2,05^2}{2} \cdot \ln(2,05) + \frac{2,05^2}{4} - \frac{(e^2 + e)(2,05)}{1-e} + \frac{2,05^2 e}{2(1-e)} \right)$$

$$A_6 = \left(\frac{3,4058}{e-1} - 0,3751 \right) \approx 0,8669$$

It can be evaluated numerically using Geogebra commands; therefore:

GeoGebra code:

$$A_6 = \text{The whole part between} \left(\left(x - \frac{1}{2}\right) \cdot \left(\ln\left(x - \frac{1}{2}\right)\right)^2 + \frac{1}{2} - \frac{ex}{1-e} - \frac{e^2 + e - \frac{1}{2}}{1-e}, 2,55, e + \frac{1}{2} \right)$$

GeoGebra Chart:

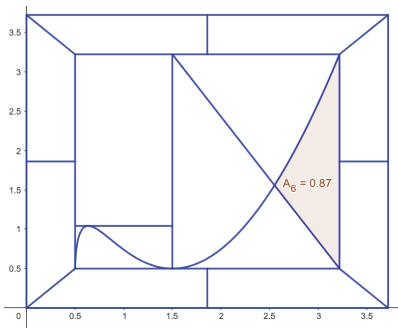


Figure 30: Determining the value and shading of area 6.

- Calculation of area 7:

Similar to area 5, this area is calculated as the sum of two integrals; The first corresponds to the area between the curves $y = \left(x - \frac{1}{2}\right) \cdot \left(\ln\left(x - \frac{1}{2}\right)\right)^2 + \frac{1}{2}$; and $y = \frac{1}{2}$, defined between the integration limits $\frac{3}{2}$ and 2,55; therefore: defined between the integration limits .

$$A_{71} = \int_{\frac{3}{2}}^{2,55} \left(\left(x - \frac{1}{2}\right) \cdot \left(\ln\left(x - \frac{1}{2}\right)\right)^2 + \frac{1}{2} - \frac{1}{2} \right) dx$$

$$A_{71} = \int_{\frac{3}{2}}^{2,55} \left(\left(x - \frac{1}{2}\right) \cdot \left(\ln\left(x - \frac{1}{2}\right)\right)^2 \right) dx$$

$$A_{71} = \int_1^{2,05} \left((z) \cdot (\ln(z))^2 \right) dz$$

$$A_{71} = \left(-\frac{z^2}{2} \cdot (\ln(z))^2 - \frac{z^2}{2} \cdot \ln(z) \right) \Big|_1^{2,05}$$

$$A_{71} \approx 0,3751$$

Meanwhile, the second corresponds to the area between the curves: $y = \frac{ex}{1-e} - \frac{e^2 + e - \frac{1}{2}}{1-e}$; and $y = \frac{1}{2}$, defined between the integration limits: and 2,55 and $(e + \frac{1}{2})$; therefore:

$$A_{72} = \int_{2,55}^{e+\frac{1}{2}} \left(\frac{ex}{1-e} - \frac{e^2 + e - \frac{1}{2}}{1-e} - \frac{1}{2} \right) dx$$

$$A_{72} = \int_{2,55}^{e+\frac{1}{2}} \left(\frac{2ex - 2e^2 - 2e + 1 - 1 + e}{2(1-e)} \right) dx$$

$$A_{72} = \int_{2,05}^{e+\frac{1}{2}} \left(\frac{e\left(x - \frac{1}{2}\right) - e^2}{(1-e)} \right) dx$$

$$A_{72} = \int_{2,05}^e \left(\frac{ez - e^2}{(1-e)} \right) dz$$

$$A_{72} = \left(\frac{ez^2}{2(1-e)} - \frac{e^2 z}{(1-e)} \right) \Big|_{2,05}^e$$

$$A_{72} = \frac{(0,60701)}{e-1} \approx 0,3532$$

Therefore, the entire area would be the sum of the two partial areas, like this:

$$A_7 = A_{71} + A_{72} \approx 0,7284$$

It can be evaluated numerically using Geogebra commands; So:

GeoGebra code:

$$A_{71} = \text{The whole part between} \\ \left(\left(x - \frac{1}{2}\right) \cdot \left(\ln\left(x - \frac{1}{2}\right)\right)^2 + \frac{1}{2}, \frac{1}{2}, 2.55, e + \frac{1}{2} \right)$$

$$A_{72} = \text{The whole part between} \left(\frac{ex}{1-e} - \frac{e^2 + e - \frac{1}{2}}{1-e}, \frac{1}{2}, \frac{3}{2}, 2.55 \right)$$

GeoGebra Chart:

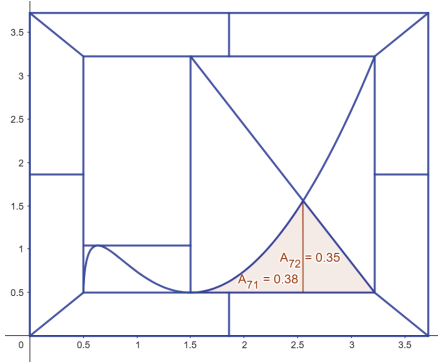


Figure 31: Determining the value and shading of area 7.

• Calculation of area 8

This area is calculated as the integral between the curves: $y = \frac{ex}{1-e} - \frac{e^2 + e - \frac{1}{2}}{1-e}$; and $y = \frac{1}{2}$, defined between the integration limits: $\frac{3}{2}$ y 2,55, therefore:

$$A_8 = \int_{\frac{3}{2}}^{2,55} \left(\frac{ex}{1-e} - \frac{e^2 + e - \frac{1}{2}}{1-e} - \left(x - \frac{1}{2}\right) \cdot \left(\ln\left(x - \frac{1}{2}\right)\right)^2 - \frac{1}{2} \right) dx$$

$$A_8 = \int_{\frac{3}{2}}^{2,55} \left(\frac{e\left(x - \frac{1}{2}\right) - e^2}{(1-e)} - \left(x - \frac{1}{2}\right) \cdot \left(\ln\left(x - \frac{1}{2}\right)\right)^2 \right) dx$$

$$A_8 = \int_1^{2,05} \left(\frac{ez - e^2}{1-e} - z \cdot (\ln(z))^2 \right) dz$$

$$A_8 = \left(\frac{(e^2 + e)z}{1-e} - \frac{ez^2}{2(1-e)} + \frac{1}{2}z^2 \cdot (\ln(z))^2 + \frac{1}{2}z^2 \cdot \ln(z) - \frac{z^2}{4} \right) \Big|_1^{2,05}$$

$$A_8 = \left(\frac{(e^2 + e)(2,05)}{1-e} - \frac{e(2,05)^2}{2(1-e)} + \frac{2,05^2}{2} \cdot (\ln(2,05))^2 + \frac{2,05^2}{2} \cdot \ln(2,05) - \frac{2,05^2}{4} \right) - \left(\frac{(e^2 + e)}{1-e} - \frac{e}{2(1-e)} - \frac{1}{4} \right)$$

$$A_8 = \left(\frac{3,4058}{e-1} - 0,3751 \right) \approx 1,6069$$

It can be evaluated numerically using GeoGebra commands; So:

GeoGebra code:

$$A_8 = \text{the whole part between} \left(\frac{ex}{1-e} - \frac{e^2 + e - \frac{1}{2}}{1-e}, \left(x - \frac{1}{2}\right) \cdot \left(\ln\left(x - \frac{1}{2}\right)\right)^2 + \frac{3}{2}, 2,55 \right)$$

GeoGebra Chart:

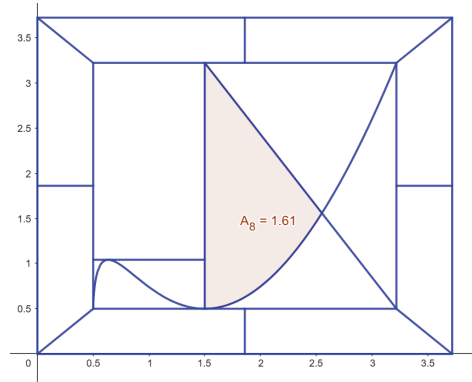


Figure 32: Determining the value and shading of area 8.

• Calculation of the areas from 9 to 16:

To obtain the value of the areas of each of the elements of the mosaic that are found inside the contours between the external square and its inscribed square; The total internal area is calculated and divided by the total number of regions, that is, 8, obtaining the following area:

$$A_i = \frac{(e + 1)^2 - e^2}{8}; \text{ to } i = 9, 10, \dots, 16$$

$$A_i = \frac{2e + 1}{8}; \text{ to } i = 9, 10, \dots, 16$$

$$A_i \approx 0,8045; \text{ to } i = 9, 10, \dots, 16$$

Equivalently, the area between the curves can be calculated: $y = x$; and $y = \left(e + \frac{1}{2}\right)$ defined between the integration limits: $e + \frac{1}{2}$ y $e + 1$; therefore:

$$A_i = \int_{e + \frac{1}{2}}^{e + 1} \left(x - \left(\frac{e + 1}{2}\right) \right) dx; \text{ to } i = 9, 10, \dots, 16$$

$$A_i = \int_{e + \frac{1}{2}}^{e + 1} \left((x - e) + \frac{e}{2} - \frac{1}{2} \right) dx; \text{ to } i = 9, 10, \dots, 16$$

$$A_i = \int_{\frac{1}{2}}^1 \left(z + \frac{e - 1}{2} \right) dz; \text{ to } i = 9, 10, \dots, 16$$

$$A_i = \left(\frac{z^2}{2} + \left(\frac{e - 1}{2}\right)z \right) \Big|_{\frac{1}{2}}^1; \text{ to } i = 9, 10, \dots, 16$$

$$A_i = \left(\frac{1}{2} + \left(\frac{e - 1}{2}\right) \right) - \left(\frac{1}{8} + \left(\frac{e - 1}{4}\right) \right); \text{ to } i = 9, 10, \dots, 16$$

$$A_i = \frac{2e + 1}{8}; \text{ to } i = 9, 10, \dots, 16$$

$$A_i \approx 0,8045; \text{ to } i = 9, 10, \dots, 16$$

It can be evaluated numerically using GeoGebra commands; So:

GeoGebra code:

$$A_9 = \text{the whole part between} \\ \left(x, \left(\frac{e+1}{2}\right), e + \frac{1}{2}, e + 1\right)$$

GeoGebra Chart:

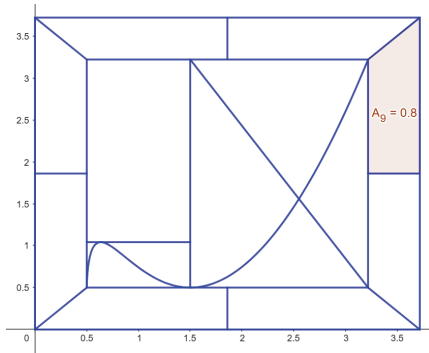


Figure 33: Determination of the value and shading of the area 9.

MOSAIC PRICE

In the process of determining the price of a good related to art, “the basic mathematics that underlies the art business follows the rules of any market economy: supply and demand, in their infinite variations” [2]; Therefore, important aspects such as the negotiation strategy, the benefit obtained by the client or the community, but implicitly marketing in the sale of the product, are necessary. For this procedure, there are different methods that rely on different formulations that help in assessing the appropriate price. Based on the above, the price of a good that is subject to art must include information on the profitability margin, as well as the attractiveness for the customer; but fundamentally, the coverage of production costs and the social and economic benefit it grants to the parties (artist who makes the work and the community or client who receives the work) must be considered.

As Michel Miró states in his blog, “Calculating the selling price of a product is a crucial task for any business. If the price is

too high, customers will turn away, but if it is too low, the business will not be profitable.” [3]; and within his blog he proposes a methodology for calculating pricing, thus there are the following steps:

1. Cost determination

Determine the cost of producing a unit; considering for this, costs of materials, labor, shipping and any other expense related to its production such as general expenses such as security and taxes.

2. Establishing the profit margin

Once the cost has been determined, a profit margin must be established; which corresponds to the percentage of profit that is desired to be obtained from the sale of the product, in the artistic field the profit margin is between 20% and 50%.

3. Calculation of the marketing price of the work

With the two inputs from the previous points, the calculation of the marketing price is proposed, considering the following formula:

$$P_v = \frac{C}{1 - M} \quad [1]$$

Where:

P_v := Sales or marketing price.

C := Cost of the good or work.

M := profit margin value.

4. Consideration of the market effect

It must be considered that the market defines the price of some products; Thus, if similar products (works) appear on the market at lower prices, this forces us to reduce the profit margin or, failing that, adjust the cost in order to maintain the market and be competitive; in contrast to the fact that, if the works are unique or of high quality, this would allow higher prices to be established.

5. Test – error and adjustment of the work

Once the marketing price of a work has been established, it becomes necessary to test and adjust the price with a trial and error

mechanism; considering for this, the response of clients when making adjustments, when carrying out a comparative analysis of the sale with respect to the profit margin, when they are at high or low levels.

In short, for this article it focuses on determining the starting price (foot or base) of the good or work with the mathematical application up to point 3 of the proposed methodology; leaving aside the subsequent steps, which include the consideration of the effect of the market, as well as the trial-error and adjustment of the price of the work that is not the subject of this article; This is how you have it:

MOSAIC COST

Given that the cost of the mosaic is based mainly on the amount of painted area as well as the cost per type of color, as well as the arrangement of colors at the time of making the mosaic, the following considerations are taken into account, namely:

MOSAIC AREAS

The following Figure shows the results of the calculations that were carried out in the previous section; and, synthesizes the arrangement and area values obtained for the regions of the mosaic, and presents the area values in descending order for the inscribed square, as follows:

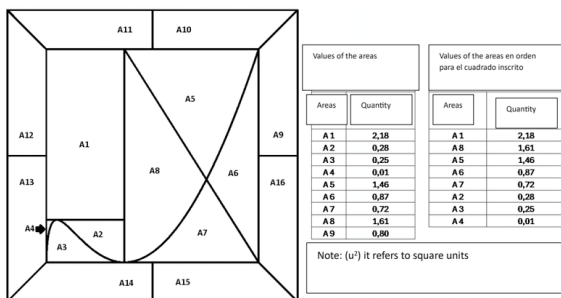


Figure 34: Location and values of each of the areas of the mosaic regions; and presentation in descending order of the area values of the inscribed square.

COSTS ASSOCIATED WITH THE TYPE OF COLOR

The following Table shows the costs per square unit of paint application, discriminated by color; So:

Color cost per square unit	
Color	Value (US\$)
Purple	\$ 10,00
Yellow	\$ 20,00
Red	\$ 40,00
Orange	\$ 80,00
Green	\$ 100,00
Pink	\$ 130,00
Purple	\$ 170,00
Sky blue	\$ 300,00

Table 03: Application costs by color type in each square unit.

DETERMINATION OF COSTING BASED ON THE PROCESS OF PAINTING MOSAIC REGIONS

Based on the instructions for filling and painting the regions of the mosaic, where it is established that the painting begins from the inscribed square, starting from the largest area until reaching the last one, considering for the effect a single different color in each region; to later continue with filling the border with the same colors but taking into account that said colors that have been used in the painting of the inscribed square are not repeated. One must consider the fact that the arrangement of the colors on the border does not influence, even when the unit costs of each color are different, but in the calculation of the final cost it becomes indifferent since each color is arranged in a region of equal area within the border that lies between the external and inscribed squares; Therefore, for ease of calculation of the mosaic, only the costs of the colors and their areas are considered, thus we have the following equation:

$$C_f = C_{CI} + C_B \quad [2]$$

Where:

C_f := Final cost

C_{CI} := Cost of the registered square.

C_B := Cost of the border between the external and inscribed square.

By maintaining a linear relationship between the amount of area and the unit cost of material for each type of paint color, we have:

$$C_{CI} = A^t * B; \text{ with } A = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \\ \alpha_6 \\ \alpha_7 \\ \alpha_8 \end{bmatrix} \text{ and } B = \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \\ \beta_5 \\ \beta_6 \\ \beta_7 \\ \beta_8 \end{bmatrix} \quad [3]$$

$$C_B = D^t * B; \text{ with } D = \begin{bmatrix} \delta_1 \\ \delta_1 \\ \delta_1 \\ \delta_1 \\ \delta_1 \\ \delta_1 \\ \delta_1 \\ \delta_1 \end{bmatrix} \text{ and } B = \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \\ \beta_5 \\ \beta_6 \\ \beta_7 \\ \beta_8 \end{bmatrix} \quad [4]$$

Where:

A:=Vector of areas of the regions of the inscribed square.

B:=Cost vector by paint color type.

D:=Constant area vector of the edge between the external and inscribed square.

Therefore, the final cost would be established as the product of the following matrices:

$$C_f = (A + D)^t * B; \text{ with } (A + D) = \begin{bmatrix} \alpha_1 + \delta_1 \\ \alpha_2 + \delta_1 \\ \alpha_3 + \delta_1 \\ \alpha_4 + \delta_1 \\ \alpha_5 + \delta_1 \\ \alpha_6 + \delta_1 \\ \alpha_7 + \delta_1 \\ \alpha_8 + \delta_1 \end{bmatrix} \text{ and } B = \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \\ \beta_5 \\ \beta_6 \\ \beta_7 \\ \beta_8 \end{bmatrix} \quad [5]$$

BASE RATE AND DETERMINATION OF MINIMUM AND MAXIMUM COST MOSAICS

Once the values that enter the matrices A, B and D have been defined, which are involved in the structuring of the minimum cost mosaics, as well as the maximum cost; The following results are obtained with their calculations, values and the pictography of the final mosaic.

MINIMUM COST MOSAIC

In the case of the minimum cost mosaic, there are the following calculations and the base pricing value (minimum cost), like this:

$$C_{Minimum} = (A + D)^t * B; \text{ with } (A + D) = \begin{bmatrix} 0,01 + 0,80 \\ 0,25 + 0,80 \\ 0,28 + 0,80 \\ 0,72 + 0,80 \\ 0,87 + 0,80 \\ 1,46 + 0,80 \\ 1,61 + 0,80 \\ 2,18 + 0,80 \end{bmatrix} \text{ and } B = \begin{bmatrix} 10 \\ 20 \\ 40 \\ 80 \\ 100 \\ 130 \\ 170 \\ 300 \end{bmatrix}$$

$$C_{Minimum} = [2,98 \quad 2,41 \quad 2,26 \quad 1,67 \quad 1,52 \quad 1,08 \quad 1,05 \quad 0,81] * \begin{bmatrix} 10 \\ 20 \\ 40 \\ 80 \\ 100 \\ 130 \\ 170 \\ 300 \end{bmatrix}$$

$$C_{Minimum} = 1.915,90 \text{ [US\$]}$$

Based on the above, and filling the regions of the mosaic with the colors established for each area, the pictograph of the final mosaic is obtained with its corresponding colors, like this:

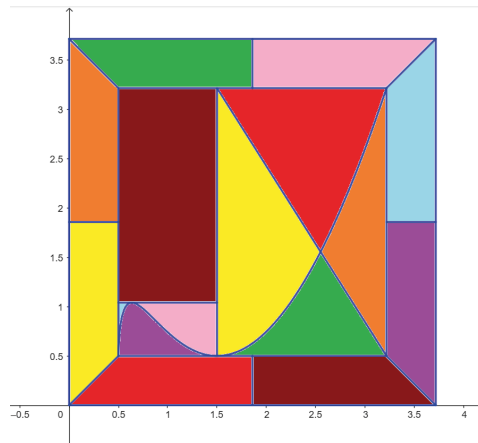


Figure 35: Minimum cost final mosaic.

MAXIMUM COST MOSAIC

In the case of the maximum cost mosaic, there are the following calculations and the base pricing value (maximum cost), like this:

$$C_{Maximum} = (A + D)^t * B; \text{ con } (A + D) = \begin{bmatrix} 2,18 + 0,80 \\ 1,61 + 0,80 \\ 1,46 + 0,80 \\ 0,87 + 0,80 \\ 0,72 + 0,80 \\ 0,28 + 0,80 \\ 0,25 + 0,80 \\ 0,01 + 0,80 \end{bmatrix} y B = \begin{bmatrix} 10 \\ 20 \\ 40 \\ 80 \\ 100 \\ 130 \\ 170 \\ 300 \end{bmatrix}$$

$$C_{Maximum} = [0,81 \quad 1,05 \quad 1,08 \quad 1,52 \quad 1,67 \quad 2,26 \quad 2,41 \quad 2,98] * \begin{bmatrix} 10 \\ 20 \\ 40 \\ 80 \\ 100 \\ 130 \\ 170 \\ 300 \end{bmatrix}$$

$$C_{Maximum} = 1.958,40 \text{ [US\$]}$$

Based on the above, and filling the regions of the mosaic with the colors established for each area, the pictograph of the final mosaic is obtained with its corresponding colors, like this:

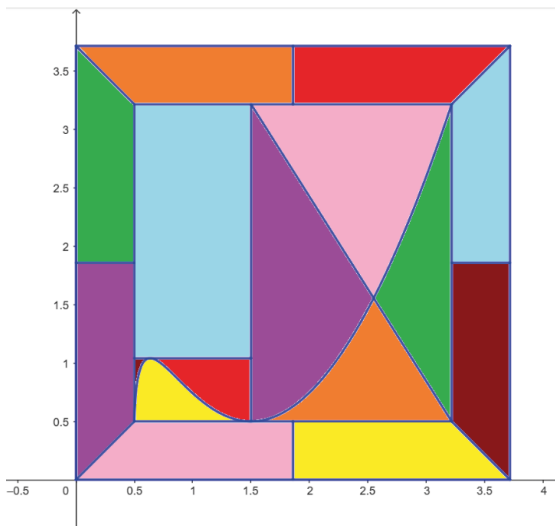


Figure 35: Max Cost Final Tile.

PRICING ANALYSIS OF OTHER MOSAICS

To answer the question of what happens to the potential colors of mosaics that meet all the conditions, but their paint filling is random, the answer is structured based on the following conditions:

1. The first condition established for the painting of the mosaic is given by

the restriction of a total of 8 different colors that are arranged in 8 regions with different areas inside the inscribed square, considering that the same color is not repeated in 2 different regions., it is therefore obtained that the number of different ways of coloring corresponds to a permutation of 8, that is, 8!, which is equivalent to 40,320 different ways of painting the internal area of the inscribed square.

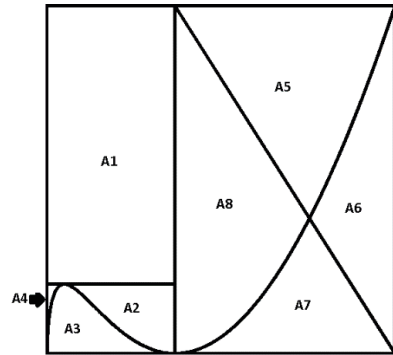


Figure 36: Location of the areas of the inscribed square of the mosaic.

2. The second condition establishes that for the painting of the 8 equal regions that are on the border of the external and internal squares, the colors must be different from those that have already been painted in the areas of the inscribed square; Therefore, the painting sequence would be reduced to only 4 different colors for 4 different areas (gray regions), without the same color being repeated; Therefore, the number of different ways of coloring corresponds to a permutation of 4, that is, 4!, which is equivalent to 24 different ways of painting the 4 regions of the established area, leaving the remaining 4 areas (white regions) to proceed. in the same way, that is to say that you would again have 24 different ways of painting.

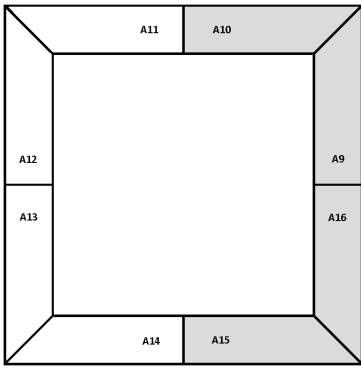


Figure 37: Location of the border regions between the external squares and their inscribed, symmetrical arrangement with respect to the axis and to paint different colors than those contained in the inscribed square.

3. Taking into consideration, the two previous considerations, it is obtained that all the mosaics are filled with different colors in each region of the inscribed square, and in turn, with the condition that the edge is filled with the colors opposite to every 4 equal regions and symmetrical to the y axis, it is obtained that all the different and viable mosaic shapes are:

$$\text{Number of different tiles} = \frac{8!}{4!4!} = \frac{8 \cdot 7 \cdot 6 \cdot 5 \cdot 4!}{4 \cdot 3 \cdot 2 \cdot 4!} = 70$$

4. Considering the phenomenon of permutation of colors, the prices of these mosaics will be established between a value of US\$ 1,015.09 and US\$ 1,958.40; Therefore, the calculations of their values are obtained in an ascending manner through a convolution between the 70 different shapes of mosaics with respect to the vector of costs per type of paint color for filling each region, obtaining the results that are presented in the following graph:

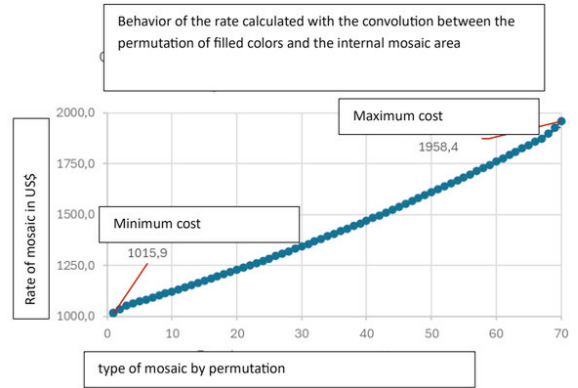


Figure 38: Plot of results of the rate behavior, calculated with the convolution between the permutation of filling colors and the area of the internal region of the mosaic.

4. Therefore, the basic pricing cost prior to marketing the abstract art mosaic is calculated with the formula [1]; Thus, when carrying out a simulation with several profit margins between 20% and 50% with a variation of 5%, the following results are obtained; So:

Basic pricing table by type of mosaic and profit margins			
Mosaic per type of cost			
Profit margins (%)	Minimum cost (US\$ 1,015.9)	Average cost (US\$ 1,487.2)	Maximum cost (US\$ 1,958.4)
20%	\$ 1.269,88	\$ 1.859,00	\$ 2.448,00
25%	\$ 1.354,53	\$ 1.982,93	\$ 2.611,20
30%	\$ 1.451,29	\$ 2.124,57	\$ 2.797,71
35%	\$ 1.562,92	\$ 2.288,00	\$ 3.012,92
40%	\$ 1.693,17	\$ 2.478,67	\$ 3.264,00
45%	\$ 1.847,09	\$ 2.704,00	\$ 3.560,73
50%	\$ 2.031,80	\$ 2.974,40	\$ 3.916,80

Table 02: Basic pricing for a mosaic by its cost type and profit margin value.

CONCLUSIONS

- In the generation of a work of abstract art such as the mosaic, the use of differential and integral calculus could be put into practice for its design, mainly the use of functions, generation of line segments, as well as arc length, the calculation of critical points and their identification in local maxima or minima, application of algorithms of numerical methods for solutions of an equation, calculation of areas of regions between curves. At the same time, the mosaic

design could be captured through the use of scientific software such as GeoGebra; and in a complementary way, it was possible to study the values of the areas and obtain the pictography of the desired mosaics based on the arrangement of colors.

- Another important aspect lies in pricing a work of abstract art, by correlating the costs associated with the size of the mosaic, as well as the dimensions of the areas of the regions that are painted and the costs associated with the typology of color. with which the work is going to be painted.

RECOMMENDATIONS

- The article could be reused to define similar or restricted problems exclusively to put into practice the application of differential and integral calculus in an engineering, architecture or art class.

- This article can be used to prepare technical notes for a pricing process for works of abstract art.

- The analysis of curves and handmade surfaces through the GeoGebra software would allow the study of a similar work of art, but can be extensive for the pricing of pictorial works of other types.

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