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MATHEMATICAL METHODS IN POPULATION DYNAMICS

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Abstract: Without a doubt, the study in mathematical modeling is current, it is of vital importance in ecological studies, naturally some arise in population dynamics, highlighting the role of mathematical analysis that allows optimizing decision making in a certain behavior. The main objective is the study of the logistic model, to give way to the classic model of the Lotka-Volterra type without diffusion. At the end, an analysis of the model case with diffusion is made, and much attention is paid to linear and nonlinear models. Algorithmic results make sense to explain natural phenomena, growth that can be: a cell, an organ, a human being, a plant or a population, better if it is a fundamental problem of biology.

Keywords: Population dynamics, model without diffusion, model with diffusion.

INTRODUCTION

Mathematics is useful in the study of certain phenomena that have important characteristics. There are linear and non-linear models, the study of which is latent in the scientific environment. To describe non-linear models, it is necessary to understand how a linear model affects and works; since it is not exact for the most part, approximations are made. In general, for non-predictable results, the theories are not firm, this allows the theories to be reinforced, and sustained work on the experimental part. The truth is that nonlinear models explain a greater number of phenomena.

A concern about this reference is found in the work of the Belgian demographer Verhulst (1845), using data on the North American population from 1970 to 1840 to predict its population until the year 1930. His hypothesis was that it would continue to satisfy the logistic equation.

An important theoretical reference is found in the study of marine biologist Umberto

D'Ancona, (Israel, et al., 2002; Margalef, 1998), in relation to the variations in fish populations, their growth was associated with competition for food,

A fundamental problem in biology is growth: a cell, an organ, a plant, a human being, a population. For this essential case we have the differential equation

$$\frac{dx}{dt} = P(t)x \quad (1)$$

whose general solution of (1) is

$$P(t) = ce^{kt} \quad (2)$$

where c behaves like any positive constant (Malthus, 1998). It can be said that growth occurs if $k > 0$, while decrease occurs for $k < 0$. A defect that is observed in equation (1) and the associated solution (2) is that when $k > 0$, then it turns out that $x \rightarrow \infty$ when $t \rightarrow \infty$, as time passes the growth is unlimited, which is in conflict with reality, above all we see that after a certain time passes a cell stops growing, the same than a human being, having reached a maximum size.

The function $P(t)$ follows an unbounded exponential growth. In most cases, this differential equation provides an unrealistic model of population growth, generating controversy between what is predicted and what is observed.

For this change, we can cite the Belgian mathematician and biologist P.F Verhulst, back in 1840, he became interested in some mathematical formulations to predict the populations of several countries, this equation is

$$\frac{dP(t)}{dt} = P(t)(k - bP(t)) \quad (3)$$

the initial condition being $P(0) = P_0$, that is, an initial population P_0 for an initial time $t = 0$, and that k and b are positive constants. Later equation (3) became known as the *logistic equation*, whose solution is called *logistics function*, and therefore its graph is

also a *logistic curve*. It turns out that when the population is very large, equation (3) does not provide a very accurate result in relation to population growth.

A little analysis of this equation, to $k > 0$, where k represents a constant average birth rate, when the average death rate is assumed, at an instant any, is proportional to the population $P(t)$, then, if $\frac{1}{P(t)} \cdot \frac{dP(t)}{dt}$ is the rate of growth per individual in a population (Boyce and DiPrima, 2005), we have

$$\frac{1}{P(t)} \cdot \frac{dP(t)}{dt} = (\text{average birth rate}) - (\text{average death rate})$$

$$\frac{1}{P(t)} \cdot \frac{dP(t)}{dt} = k - bP(t) \quad (4)$$

In this case b is considered a positive constant of proportionality, where multiplying (3) by $P(t)$ gives equation (3), (Gutiérrez, 2019). On the other hand, if equation (4) is written as

$$\frac{dP(t)}{dt} = kP(t) - bP^2(t)$$

The term $-bP^2(t)$ with $b > 0$ is interpreted as a term of inhibition or competition. What is k like with respect to b ? The question remains open.

ESTIMATION EQUATION

We will show that the solution of (4) is bounded when $t \rightarrow \infty$. The equation is nonlinear, but with a separable variable

$$\frac{dP}{kP - bP^2} = dt, P(0) = P_0$$

it is resolvable to see (Gutiérrez, 2019; Scudo, 1971)

$$P(t) = \frac{kP_0}{bP_0 + (k - bP_0)e^{-kt}} \quad (8)$$

in equation (8), given that time can pass without limit, when $t \rightarrow \infty$, with $k > 0$, we obtain

$$P_{\max} = \lim_{t \rightarrow \infty} P(t) = \lim_{t \rightarrow \infty} \left[\frac{kP_0}{bP_0 + (k - bP_0)e^{-kt}} \right]$$

where from

$$P_{\max} = \frac{k}{b} \quad (9)$$

The result (9) confirms that a growth limit of $P(t)$ occurs, as required by biological facts, and only indicates the validity of this mathematical model. In order to analyze the result (8), it would be convenient to set two times for $P(t)$, let us consider these times $t = 1$ and $t = 2$, with some unit of time, and result in the populations P_1 and P_2 respectively. Using equation (7) we obtain, for $t = 1$

$$P_1 = \frac{\frac{k}{b}}{1 + \left(\frac{k}{bP_0} - 1\right)e^{-k}}$$

where from

$$\frac{k}{b} (1 - e^{-k}) = \frac{1}{P_1} - \frac{e^{-k}}{P_0} \quad (a)$$

Similarly when $t = 2$, we have

$$P_2 = \frac{\frac{k}{b}}{1 + \left(\frac{k}{bP_0} - 1\right)e^{-2k}}$$

where from,

$$\frac{k}{b} (1 - e^{-2k}) = \frac{1}{P_2} - \frac{e^{-2k}}{P_0} \quad (b)$$

Clearly the idea is to obtain $\frac{k}{b}$ and k that depend on P_0, P_1 and P_2 , we proceed from (a) and (b) dividing each member respectively,

$$\frac{\frac{k}{b}(1 - e^{-2k})}{\frac{k}{b}(1 - e^{-k})} = \frac{\frac{1}{P_2} - \frac{e^{-2k}}{P_0}}{\frac{1}{P_1} - \frac{e^{-k}}{P_0}}$$

$$\frac{(1 - e^{-2k})}{(1 - e^{-k})} = \frac{\frac{1}{P_2} - \frac{e^{-2k}}{P_0}}{\frac{1}{P_1} - \frac{e^{-k}}{P_0}}$$

$$1 + e^{-k} = \frac{P_1(P_0 - P_2e^{-2k})}{P_2(P_0 - P_1e^{-k})}$$

Thus, it turns out

$$e^{-k} = \frac{P_0(P_2 - P_1)}{P_2(P_1 - P_0)} \quad (10)$$

Substituting equation (10) into result (a), it turns out

$$\frac{k}{b} = \frac{P_1^2 - P_0P_2}{P_1(P_0P_1 - 2P_0P_2 + P_1P_2)} \quad (11)$$

This way, the maximum population is a limit value of $P(t)$, and depends on P_0 , P_1 and P_2 , that is,

$$P(t)_{\max} = \lim_{t \rightarrow \infty} P(t) = \frac{k}{b} = \frac{P_1^2 - P_0P_2}{P_1(P_0P_1 - 2P_0P_2 + P_1P_2)}$$

DIFFUSIONLESS MODEL

A model is established that is based on the following assumptions:

(1) A population of fish is divided into two classes that are exclusive: on the one hand, the *predators* (selacians) and on the other, the prey, with $u(t)$ and $v(t)$ being the number of fish of each class at time t .

(2) During the growth of each of the classes, only the number of their respective individuals and the number of contacts in unit time between the two species intervene.

Let us specify, the growth of predators is affected proportionally, negatively, by their number (due to the limitation of food) and positively proportionally by the number of the aforementioned prey. On the other hand, the growth of prey is affected proportionally and positively by their number (reproductive effect), and proportionally and negatively by the number of contacts mentioned. Considering that there is an absence of fishing intensity and under the interpretation of the derivative (Edwards and Penney, 2009), Vito Volterra proposed the model

$$\begin{aligned} \frac{du}{dt} &= u(t)(m - nv(t)) \\ \frac{dv}{dt} &= v(t)(-p + qu(t)) \end{aligned} \quad (12)$$

In equation (12) t indicates the time, while m , n , p and q are positive constants and explain the indicated proportionalities; That is, these constants positively or negatively affect the growth of the species. It turns out that this same model was proposed by Lotka for the case of some problems of concentrations and chemical reactions (Lotka, 1925). Therefore, system (12) is also called as *Lotka-Volterra model*.

MODEL ANÁLISIS

Equation (12) is a system of first-order ordinary differential equations, according to the data provided it presents different types of solutions where the initial condition is fundamental, $(u(0), v(0)) = (u_0, v_0)$. We will give a biological interpretation of this system, with a primary interest being the solutions $u(t)$, $v(t)$ both positive whatever $t \geq 0$.

With these limitations, the following solutions are distinguished:

(i) Equilibrium solutions. Those that change with the passage of time. We have when $u_0 = 0$, $v_0 = 0$, then $u(t) = 0$, $v(t) = 0$, that is, it is the null solution. For the case where $u_0 = \frac{p}{q}$, $v_0 = \frac{m}{n}$, also the solution is constant.

(ii) Semi-trivial solutions. It is when the species does not have individuals. In this case you could have,

If $u_0 = u_0$, $v_0 = 0$ with $u_0 > 0$ the system turns out

$$\begin{aligned} \frac{du(t)}{dt} &= mu(t) \\ \frac{dv(t)}{dt} &= 0 \end{aligned}$$

It is a separable variable equation whose solution is written as $u(t) = ce^{mt}$, si $u_0 = u_0$, then $c = u_0$, then $u(t) = u_0 e^{mt}$. For $v(t) = c_1$ con $v_0 = 0$ results $c_1 = 0$, then $v(t) = 0$. Therefore, the only solution is

$$(u(t), v(t)) = (u_0 e^{mt}, 0).$$

If $(u_0, v_0) = (0, v_0)$ with $v_0 > 0$, in this case the system is

$$\begin{aligned} \frac{du}{dt} &= 0 \\ \frac{dv(t)}{dt} &= -pv(t) \end{aligned}$$

Therefore the solution is

$$(u(t), v(t)) = (0, v_0 e^{-pt}).$$

(iii) State of non-trivial coexistence. It occurs when both components are non-trivial and positive. In this case, in system (12), if the initial condition satisfies $u_0 > 0, v_0 > 0$, then the only solution originated $(u(t), v(t))$ satisfies $u(t) > 0, v(t) > 0, \forall t > 0$, and therefore the set of points of the form

$$\{(u(t), v(t)) / t \geq 0\}$$

is a closed curve at \mathbb{R}^2 , this implies that the solution is periodic.

Precisely, if $(u(t), v(t))$ is a coexistence state for (12), then in application of the inverse function theorem, it turns out

$$\frac{dv}{du} = \frac{v(-p+qu)}{u(m-nv)},$$

Being an equation of separable variables, the solution is given by

$$\begin{aligned} \int \frac{m-nv}{v} dv &= \int \frac{-p+qu}{u} du + c, \\ \ln(|v|^m |u|^n) &= nv + qu + c, \end{aligned}$$

that exponentiating and making $c_1 = e^c$ results,

$$v^m u^p = c_1 e^{nv} e^{qu} \quad (13)$$

for some constant c_1 , we will show that the curve of equation (13) is closed at \mathbb{R}^2 (Lee, 1967).

Lemma 1. For $u > 0, v > 0$ the equation given in (13) describes a closed curve.

Demonstration. Being $u > 0, v > 0$, let us consider the functions

$$f(u) = \frac{u^p}{e^{qu}}, g(v) = \frac{v^m}{e^{nv}}$$

The behavior of these functions will be determined. It is observed that $g(0) = 0, g(\infty) = 0$, since $v > 0$ and $e^{nv} > 0$ means that $g(v)$ is positive. Differentiating the function g ,

$$\frac{dg(v)}{dv} = \frac{v^{m-1}(m-nv)}{e^{nv}},$$

when doing $\frac{dg(v)}{dv} = 0$, we obtain that $v = \frac{m}{n}$ so $g(v)$ presents a critical point and reaches a maximum value in

$$g\left(\frac{m}{n}\right) = \frac{\left(\frac{m}{n}\right)^m}{e^m} = M_v.$$

The graph is a curve with a maximum end and asymptotic with the X axis, as shown in Figure 1.

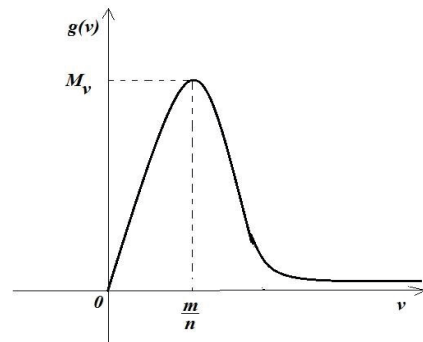


Figure 1. Graph of $g(v) = v^m e^{-nv}$.

For the case of the function $f(u)$, it is done in a similar way. It is observed that $f(0) = 0, f(\infty) = 0$, as $u > 0$ and $e^{qu} > 0$ we have that $f(u)$ is positive. Differentiating the function f ,

$$\frac{df(u)}{du} = \frac{u^{q-1}(p-qu)}{e^{qu}},$$

when doing $\frac{df(u)}{du} = 0$, we obtain that $u = \frac{p}{q}$, so $f(u)$ presents a critical point and reaches a maximum value in

$$f\left(\frac{p}{q}\right) = \frac{\left(\frac{p}{q}\right)^p}{e^p} = M_u.$$

The graph is a curve with a maximum end and asymptotic with the horizontal axis, as shown in Figure 2.

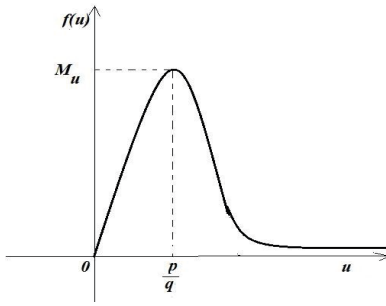


Figure 2. Graph of $f(u) = u^p e^{qu}$.

Therefore, according to the previous analysis, it is concluded that equation (12) does not admit solutions $u > 0, v > 0$ for some constant $c_1 > M_u M_v$, and for $c_1 = M_u M_v$ has the solution $u = \frac{p}{q}, v = \frac{m}{n}$. So, it would only be necessary to take the case when $c_1 = \delta M_v$, where δ is not positive number and $\delta < M_u$. It can be seen that the equation $\delta = \frac{p}{e^{qu}}$ admit a solution $u = u_R < \frac{p}{q}$ or other solution $u = u_r > \frac{p}{q}$. Therefore,

$$g(v) = \frac{q}{v} \frac{m}{e^{-nv}} = \left(\frac{\delta}{u^p e^{-qu}} \right) \frac{M_v}{v}$$

It has no solution and if u is less than u_r , or greater than u_R . If $u = u_r$ or $u = u_R$ only the solution results $v = \frac{m}{n}$ while for each u between u_r and u_R it has two solutions $v_1(u)$ and $v_2(u)$. In this case, the smallest solution $v_1(u)$ is always less than and the solution $v_2(u)$ is always greater than $\frac{m}{n}$. When u approaches u_r or u_R , then both $v_1(u)$ as $v_2(u)$ they have to $\frac{m}{n}$, (Blat, 1984; Logan, 1987).

Therefore, the curves defined by (13) are closed for positive u and v , and take the form as shown in Figure 3.

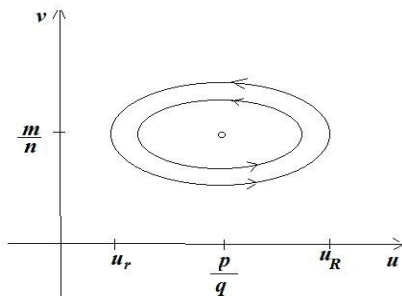


Figure 3. Orbit of (12) for positive u, v

Lemma 2. Let $(u(t), v(t))$ be any coexistence state of (12), with period $T > 0$, then their average values are

$$\bar{u} = \frac{1}{T} \int_0^T u(t) dt, \quad \bar{v} = \frac{1}{T} \int_0^T v(t) dt,$$

that is to say

$$\bar{u} = \frac{p}{q}, \quad \bar{v} = \frac{m}{n}$$

Demonstration. This lemma specifies that the average values of $u(t)$ and $v(t)$ are the equilibrium values. From equation (12), we have

$$\frac{du(t)}{dt} = mu(t) - nu(t)v(t)$$

$$\frac{1}{u} \frac{du(t)}{dt} = m - nv(t)$$

so that, integrating in both members

$$\frac{1}{T} \int_0^T \frac{u^F(t)}{u(t)} dt = \frac{1}{T} \int_0^T (m - nv(t)) dt,$$

you have,

$$\frac{1}{T} (Lnu(T) - Lnu(0)) = \frac{1}{T} \int_0^T m dt - \frac{n}{T} \int_0^T v(t) dt$$

but since $u(T) = u(0)$, it turns out

$$0 = m - n \int_0^T v(t) dt$$

that is to say,

$$\frac{1}{T} \int_0^T v(t) dt = \frac{m}{n}$$

Therefore, $\bar{v} = \frac{m}{n}$.

Similarly, from equation (12), we have

$$\frac{dv(t)}{dt} = -pv(t) + qu(t)v(t),$$

$$\frac{1}{v} \frac{dv(t)}{dt} = -p + qu(t),$$

so that, integrating in both members

$$\frac{1}{T} \int_0^T \frac{v^F(t)}{v(t)} dt = \frac{1}{T} \int_0^T (-p + qu(t)) dt$$

one has

$$\frac{1}{T} (Lnv(T) - Lnv(0)) = \frac{1}{T} \int_0^T p dt - \frac{q}{T} \int_0^T u(t) dt,$$

but since $v(T) = v(0)$, it turns out

$$0 = -p - q \int_0^T u(t) dt,$$

that is to say,

$$\int_0^T u(t) dt = \frac{p}{q}$$

Therefore, $\bar{u} = \frac{p}{q}$

This analysis indicates that even when it is not possible to obtain $(u(t), v(t))$ explicitly, it is possible to calculate (\bar{u}, \bar{v}) , resulting in that, regardless of the chosen coexistence state, we always have,

$$(\bar{u}, \bar{v}) = \left(\frac{p}{q}, \frac{m}{n} \right) \quad (14)$$

CONCLUSIONS

We can observe that in model (12), it has not been foreseen how the intensity of fishing of the species can affect their growth. To execute it, the following hypothesis is proposed: “the fishing intensity of species u is directly proportional to the total population at time t and negatively affects its growth”, this occurs in a similar way for the species v .

Then the new corrected model is written

$$\begin{aligned} \frac{du}{dt} &= u(t)(m - nv(t)) - \varepsilon u(t) \\ \frac{dv}{dt} &= v(t)(-p + qu(t)) - \varepsilon v(t) \end{aligned} \quad (15)$$

where $\varepsilon > 0$ indicates the intensity of fishing. The system given in (15) is the same as the system (12), provided that $m < \varepsilon$, i.e.

$$\begin{aligned} \frac{du}{dt} &= u(t)(m - \varepsilon - nv(t)) \\ \frac{dv}{dt} &= v(t)(-p - \varepsilon + qu(t)) \end{aligned}$$

However, it must be noted that fishing reduces the population of edible fish at a rate $\varepsilon u(t)$, in the same way as selacians, at a rate $\varepsilon v(t)$. It can be deduced that the constant ε indicates the intensity of fishing; that is, the number of fishing boats in operation and the number of nets in water. Therefore, the average values of the corresponding solutions are,

$$\bar{u} = \frac{p + \varepsilon}{q}, \bar{v} = \frac{m - \varepsilon}{n}$$

Finally, the decrease in fishing intensity; That is, decreasing ε causes a decrease in prey u and an increase in predators v , this result corresponds to the Volterra principle (Volterra, 1931).

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