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THEORY OF VON KARMA AND POHLHA. E. USED TO OBTAIN PRANDTL NUMBERS, NUSSELT AND REYNOLDS

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Abstract: Theodore von Kármán and Pohlhausen E. developed an integral method for solving partial differential equations. This method is easier to apply than classical methods, such as Fourier, Laplace or Vito Volterra. The von Kármán and Pohlhausen method is based on the basic momentum equation for constant and incompressible flows. The equation is integrated over the thickness of the boundary layer. The result is an integral equation that relates the velocity, pressure, and temperature in the boundary layer.

The method can be applied to a variety of flow problems. In the article, it is applied to the hydrodynamic and thermal boundary layer. The results are compared with those published in the literature and are in good agreement. To obtain a better approximation to the solution of the integral equation, the degree of the algebraic polynomial used in the integration can be increased.

Keywords: Von Karman, Fourier, Laplace, Volterra, Reynols

INTRODUCTION

In the application of the integral method for the solution of partial differential equations it is very important and is highlighted in the work of Van Karman and Pohlhausen[1,2,3] who applied the method for the approximate analysis of the boundary layer, momentum and energy equation, as well as in fluid mechanics. Landah[3,4,5] He used it in the field of biophysics to solve the diffusion equation.

METHODOLOGY

The procedure[5,6,7] will be used to approximate the value of Reynolds, Prandtl and Nusselt, these will be compared with those already obtained in the literature.

This is simple, straightforward, and easy to apply to heat conduction problems with linear and nonlinear boundary values. These basic concepts related to this methodology are used to solve problems in semi-infinite spaces. It can be applied to stable and transitory cases, for the general case an algebraic polynomial of degree n is proposed, in particular a polynomial of degree three is used, in an application of a transient homogeneous problem a polynomial of degree two is proposed[5,6,7].

To generalize this, a modification is made to its procedure, the methodology will be reformulated in the following repetitive way to reach a better approximation.

The following algorithm is proposed in pseudocode.

1. Start.

2. Read ε

3. The partial differential equation integrates over a distance, this is the distance from the wall at which the temperature is affected by the boundary conditions. $\delta(t)$

4. Two appropriate profiles are selected on the thermal film, which is an algebraic polynomial for the thermal film, it is proposed that they be greater than or equal to the third degree. also (approximation error or tolerance) $\delta(t)$ $\delta_1(t)$ $\delta_2(t)$ $\varepsilon > 0$

5. This is substituted into the integral equation, performing the operations we obtain an ordinary differential equation, for the film, time is an independent variable. The solution is determined subject to initial conditions as a function of time. $\delta(t)_{1,2}$

6. If then increase the degree of the polynomial, return to point four. $\| \delta_2(t) - \delta_{1(t)} \| > \varepsilon$ $\delta_1(t) = \delta_2(t)$

7. When the error is met $\delta(t)_1$ $T(x,t)$, we have which is a distribution, for example of the temperature of the problem, which is a solution that depends on position

and time.

8. End.

We have the following partial differential equation:

$$\frac{\partial^2 T(x,t)}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T(x,t)}{\partial t} \text{in } 1) x > 0, t > 0$$

The initial boundary (boundary) conditions given by:

$$T(x,t) = T_0 \qquad \text{For} x = 0, t > 0 \tag{2}$$

$$T(x,t) = T_i \qquad t = 0, x \geq 0 \tag{3}$$

Developing the integral method for (a single algebraic polynomial) the application for equation (1) is solved with the boundary conditions equation (2-3).

The Equation is integrated. (1) with respect to the variable since

xx=0 a xt=δt

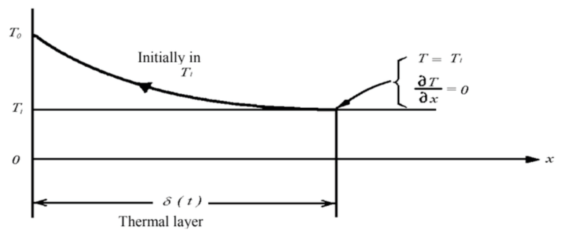

Figure. (2) definition of the Thermal layer 5,6,7

For this type of integration, the fundamental theorem for line integral is used, which says that it is fx,y a function in the open region, which contains the points, Ax_0, y_0, Bx_1, y_1 *if* $fx,y = \nabla\phi(x,y)$ the points are in the region, then for any piecewise smooth curve in a region C, which begins in A and ends in B, these are completely included in the region, this way we have the line integral $\int_C f(x,y)dr = \phi(x_1,y_1) - \phi(x_0,y_0)$. The following identity will be used that relates the derivative under the integration operator:

$$\frac{d}{dx}\left(\int_{a(x)}^{b(x)} F(x,m)dm\right) = \int_{a(x)}^{b(x)} \frac{\partial F(x,m)}{\partial x} dm + f(x,b(x))\frac{db(x)}{dx} - f(x,a(x))\frac{da(x)}{dx} \; asi \; \frac{\partial T}{\partial x}\Big|_{x=\delta(t)} - \frac{\partial T}{\partial x}\Big|_{x=0} = \frac{1}{\alpha}\int_{x=0}^{\delta(t)} \frac{\partial T}{\partial t} dx \tag{4}$$

The integral of equation (4) is applied to the right side with the differentiation rules under integration, we have:

$$\frac{d}{dt}\left[\int_{x=0}^{x=\delta(t)} T(x,t)dx\right] = \int_{x=0}^{x=\delta(t)} \frac{\partial T(x,t)}{\partial t} dx + T(x,\delta(t))\frac{d\delta(t)}{dt} - T(x,0)\frac{dx}{dt}$$

where like this

$$\frac{dx}{dt} = 0$$

$$\frac{\partial T}{\partial x}\Big|_{x=\delta(t)} - \frac{\partial T}{\partial x}\Big|_{x=0} = \frac{\partial T}{\partial x}\Big|_{x=\delta} - \frac{\partial T}{\partial x}\Big|_{x=0} = \frac{1}{\alpha}\left[\frac{d}{dt}\left(\int_{x=0}^{\delta(t)} T\,dx\right) - T\Big|_{x=\delta}\frac{d\delta}{dt}\right] \tag{5}$$

By the definition of the thermal boundary layer as seen in the figure. (1), the conditions are obtained:

$$\frac{\partial T}{\partial x}\Big|_{x=\delta(t)} = 0 \; y \; T_{=\delta(t)} = T_i \tag{6}$$

To facilitate the development, a variable change is made

$$\theta = \int_{x=0}^{\delta(t)} T(x,t)\,dx \tag{7}$$

the analysis is defined:

Introducing the values of equations (6-7) into equation (5) we obtain:

$$-\alpha\frac{\partial T(x,t)}{\partial x}\Big|_{x=0} = \frac{d}{dt}(\theta - T(x,t)_i\delta(t)) \tag{8}$$

Equation (8) is known as the integral energy equation7. As an application, this method will be used in the determination of the Reynolds, Prandtl and Nuselt number relationships.

$$\delta(t)\frac{dp}{dx} - \tau_w = \frac{d}{dx}\int_{x=0}^{x=\delta(t)} (v_x - v_\infty)\rho v_x\,dy \tag{9}$$

If then equation (9) is obtained $\frac{dp}{dx} = 0$

$$\tau_w = \frac{d}{dx}\int_{x=0}^{x=\delta(t)} (v_x - v_\infty)\rho v_x\,dy \tag{10}$$

The Von Karma method AND POHLHA. E. It requires assuming velocity profiles so that the thickness of the displacement for the velocity field can be evaluated at zero, in this case it can be initialized with a polynomial of degree two. Therefore, we normally assume a simple velocity profile and obtain

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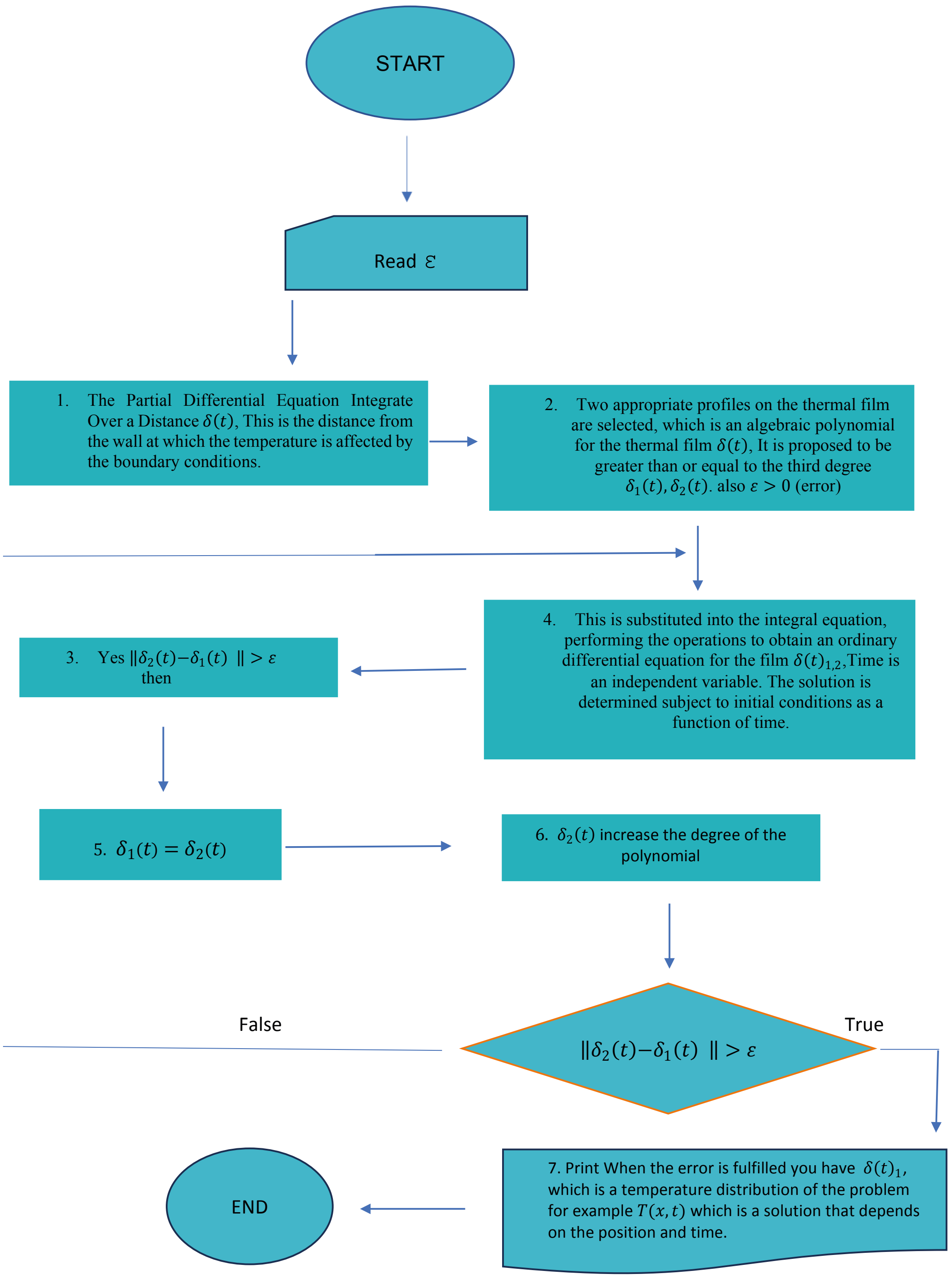

Figure 1). Algorithm flowchart

the displacement thickness, this moment thickness and the shear stress. For this, a cubic polynomial is proposed as a test function.

$$v_x = a + by + cy^2 + dy^3 \tag{11}$$

The boundary conditions are:

$$v_x = 0 \; en \; y = 0 \tag{12 a}$$

$$v_x = v_\infty \; en \; y = \delta(t) \tag{12b}$$

$$\frac{dv_x}{dy} = 0 \; en \; y = \delta(t) \tag{12c}$$

of the Navier-Stokes equation $\rho\delta_x \frac{dT(x,t)}{dt} = f$ and the Continuity equation.

$\frac{\partial \rho}{\partial t} = -\nabla\rho v$ one has

$$\rho\delta_x \frac{dT(x,t)}{dt} = f\frac{\partial\rho}{\partial t} = -(\nabla\rho v)$$

$$v_x\frac{\partial v_x}{\partial x} + v_y\frac{\partial v_y}{\partial y} = \gamma\frac{\partial^2 v_x}{\partial x^2} \tag{13}$$

Applying the boundary condition equations (12 a, b and c)

$$y = 0,\; v_x, v_y = 0 \,\&\, \gamma\frac{\partial^2 v_x}{\partial x^2} = 0 \tag{14}$$

$$\frac{\partial^2 v_x}{\partial y^2} = 0 \;\; en \; y = 0 \tag{15}$$

From equation (11) the first and second partial derivative are calculated.

$$\frac{\partial v_y}{\partial y} = b + 2cy + 3dy^2 \tag{16}$$

$$\frac{\partial^2 v_x}{\partial y^2} = 2c + 6dy \tag{17}$$

Applying the boundary condition (12 a) in equation (11) we have:

$$a = 0 \tag{18}$$

Substituting equation (15) into equation (17)

$$c = 0 \tag{19}$$

substituting the following boundary condition (12c) into equation (16)

$$b = -3d\delta(t)^2 \tag{20}$$

Substitute equations (18-19) into equation (11)

$$v_x = (-3d\delta(t)^2) + dy^3 \tag{21}$$

$$v_x = d[y^3 - 3\delta(t)^2] \tag{22}$$

Using the boundary condition (12b) in equation (22)

$$d = -\frac{v_\infty}{2\delta(t)^3} \tag{23}$$

$$v_\infty = d(-2\delta(t)^3) \tag{24}$$

Substituting equation (23) into equation (22)

$$v_x = v_\infty\left[\frac{3}{2}\left(\frac{y}{\delta(t)}\right) - \frac{1}{2}\left(\frac{y}{\delta(t)}\right)^3\right] \tag{25}$$

partially deriving equation (25) with respect to “y” evaluating in y=0

$$\frac{dv_x}{dy}\Big|_{y=0} = \frac{3}{2}\frac{v_\infty}{\delta(t)} \tag{26}$$

analyzing the integral of equation (10)

$$\int_{x=0}^{x=\delta(t)} (v_x - v_\infty)\rho v_x \, dy = \rho\left[v_\infty \int_{x=0}^{x=\delta(t)} v_x dy - \int_{x=0}^{x=\delta(t)} v_x{}^2 dy\right] \tag{27}$$

Substituting equation (25) into equation (27)

$$\int_{x=0}^{x=\delta(t)} (v_\infty - v_x)\rho v_\infty \, dy = \frac{39}{280}\rho\, v_\infty{}^2\delta(t) \tag{28}$$

substituting equation (28) into equation (10)

$$\tau_w = \frac{29}{280}\rho\, v_\infty{}^2\frac{d\delta(t)}{dx} \tag{29}$$

Since:

$$\tau_w = \mu\frac{dv_x}{dy}\Big|_{y=0} \tag{30}$$

and substituting equation (26) into equation (30)

$$\tau_w = \mu\left[\frac{3}{2}\frac{v_\infty}{\delta(t)}\right] \tag{31}$$

Equating equation (29) and equation (31), separating variables and integrating

$$\delta(t)^2 = \frac{280}{13}\left(\frac{\gamma}{v_\infty}\right)x + c \tag{32}$$

In equation (32) the value of c = 0

$$\delta(t)^2 = \frac{280}{13}\left(\frac{\gamma}{v_\infty}\right)x \quad (33)$$

where finally the hydrodynamic boundary layer is given by:

$$\delta(t) \approx \frac{4.641}{\sqrt{Re_x}}x \quad (34)$$

Now the thermal boundary layer is calculated, where the same polynomial is proposed as a test function by performing a procedure analogous to the previous one.

$$\frac{T-T_s}{T_\infty - T_s} = a' + b'y + c'y^2 + d'y^3 \quad (35)$$

$$\frac{\partial T}{\partial y} = b' + 2c'y + 3d'y^2 \quad (36)$$

$$\frac{\partial^2 T}{\partial y^2} = 2c' + 6d'y \quad (37)$$

boundary condition

$$\frac{T-T_s}{T_\infty - T_s} = 0 \; en \; y = 0 \quad (38\text{ a})$$

$$\frac{\partial^2 T}{\partial y^2} = 0 \; en \; y = 0 \quad (38\text{b})$$

$$\frac{\partial T}{\partial y} = 0 \; en \; y = \delta th \text{or} \quad (38\text{ c})$$

$$\frac{\partial T}{\partial y}\Big|_{y=\delta th} = 0$$

$$\frac{T-T_s}{T_\infty - T_s} = 1 \; en \; y = \delta th \quad (38\text{d})$$

From equation (35) and (38 a)

$$a' = 0 \quad (39)$$

Using condition (38b) in equation (37)

$$c' = 0 \quad (40)$$

Substituting condition (38c) into equation (36)

$$b' = -3d'(\delta th)^2 \quad (41)$$

Substituting condition (38d) into equation (35)

$$d' = -\frac{1}{2\delta th^3} \quad (42)$$

substituting equation (42) into equation (41)

$$b' = \frac{3}{2}\left(\frac{1}{\delta th}\right) \quad (43)$$

Using equations (39-43) in equation (35)

$$\frac{T-T_s}{T_\infty - T_s} = \frac{3}{2}\left(\frac{y}{\delta th}\right) - \frac{1}{2}\left(\frac{y}{\delta th}\right)^3 \quad (44)$$

making a change of variable in equation (44)

$$\frac{\theta}{\theta_\infty} = \frac{T-T_s}{T_\infty - T_s} = \frac{3}{2}\left(\frac{y}{\delta th}\right) - \frac{1}{2}\left(\frac{y}{\delta th}\right)^3 = \frac{3}{2}\left(\frac{1}{\delta th}\right)y - \frac{1}{2}\left(\frac{1}{\delta th}\right)^3 y^3 \quad (45)$$

$$\theta = \theta_\infty\left[\frac{3}{2}\left(\frac{1}{\delta th}\right)y - \frac{1}{2}\left(\frac{1}{\delta th}\right)^3 y^3\right] \quad (46)$$

from equation (8)

$$\alpha\frac{\partial T}{\partial y}|_{y=0} = \frac{d}{dx}\int_0^{\delta t} \quad v_x(T_\infty - T)dy = \frac{d}{dx}\int_0^{\delta t} \quad v_x(\theta_\infty - \theta)d\theta \quad (47)$$

$$\alpha\frac{\partial T}{\partial y}|_{y=0} = \frac{d}{dx}\int_0^{\delta t} \quad v_x(\theta_\infty - \theta)d\theta \quad (48)$$

Evaluating the term on the right side of equation (48) using equation (44)

$$\alpha\frac{\partial T}{\partial y}|_{y=0} = \frac{3}{2}\left[\frac{\theta_\infty}{\delta th}\right] \quad (49)$$

Multiply equation (49) by α:

$$\alpha^2\frac{\partial T}{\partial y}\Big|_{y=0} = \frac{3}{2}\alpha\left[\frac{\theta_\infty}{\delta th}\right] \quad (50)$$

Substitute equations (46,50) into equation (48)

$$\frac{\alpha}{\delta th} = \frac{1}{10}\upsilon_\infty\frac{d}{dx}\left[\delta\left(\frac{\delta t^2}{\delta^2} - \frac{1}{14}\frac{\delta t^4}{\delta^4}\right)\right] \quad (51)$$

performing a variable change

$$\xi = \frac{\delta t}{\delta} \quad (52)$$

substituting equation (52) into equation (51)

$$\frac{\alpha}{\xi\delta} = \frac{1}{10}\upsilon_\infty\frac{d}{dx}\left[\delta\left(\xi^2 - \frac{1}{14}\xi^4\right)\right] \quad (52)$$

Since $\varepsilon^4 \approx 0$ it is very smal

$$\frac{\alpha}{\xi\delta} = \frac{1}{10}\upsilon_\infty\frac{d}{dx}\left[\delta\left(\xi^2\right)\right] \quad (53)$$

$$\frac{v_\infty}{10}\left[2\delta^2\xi^2\frac{d\xi}{dx} + \xi^3\delta\frac{d\delta}{dx}\right] = \alpha \quad (54)$$

$$\delta d\delta = \frac{140}{13}\frac{\gamma}{v_\infty}dx \tag{55}$$

$$\delta\frac{d\delta}{dx} = \frac{140}{13}\frac{\gamma}{v_\infty} \tag{56}$$

From equation (33) and equation (56) substituted into equation (54) simplifying

$$\xi^3 + 4x\xi^2\frac{d\xi}{dx} = \frac{13}{14}\frac{\alpha}{\gamma} \tag{57}$$

Taking into consideration

$$\xi^2\frac{d\xi}{dx} = \frac{1}{3}\frac{d}{dx}(\xi^3) \tag{58}$$

substituting equation (58) into equation (57)

$$\xi^3 + 4x\left(\frac{1}{3}\frac{d}{dx}(\xi^3)\right) = \xi^3 + \frac{4}{3}x\frac{d}{dx}(\xi^3) = \frac{13}{14}\frac{\alpha}{\gamma} \tag{59}$$

using the following variable change

$$y = \xi^3 \tag{60}$$

substituting equation (60) into equation (59)

$$\left(y - \frac{13}{14}\frac{\alpha}{\gamma}\right)dx + \frac{4}{3}dy = 0 \tag{61}$$

solving (61)

$$y = C\left(x^{-\frac{3}{4}}\right) + \frac{13}{14}\frac{\alpha}{\gamma} \tag{62}$$

replace equation (60) with equation (62)

$$\xi^3 = C\left(x^{-\frac{3}{4}}\right) + \frac{13}{14}\frac{\alpha}{\gamma} \tag{63}$$

using the relationship

$$\frac{1}{Pr} = \frac{\alpha}{\gamma} \tag{64}$$

Substituting equation (64) into equation (63)

$$\xi^3 = C\left(x^{-\frac{3}{4}}\right) + \frac{13}{14}\frac{1}{Pr} \tag{65}$$

as the thermal and hydrodynamic layer coincide at $x = x_0 = 0$ *for* $0\ \varepsilon = 0$

$$C = -\frac{13}{14}\frac{1}{Pr}x_0^{\frac{3}{4}} \tag{66}$$

Substituting equation (66) into equation (65)

$$\xi = \frac{\delta t}{\delta} \approx \frac{1}{1.023}Pr^{-\frac{1}{3}}\left[1 - \left(\frac{x_0}{x}\right)^{\frac{3}{4}}\right]^{\frac{1}{3}} \tag{67}$$

If then $x_0 = 0$

$$\frac{\delta t}{\delta} \approx 0.977Pr^{-\frac{1}{3}} \tag{68}$$

of (49)

$$\frac{\partial T}{\partial y}\bigg|_{y=0} = \frac{3}{2}\left[\frac{\theta_\infty}{\delta th}\right] = \frac{3}{2}\frac{1}{\delta th}\theta_\infty = \frac{3}{2}\frac{1}{\delta th}(T_\infty - T_s) \tag{69}$$

using the heat conduction equation and substituting equation (69)

$$\frac{q}{A} = -k\frac{\partial T}{\partial y}\bigg|_{y=0}$$

$$= -k\left[\frac{3}{2}\frac{1}{\delta th}(T_\infty - T_s)\right] = \frac{3}{2}\frac{k}{\delta th}(T_s - T_\infty) \tag{70}$$

now using the convection equation and equation (70)

$$h(T_s - T_\infty) = \frac{3}{2}\frac{k}{\delta th}(T_s - T_\infty) \tag{71}$$

$$h = \frac{3}{2}\frac{k}{\delta ht} \tag{72}$$

using the inverse of equation (68)

$$\frac{1}{\delta t} = \left(1.026Pr^{\frac{1}{3}}\right)\frac{1}{\delta} \tag{73}$$

Substituting equation (73) into equation (72)

$$h = \frac{3}{2}k\left[\left(1.026Pr^{\frac{1}{3}}\right)\frac{1}{\delta}\right] \tag{74}$$

using the reciprocal of (34)

$$\frac{1}{\delta} = \frac{Re_x^{\frac{1}{2}}}{4.62x} \tag{75}$$

substituting in equation (75) into equation (74)

$$h = \frac{3}{2}k\left[\left(1.026Pr^{\frac{1}{3}}\right)\left(\frac{Re_x^{\frac{1}{2}}}{4.62x}\right)\right] \tag{76}$$

$$h = \frac{1.026}{4.64}\left(\frac{3}{2}\right)\frac{k}{x}Pr^{\frac{1}{3}}Re_x^{\frac{1}{2}} \tag{77}$$

$$\frac{hx}{k} = \frac{1.026}{4.64}\left(\frac{3}{2}\right)Pr^{\frac{1}{3}}Re_x^{\frac{1}{2}} \tag{78}$$

$$Nu_x = \frac{1.026}{4.64}\left(\frac{3}{2}\right)Pr^{\frac{1}{3}}Re_x^{\frac{1}{2}} \tag{79}$$

$$Nu_x = 0.331 Pr^{\frac{1}{3}} Re_x^{\frac{1}{2}} \tag{80}$$

$$\frac{hx}{k} 0.331 Pr^{\frac{1}{3}} Re_x^{\frac{1}{2}} \tag{81}$$

$$h = 0.331 Pr^{\frac{1}{3}} Re_x^{\frac{1}{2}} \left(\frac{k}{k}\right) \tag{82}$$

Value obtained by the integral method	Values reported in the literature
1. hydrodynamic boundary layer $\delta = \frac{5}{\sqrt{Re_x}} x$	$\delta \approx \frac{4.64}{\sqrt{Re_x}} x$
2.Prandtl $\frac{\delta t}{\delta} \approx 0.98 Pr^{-\frac{1}{3}}$	$\frac{\delta t}{\delta} = (1) Pr^{-\frac{1}{3}}$
3.Nusselt $Nu_x \approx 0.331 Re_x^{\frac{1}{2}} Pr^{\frac{1}{3}}$	$Nu_x = 0.332 Re_x^{\frac{1}{2}} Pr^{\frac{1}{3}}$
4. Reynolds $h \approx 0.331 \frac{k}{x} Re_x^{\frac{1}{2}} Pr^{\frac{1}{3}}$	$h = 0.332 \frac{k}{x} Re_x^{\frac{1}{2}} Pr^{\frac{1}{3}}$

Table 1. Shows the values obtained by the integral method with those published: 8 and 9.

Value by the integral method	Percent error
1 hydrodynamic boundary layer number	7.2%
2 Prandtl number	2%
3 number of Nusselt	0.301204%
4 numberReynolds	0.301204%

Table 2: Errors of the method taking the reported value as true.

RESULTS

In the table. (1) the results are shown using the Von Karman-Pohlhasuen method, they were obtained in their approximate analytical form and those in the literature. In the table. (2) shows the percentage error using table (1) that the values of Prandtl, N.usselt and Reynolds, it can be said that they are acceptable for engineering where the tolerance taken as a reference is 5% tolerance.

DISCUSSION

A better approximation can be achieved by taking an algebraic polynomial of a higher degree, the one proposed gave a good approximation compared to those reported.

in table (1); It is observed that the solutions obtained are close to those published.

CONCLUSIONS

The von Kármán and Pohlhausen method is a viable alternative to solve hydrodynamic and thermal boundary layer problems. The method is simpler to apply than classical methods, and provides comparable results.

Table (1) shows that the von Kármán and Pohlhausen method produces accurate results for a variety of flow parameters. Table (2) shows that the percentage errors are reduced by increasing the degree of the algebraic polynomial used in the integration.

Overall, the von Kármán and Pohlhausen method is a useful tool for engineers and scientists who need to solve boundary layer problems.

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