v. 3, n. 35, 2023

Journal of
Engineering
Research

A MATHEMATICAL ANALYSIS OF INCOMPRESSIBLE TURBULENT FLOW: EXPLORING SMAGORINSKY SUB-GRID MODEL WITH ASYMPTOTIC BEHAVIOR

Rômulo Damasclin Chaves dos Santos
Postdoctoral Researcher, Department of Physics,Technological Institute of Aeronautics, São Paulo, Brazil

Alice Rosa da Silva
Prof. Associate of the Department of Civil Engineering at the Federal University of Uberlândia, UFU

Abstract: This study deepens the Smagorinsky model, known for effectively simulating large turbulent eddies in incompressible flows, a model grounded in both mathematical and physical rigor. Within this context, we investigate the Large Scale Simulation (LES) model in the context of the Smagorinsky model, with a meticulous mathematical analysis, we unveil subtle prin- ciples that guide the ideal construction of sub-mesh representations. The mathematical analysis employed here invites further exploration, especially in relation to the persistent conundrum of regularity in the Navier-Stokes equations.
Keywords: Smagorinsky model, Asymptotic behavior, Turbulent flow.

INTRODUCTION

THE TURBULENT FLOW

Turbulent patterns manifest in both natural phenomena and human activities, like the current of a rivers or the plumes rising from chimneys. Analyzing the dynamics motion holds significance in fields like aeronautics, meteorology, and engineering. The parameter known as the Reynolds number

$$Re = \frac{UL}{\nu} = \frac{\rho UL}{\mu} \tag{1}$$

(with characteristic velocity *U*, characteristic length *L*, kinematic viscosity ν, density ρ and dynamic viscosity μ) is a measure for turbulence of a flow. As demonstred by Reynolds' experiment with piper-flow, a fluid motion featuring a Reynolds number exceeding 4×10^3 displays turbulence, see more in [1].

NAVIER-STOKES EQUATIONS

Within the scope of this study, the Navier-Stokes equations (NSE) assume a central role as they provide a comprehensive description of fluid motion. Specifically, for fluids that are both incompressible and homogeneous, these equations manifest as follows

$$\partial_t u_j + u_i \partial_i u_j = 2\nu \partial_i S_{ij} - \frac{1}{\rho}\partial_j P + f_j \tag{2}$$

$$\text{in} \quad \Omega \times (0,T]\,, j = 1, 2.3$$

$$\partial_t u_j = 0 \text{ in } \Omega \times [0,T]\,, \tag{3}$$

where $\mathbf{u}\,(x,\,t) = (u_1,\,u_2,\,u_3)\,(x,\,t)$ is the velocity field depending on the position in space and time, ν the kinematic velocity,

$$S_{ij} = S_{ij}(\mathbf{u}) := \frac{1}{2}(\partial_i u_j + \partial_j u_i)\;, \tag{4}$$

the rate of strain tensor, representing friction between particles (see, [2]), and $f = (f_1, f_2, f_3)$ are forces per unit mass acting on the fluid, ρ is the density of the fluid, *P* is the pressure, (0, *T*] and [0, *T*] are the time intervals and Ω ⊆ R3 the domain. The momentum (Eq. (2)) is based on the consevation of momentum and continuity (Eq. (3)) on the conservation of mass.

CHALLENGES IN SIMULATION TURBULENT FLUID FLOW

The premises underpinning the derivation of the NSE are solid; however, we are engaging with a model nonetheless. One challenge arises from the interdependence of velocity and pressure, while another stems from the nonlinearity of the convective term $u_i \partial_i u_j$, in Eq. (2).

While the focus of this study does not revolve around conducting or show- casing computational simulations, tackling the numerical solution of the NSE for turbulent flows remains intricate due to the sheer volume of information encapsulated within the velocity field. The equations can be addressed through direct numerical simulations (DNS); however, the computational expenditure demonstrates a swift escalation, following a polynomial pattern relative to the Reynolds number. For example, a DNS of a turbulent flow at $Re = 10^6$ would require $Re^3 = 10^{18}$ uniformly distributed grid points in space-time [2].

Hence, computations involving exceedingly high Reynolds numbers re-main impractical in the foreseeable future, despite the progression of Moore's Law. An alternative to the direct numerical simulation (DNS) of non- averaged quantities involves a shift towards mean values, adopting a statistical methodology. This shift is exemplified by large-eddy simulation (LES), which can be effectively executed through the implementation of the Smagorinsky model. Unlike DNS, LES presents a more cost-effective approach, mitigating the constraints of DNS by explicitly calculating the dynamics of larger-scale motions while approximating the impact of smaller scales using simplified models, more details at [1] and [2].

FUNDAMENTALS OF LES

INTRODUCTION TO LARGE-EDDY SIMLATION

Within the realm of large-eddy simulations (LES), prominent macroscopic movements find direct representation, whereas diminutive-scale motions are subject to modeling. Pope [1], mentions four conceptual steps:

1. The velocity **u** is split between a filtered component **u** and a residual (subgrid-scale) component $\mathbf{u}' = \mathbf{u} - \bar{\mathbf{u}}$. Ths former represents the motion of large eddies.

2. To ascertain the progression of the filtered velocity field, one derives the filtered Navier-Stokes equations from the original Navier-Stokes equations (NSE). These filtered equations mirror the structure of the unfiltered Navier-Stokes equations, with the inclusion of a residual stress tensor that emerges from the unresolved motions.

3. Modeling of the residual stress tensor becomes necessary to attain equation closure.

4. Subsequently, the filtered equations are numerically solved to determine the filtered velocity.

The filtering operation is characterized as

$$\overline{\mathbf{u}}\left(x,t\right) := \int G_{\Delta}\left(\mathbf{r},\mathbf{x}\right)\mathbf{u}\left(\mathbf{x}-\mathbf{r},t\right)d\mathbf{r} \tag{5}$$

involving integration across the flow domain and the filter function G_{Δ} (frequently contingent on the filter width) that adheres to the normalization condition

$$\int G_{\Delta}\left(\mathbf{r},\mathbf{x}\right)d\mathbf{r} = 1\,, \tag{6}$$

according to [1]. Unless explicitly stated otherwise, an overline atop a variable indicates its filtered value.

A filter is called uniform if G_{Δ} does not depend on **x**, and isotropic if G_{Δ} depends on **r** only through $r = [\mathbf{r}]$. Evidently, the filtering process maintains constants and adheres to linearity. Moreover, filtering demonstrates commutativity with both temporal differentiation and the computation of means, [1]. Nonetheless, only specific filters exhibit commutativity when subjected to differentiation in relation to x_j, (see more at [2]).

An often encountered isotropic filter comes in the form of a Gaussian

$$G_{\Delta}\left(\mathbf{r}\right) = \left(\frac{6}{\pi\Delta^2}\right)^{\frac{1}{2}}\exp\left(-\frac{6\left[\mathbf{r}\right]^2}{\Delta^2}\right), \tag{7}$$

according to [3] and [1].

Journal of Engineering Research ISSN 2764-1317 DOI 10.22533/at.ed.3173352319109

There are numerous filter functions with varying properties. We solely examine filters that commute with differentiation. Filtering the NSE Eq. (2) and Eq. (3), yields:

$$\partial_t \overline{u}_j + \overline{u}_i \partial_i \overline{u}_j = \partial_i \left(2\nu \overline{S}_{ij} - \tau^{\mathbf{r}}_{ij}\right) - \partial_j \overline{p} + \overline{f}_j \tag{8}$$

$$\text{in} \quad \Omega \times (0, T], j = 1, 2.3,$$

$$\partial_t \overline{u}_j = 0 \text{ in } \Omega \times [0, T] \ . \tag{9}$$

While deriving the filtered continuity equation is straightforward, obtaining the filtered momentum equation necessitates some effort. The anisotropic residual-stress tensor τ^R_{ij}, is obtained by calculating the derivation of the filtered equation for momentum, performed by adapting what was done in the work of [1], we get, given that differentiation and filtering commute, and linearity is applicable in

$$\partial_t \overline{u}_j + \overline{u_i \partial_i u_j} = 2\nu \partial_i \overline{S}_{ij} - \frac{1}{\rho} \partial_j \overline{P} + \overline{f}_j \tag{10}$$

$$\text{in} \ \ \Omega \times (0, T], j = 1, 2.3.$$

We establish the residual stress tensor, the anisotropic residual-stress tensor, and the adjusted filtered pressure:

$$\tau^R_{ij} := \overline{u_i u_j} - \overline{u}_i \overline{u}_j \ , \tag{11}$$

$$\tau^r_{ij} := \tau^R_{ij} - \frac{1}{3} \tau^R_{kk} \delta_{ij} \ , \tag{12}$$

$$\overline{p} := \frac{1}{\rho} \overline{P} + \frac{1}{3} \tau^R_{kk} \ . \tag{13}$$

Using the continuity Eqs. (3) and (9), we get:

$$\partial_i (u_i u_j) = (\partial_i u_i) u_j + u_i \partial_i u_j = u_i \partial_i u_j \ , \tag{14}$$

$$\partial_i (\overline{u}_i \overline{u}_j) = (\partial_i \overline{u}_i) \overline{u}_j + \overline{u}_i \partial_i \overline{u}_j = \overline{u}_i \partial_i \overline{u}_j \ . \tag{15}$$

Employing the preceding two equations along with the definition of the residual-stress tensor τ^R_{ij}, we obtain:

$$\begin{aligned} \partial_t \overline{u}_j \ + \ \overline{u_i \partial_i u_j} \ &= \ \partial_t \overline{u}_j \ + \ \overline{\partial_i (u_i u_j)} \\ &= \ \partial_t \overline{u}_j \ + \ \partial_i (\overline{u_i u_j}) \\ &= \ \partial_t \overline{u}_j \ + \ \partial_i (\overline{u}_i \overline{u}_j) + \partial_i \tau^R_{ij} \\ &= \ \partial_t \overline{u}_j \ + \ \overline{u}_i \partial_i \overline{u}_j + \partial_i \tau^R_{ij} \ . \end{aligned} \tag{16}$$

Now, with the incorporation of all three aforementioned definitions, we obtain:

$$\begin{aligned} -\partial_t \tau^R_{ij} - \frac{1}{\rho} \partial_j \overline{P} \ &= \ -\partial_t \tau^r_{ij} \ - \ \partial_i \frac{1}{3} \tau^R_{kk} \delta_{ij} - \frac{1}{\rho} \partial_j \overline{P} \\ &= \ -\partial_t \tau^r_{ij} \ - \ \partial_j \frac{1}{3} \tau^R_{kk} - \frac{1}{\rho} \partial_j \overline{P} \\ &= \ -\partial_t \tau^r_{ij} \ - \ \partial_j \left(\frac{1}{\rho} \overline{P} + \frac{1}{3} \tau^R_{kk} \right) \\ &= \ -\partial_t \tau^r_{ij} \ - \ \partial_j \overline{p} \ . \end{aligned} \tag{17}$$

Henceforth, we find ourselves at the juncture where the distilled equation of momentum unveils its form

$$\partial_t \overline{u}_j + \overline{u}_i \partial_i \overline{u}_j = \partial_i \left(2\nu \overline{S}_{ij} - \tau^{\mathbf{r}}_{ij}\right) - \partial_j \overline{p} + \overline{f}_j, \ \ j = 1, 2, 3. \tag{18}$$

THE SMAGORINKY MODEL

To conclude the equations and consequently determine the filtered velocity field **u** (**x**, *t*) along with the adjusted filtered pressure *p* (**x**, *t*), it is imperative to formulate the anisotropic residual stress tensor τ^R_{ij}, (**x**, *t*). Among the available models, the Smagorinsky model stands out due to its simplicity and its demonstrated capability to yield satisfactory performance (more details at [1]).

In the Smagorinsky model, the anisotropic residual-stress tensor τ^R_{ij}, (**x**, *t*) correlates with the filtered strain rate

$$\overline{S}_{ij}=\overline{S}_{ij}\left(\mathbf{u}\right):=S_{ij}\left(\overline{\mathbf{u}}\right):=\frac{1}{2}\left(\partial_{j}\overline{u}_{i}+\partial_{i}\overline{u}_{j}\right)\,, \tag{19}$$

as

$$\tau_{ij}^{r}\left(\mathbf{x},t\right)=-2\nu_{r}\overline{S}_{ij}\,. \tag{20}$$

This constitutes the mathematical embodiment of the Boussinesq hypothesis, which postulates that turbulent fluctuations exhibit dissipative behavior on average. The mathematical arrangement bears resemblance to that of molecular diffusion, (see more details at [4]). Substituting Eq. (20) into Eq. (8), the filtered momentum equation can be written as

$$\partial_{t}\overline{u}_{j}+\overline{u}_{i}\partial_{i}\overline{u}_{j}=2\partial_{i}\left(\left(\nu+\nu_{r}\right)\overline{S}_{ij}\right)-\partial_{j}\overline{p}+\overline{f}_{j},\;\;j=1,2,3. \tag{21}$$

The residual subgrid-scale eddy-viscosity ν_r acts as an artificial viscosity [4] and represents the eddy-viscosity of the residual motions. It is modeled as

$$\nu_{r}=\ell_{S}^{2}\sqrt{\left(2\overline{S}_{lk}\overline{S}_{lk}\right)}=\left(C_{S}\Delta\right)^{2}\sqrt{\left(2\overline{S}_{lk}\overline{S}_{lk}\right)}\,. \tag{22}$$

In this context, we encounter the Smagorinsky length scale $\ell_S = C_S\Delta$, the Smagorinsky coefficient C_S an the filter width Δ. Lastly, we can express the filtered momentum equation as follows

$$\partial_{t}\overline{u}_{j}+\overline{u}_{i}\partial_{i}\overline{u}_{j}=2\partial_{i}\left(\left(\nu+\ell_{S}^{2}\sqrt{\left(2\overline{S}_{lk}\overline{S}_{lk}\right)}\right)\overline{S}_{ij}\right)-\partial_{j}\overline{p}+\overline{f}_{j},\;\;j=1,2,3. \tag{23}$$

The model for the eddy-viscosity, Eq. (22), is called Smagorinsky model. The Smagorinsky model comes with certain limitations. They are summarized as follows in:

1. The Smagorinsky model constant C_S is an a priori input. The single contant is incapable to represent correctly various turbulent flows;

2. The eddy-viscosity does not vanish for a laminar flow;

3. The backscatter of energy is prevented completely since

$$\left(C_{S}\Delta\right)^{2}\sqrt{\left(2\overline{S}_{lk}\overline{S}_{lk}\right)}\geq 0\,.$$

4. The Smagorinsky model typically introduces excessive diffusion into the flow.

DERIVATION AND SPACETIME DYNAMICS IN THE SMAGORINSKY MODEL

DERIVATION OF THE SMAGORINSKY MODEL

Based on [5], the Smagorinsky model's derivation can occur through var- ious approaches, such as heuristic techniques. For instance, one method involves equating the production and dissipation of subgrid-scale turbulent kinetic energy. Alternatively, the model can be derived using turbulence theories. The formulation (derivation) presented here has been adapted from [4]. Both heuristic approaches and turbulence theories are given consideration. Kolmogorov [6], (cited in [5]) attained the generalized expression for the energy spectrum function

$$E\left(k\right)=K\left\langle \varepsilon\right\rangle ^{\frac{2}{3}}k^{-\frac{5}{3}}\,,\quad K\approx 1.4\,, \tag{24}$$

where

$$\varepsilon\left(t\right):=\left|\Omega\right|^{-1}\int_{\Omega}\nu\left|\nabla\mathbf{u}\right|^{2}\left(\mathbf{x},t\right)d\mathbf{x}\,, \tag{25}$$

is the energy dissipation rate, K a constant and angular brackets indicate a statistical mean. In other words, this signifies an energy cascade from the larger scales to the smaller scales. This has been famously summarized in a poem by mathematician and meteorologist L. F. Richardson, as quoted in [5]: *"Big whirls have little whirls what feed on their velocity, little whirls have smaller whirls, and so on to viscosity."*

Dimensional investigation reveals that

$$\partial_i \tau_{ij}^r \left[\frac{m}{s^2}\right] \Longleftrightarrow \tau_{ij}^r = -2\nu_r \overline{S}_i \left[\frac{m^2}{s^2}\right] \Longleftrightarrow \nu_r \left[\frac{m^2}{s}\right] .$$

Therefore, it is assumed that the residual subgrid-scale eddy viscosity ν_r

is proportional to $\varepsilon^{\sim\frac{1}{3}}\Delta^{\frac{4}{3}}$ the kinetic energy transfer rate (see more in [4]). Using Eq. (24) and the so-called two-fluid model or eddy-damped quasinormal Markovian model, we get

$$\langle \nu_r \rangle = \frac{A}{\pi^{\frac{4}{3}} K} \langle \tilde{\varepsilon} \rangle^{\frac{1}{3}} \Delta^{\frac{4}{3}} , \tag{26}$$

where A is a constant, wich is 0.438 according to the two-fluid model and 0.441 according to the eddy-damped quasinormal Markovian model, both cited in [4].

Furthermore, in the isotropic homogeneous case,

$$\left\langle 2\overline{S}_{lk}\overline{S}_{lk} \right\rangle = \int_0^{\frac{\pi}{\Delta}} 2k^2 E(k)\, dk \tag{27}$$

is true, according [4]. Substituting Eq. (24) into Eq. (27), yields

$$\begin{aligned} \left\langle 2\overline{S}_{lk}\overline{S}_{lk} \right\rangle &= \int_0^{\frac{\pi}{\Delta}} 2k^2 K (\varepsilon)^{\frac{2}{3}} k^{-\frac{5}{3}} dk \\ &= 2K \langle \varepsilon \rangle^{\frac{2}{3}} \int_0^{\frac{\pi}{\Delta}} k^{\frac{1}{3}} dk \\ &= \frac{3}{2} K \langle \varepsilon \rangle^{\frac{2}{3}} \pi^{\frac{4}{3}} \Delta^{-\frac{4}{3}} . \end{aligned} \tag{28}$$

This is equivalent to

$$\begin{aligned} \left(\frac{3K}{2}\right)^{\frac{3}{2}} \langle \varepsilon \rangle \pi^2 \Delta^{-2} &= \left\langle 2\overline{S}_{lk}\overline{S}_{lk} \right\rangle^{\frac{3}{2}} \Longleftrightarrow \\ \langle \varepsilon \rangle &= \pi^{-2} \left(\frac{3K}{2}\right)^{-\frac{3}{2}} \Delta^2 \left\langle 2\overline{S}_{lk}\overline{S}_{lk} \right\rangle^{\frac{3}{2}} . \end{aligned} \tag{29}$$

This formulation means to say that, the local equilibrium hypothesis states that the flow is in a constant spectral equilibrium. As a result, energy does not accumulate at any frequency, and the shape of the energy spectrum remains unchanged over time. This implies that the production, dissipation, and energy flux through the cutoff are all equal

$$\langle \varepsilon_I \rangle = \langle \tilde{\varepsilon} \rangle = \langle \varepsilon \rangle . \tag{30}$$

Using the last equation, we can insert Eq. (29) into Eq. (26) and get

$$\begin{aligned} \langle \nu_r \rangle &= \frac{A}{\pi^{\frac{4}{3}} K} \langle \tilde{\varepsilon} \rangle^{\frac{1}{3}} \Delta^{\frac{4}{3}} \\ &= \frac{A}{\pi^{\frac{4}{3}} K} \langle \varepsilon \rangle^{\frac{1}{3}} \Delta^{\frac{4}{3}} \\ &= \frac{A}{K} \pi^{-\frac{4}{3}} \left(\pi^{-2} \left(\frac{3K}{2}\right)^{-\frac{3}{2}} \Delta^2 \left\langle 2\overline{S}_{lk}\overline{S}_{lk} \right\rangle^{\frac{3}{2}} \right) \Delta^{\frac{4}{3}} \\ &= \frac{A}{K} \pi^{-2} \left(\frac{3K}{2}\right)^{-\frac{3}{2}} \Delta^2 \left\langle 2\overline{S}_{lk}\overline{S}_{lk} \right\rangle^{\frac{1}{2}} . \end{aligned}$$

Defining the Smagorinsky coeffcient as

$$C_S := A^{1/2} \left(\pi^{-1} K^{-\frac{1}{2}}\right) \left(\frac{3K}{2}\right)^{-\frac{1}{4}} \approx 0.148 , \tag{31}$$

we can write,

$$\langle \nu_r \rangle = (C_S \Delta)^2 \left\langle 2\overline{S}_{lk}\overline{S}_{lk} \right\rangle^{\frac{1}{2}} . \tag{32}$$

The Smagorinsky model is then expressed as

$$\nu_r(\mathbf{x}, t) = (C_S \Delta)^2 \left(2\overline{S}_{ij}(\mathbf{x}, t)\, \overline{S}_{ij}(\mathbf{x}, t) \right)^{\frac{1}{2}} . \tag{33}$$

Sagaut acknowledges that this proposition lacks specific justification, other than its observed average validity as demonstrated in

Eq. (32) (cited in [4]). The model's validation stems from its performance. Pope at [1], at the very least, deems it satisfactory, although he highlights subpar outcomes in spe- cific scenarios. It is important to note that the Smagorinsky coefficient C_S was evaluating in Eq. (31), but is adjusted to improve results. Through dif- ferent analysis, the values 0.17 ([1]), 0.18 ([6]) and 0.15 were obtained as well. Opting for C_S to vary with both space and time, rather than remaining a constant, could potentially yield even more favorable results. This will be addressed in the following section.

DYNAMIC SMAGORINSKY MODEL

To consider the Smagorinsky coefficient as a function of space and time (based on what was done in [7] and [8]), we propose an idea that is presented here. Using a so-called test filter $\hat{\Delta} > \Delta$, the filtered NSE (Eq. (2) and (3)) are filtered again:

$$\partial_t \hat{\bar{u}}_j + \partial_i \left(\widehat{\overline{u}_i \overline{u}_j} \right) = \partial_i \left(2\nu \hat{\bar{S}}_{ij} - \hat{\tau}^{\mathrm{r}}_{ij} \right) - \partial_j \hat{\bar{p}} + \hat{\bar{f}}_j, \quad j = 1, 2.3, \quad \text{in } \Omega \times (0, T],$$

$$\partial_t \hat{\bar{u}}_j = 0, \qquad \text{in } \Omega \times [0, T],$$

with hats indicating the second filtering. Similar to the residual-stress tensor τ^R_{ij}, is defined as

$$\tau^R_{ij} := \overline{u_i u_j} - \bar{u}_i \bar{u}_j \,,$$

the subtest-scale stress-tensor ${}_s\mathrm{T}_{ij}$ is defined as

$${}_s\mathbb{T}_{ij} = \widehat{\overline{u_i u_j}} - \hat{\bar{u}}_i \hat{\bar{u}}_j \,, \tag{34}$$

such that

$$\mathbb{G}_{ij} = {}_s\mathbb{T}_{ij} - \widehat{\tau^R_{ij}} = \widehat{\overline{u}_i \overline{u}_j} - \hat{\bar{u}}_i \hat{\bar{u}}_j \tag{35}$$

G_{ij} is called the Germano identidy, [7]. We denote the Smagorinsky parameter with $\widetilde{C}_S$ (instead of C_S). It is formulated without an exponent in the assumption, unlike the Smagorinsky coefficient in the Smagorinsky model (Eq. (22)).

The approach taken is (c. f. Eqs. (20) and (22)):

$$\tau^r_{ij} := \tau^R_{ij} - \frac{1}{3}\tau^R_{kk}\delta_{ij} = -2\widetilde{C}_S\left(\mathbf{x}, t\right)\Delta^2 \sqrt{\left(2\bar{S}_{lk}\bar{S}_{lk}\right)}\bar{S}_{ij}\,,$$

$${}_s\mathbb{T}_{ij} - \frac{1}{3}\,{}_s\mathbb{T}_{kk}\delta_{ij} = -2\widetilde{C}_S\left(\mathbf{x}, t\right)\Delta^2 \sqrt{\left(2\bar{S}_{lk}\bar{S}_{lk}\right)}\bar{S}_{ij}\,.$$

So, to get an equation for $C\tilde{~}S$, it is necessary to approximate

$$\tau^r_{ij} \approx -2\widetilde{C}_S\left(\mathbf{x}, t\right)\Delta^2 \left[\widehat{\sqrt{\left(2\bar{S}_{lk}\bar{S}_{lk}\right)}\bar{S}_{ij}} \right]. \tag{36}$$

Equality is achieved when the Smagorinsky parameter is not dependent on **x**, [6]. For $i, j \in \{1, 2, 3\}$, the system

$$\begin{aligned} \mathbb{G}_{ij} - \frac{1}{3}\mathbb{G}_{kk}\delta_{ij} &= {}_s\mathbb{T}_{ij} - \frac{1}{3}\,{}_s\mathbb{T}_{kk}\delta_{ij} - \widehat{\tau^r_{ij}} \\ &\approx -2\widetilde{C}_S\left(\hat{\Delta}^2\sqrt{\left(2\hat{\bar{S}}_{lk}\hat{\bar{S}}_{lk}\right)}\hat{\bar{S}}_{lk} - \Delta^2\widehat{\sqrt{\left(2\bar{S}_{lk}\bar{S}_{lk}\right)}\bar{S}_{ij}}\right) \\ &= -2\widetilde{C}_S\mathbb{M}_{ij}\,, \end{aligned}$$

with

$$\mathbb{M}_{ij} := \hat{\Delta}^2\sqrt{\left(2\hat{\bar{S}}_{lk}\hat{\bar{S}}_{lk}\right)}\hat{\bar{S}}_{lk} - \Delta^2\widehat{\sqrt{\left(2\bar{S}_{lk}\bar{S}_{lk}\right)}\bar{S}_{ij}}\,,$$

is an overdetermined system which C_S cannot satisfy exactly [9].

D. K. Lilly [9], therefore, propose a least-square method, minimizing the square of the error

$$Q = \left(\mathbb{G}_{ij} - \frac{1}{3}\mathbb{G}_{kk}\delta_{ij} + 2\widetilde{C}_S\mathbb{M}_{ij}\right)^2,$$

meaning the sum over all i, j. Since

$$\frac{\partial Q}{\partial \widetilde{C}_S} = 2\left(\mathbb{G}_{ij} - \frac{1}{3}\mathbb{G}_{kk}\delta_{ij} + 2\widetilde{C}_S\mathbb{M}_{ij}\right)2\mathbb{M}_{ij}$$

$$= 4\mathbb{G}_{ij}\mathbb{M}_{ij} - \frac{4}{3}\mathbb{G}_{kk}\delta_{ij}\mathbb{M}_{ij} + 8\widetilde{C}_S\mathbb{M}_{ij}\mathbb{M}_{ij}$$

$$= 4\mathbb{G}_{ij}\mathbb{M}_{ij} - \frac{4}{3}\mathbb{G}_{kk}\delta_{ij}\mathbb{M}_{ll} + 8\widetilde{C}_S\mathbb{M}_{ij}\mathbb{M}_{ij}$$

$$= 4\mathbb{G}_{ij}\mathbb{M}_{ij} + 8\widetilde{C}_S\mathbb{M}_{ij}\mathbb{M}_{ij} ,$$

with

$$\frac{\partial^2 Q}{\partial \widetilde{C}_S^2} = 8\mathbb{M}_{ij}\mathbb{M}_{ij} > 0 ,$$

note that $\mathbb{M}_{ll} = 0$ because $\hat{\bar{S}}_{ll} = 0$ and $\bar{S}_{ll} = 0$, see Eq. (9).The Smagorinsky parameter minimizes the error when we establish

$$\widetilde{C}_S(\mathbf{x}, t) = -\frac{\mathbb{G}_{ij}\mathbb{M}_{ij}}{2\mathbb{M}_{ij}\mathbb{M}_{ij}}(\mathbf{x}, t) \ . \qquad (37)$$

MATHEMATICAL ANALYSIS OF THE SMAGORINSKY MODEL

In order to conduct a mathematical analysis of the Smagorinsky model, it is essential that the problem is clearly and precisely defined.

VECTOR SPACES

The Lebesgue space $L^p(\Omega)$, $p \in [1, \infty]$, is the Banach space of measurable functions $\mathbf{v}$ on Ω which satisfy

$$\|\mathbf{v}\|_{L^{m,p}(\Omega)} := \left(\int_\Omega |\mathbf{v}(\mathbf{x})|\, d\mathbf{x}\right)^{\frac{1}{p}} < \infty, \quad \text{if} \quad p \in [1, \infty) ,$$

$$\|\mathbf{v}\|_{L^{m,p}(\Omega)} := \text{ess sup} |\mathbf{v}(\mathbf{x})| < \infty , \quad \text{if} \quad p = \infty , \qquad (38)$$

For $p = 2$, the Labesgue space is also a Hilbert space with the scalar product

$$(\mathbf{x}, \mathbf{v}) = \int_\Omega \mathbf{v}(\mathbf{x}) \cdot \mathbf{w}(\mathbf{x}) d\mathbf{x} .$$

in the case of one-dimensional functions, the dot signifies straightforward multiplication; however, when dealing with vectors or matrices, it denotes the dot product for vectors or the Frobenius inner product for matrices. For two matrices $A = (a_{ij})1 \le i,j \le 3$ and $B = (b_{ij})1 \le i,j \le 3$, the Frobenius inner product is

$$A : B := a_{ij}b_{ij} .$$

We write $L^p(a, b; V)$ for the Lebesgue space of functions from the interval (a, b) to the Banach space V. The identical notation is employed for the corresponding Sobolev spaces.

The Sobolev space $W^{m,p}$ is the Banach space of functions for wich

$$\|\mathbf{v}\|_{W^{m,p}(\Omega)} := \left(\sum_{0 \le |\alpha| \le m} \|D^\alpha \mathbf{v}\|^p_{L^p(\Omega)}\right)^{\frac{1}{p}} < \infty , \quad \text{if} \quad p \in [1, \infty) ,$$

$$\|\mathbf{v}\|_{W^{m,p}(\Omega)} := \max_{0 \le |\alpha| \le m} \|D^\alpha \mathbf{v}\|^p_{L^p(\Omega)} < \infty , \quad \text{if} \quad p = \infty ,$$

remains valid, i.e., it can be defined as

$$W^{m.p}(\Omega) = \{\mathbf{v} \in L^p(\Omega) : D^\alpha \mathbf{v} \in L^p(\Omega)\ \forall |\alpha| \le m\} . \qquad (39)$$

be the divergence-free Sobolev space where functions vanish on the boundary $\Gamma = \partial\Omega$,

Let

$$W^{1.3}_{0,\text{div}}(\Omega) = \left\{\mathbf{v} \in W^{1,3}(\Omega) : \mathbf{v}|_\Gamma = 0, \nabla \cdot \mathbf{v} = 0 \,\text{in}\, \Omega\right\} , \qquad (40)$$

be the divergence-free Sobolev space where functions vanish on the boundary $\Gamma = \partial\Omega$,

$$H^1\left(0, T; L^2(\Omega)\right) := W^{1,2}\left(0, T; L^2(\Omega)\right)$$

a Sobolev space that is also a Hilbert space and

$$V := H^1\left(0,T;L^2\left(\Omega\right)\right)\cap L^3\left(0,T;W^{1.3}_{0,\mathrm{div}}\left(\Omega\right)\right) \tag{41}$$

a Banach space with the norm

$$\|\mathbf{v}\|_V = \|\nabla\mathbf{v}\|_{L^3(0,T;L^3(\Omega))} + \|\partial_t\mathbf{v}\|_{L^2(0,T;L^2(\Omega))}\,.$$

STRONG AND WEAK FORMULATION OF THE PROBLEM

Consider the NSE with the conditions

$$\begin{aligned}
&\partial_t\mathbf{u} + (\mathbf{u}\cdot\nabla)\,\mathbf{u} = \nu\nabla\cdot\nabla\mathbf{u} - \frac{1}{\rho}\nabla P + f \quad \text{in}\quad \Omega\times(0,T]\,,\\
&\nabla\cdot\mathbf{u} = 0 \quad \text{in}\quad \Omega\times[0,T]\,,\\
&\mathbf{u}\left(\mathbf{x},0\right) = \mathbf{u}_0\left(\mathbf{x}\right) \quad \text{in}\quad \Omega,\\
&\mathbf{u} = 0 \quad \text{on}\quad \Gamma\times[0,T]\,,\\
&\int_\Omega P\,d\mathbf{x} = 0 \quad \text{in}\quad (0,T]\,.
\end{aligned} \tag{42}$$

with Γ = ∂Ω. Note that

$$2\partial_i S_{ij} = \partial_i\left(\partial_i u_j + \partial u_i\right) = \partial_i\partial_i u_j + \partial_i\partial_j u_i = \partial_i\partial_i u_j + \partial_j\left(\partial_i u_i\right) = \partial_i\partial_i u_j = \nabla\cdot\nabla\mathbf{u}\,.$$

The first and second equations correspond to the momentum equation (Eq. (2)) and continuity equation (Eq. 3) from above. The initial flow field $\mathbf{u}_0$ ($\mathbf{x}$) is also divergence-free, i.e., $\nabla\cdot\mathbf{u}_0 = 0$ in Ω. The fourth equation is the no slip boundary condition. It relies on the supposition that the fluid does not permeate or slide along the wall. Without the last equation, the pressure *P* would only be determined up to a constant, [2].

Filtering Eqs. (42) and using a similar condition for the modified filtered pressure, we get

$$\begin{aligned}
&\partial_t\bar{\mathbf{u}} + (\bar{\mathbf{u}}\cdot\nabla)\,\bar{\mathbf{u}} = \nabla\cdot(\nu+\nu_r)\,\nabla\bar{\mathbf{u}} - \nabla\bar{p} + \bar{\mathbf{f}} \quad \text{in}\quad \Omega\times(0,T]\,,\\
&\nabla\cdot\bar{\mathbf{u}} = 0 \quad \text{in}\quad \Omega\times[0,T]\,,\\
&\bar{\mathbf{u}}\left(\mathbf{x},0\right) = \bar{\mathbf{u}}_0\left(\mathbf{x}\right) \quad \text{in}\quad \Omega,\\
&\bar{\mathbf{u}} = 0 \quad \text{on}\quad \Gamma\times[0,T]\,,\\
&\int_\Omega \bar{p}\,d\mathbf{x} = 0 \quad \text{in}\quad (0,T]\,.
\end{aligned} \tag{43}$$

By multiplying the first equation with **v** ∈ *V* and integrating over time and space, we achieve a weak formulation. Let $\bar{\mathbf{f}} \in L^2\,(0,\ T\ ;\ L^2\ (\Omega))$. Find $\bar{\mathbf{u}}$ ∈ V that satisfies $\bar{\mathbf{u}} = (0,\mathbf{x}) = \bar{\mathbf{u}}_0 \in W^{1.3}_{0,\mathrm{div}}$ and

$$\int_0^T \left(\partial_t\bar{\mathbf{u}} + (\bar{\mathbf{u}}\cdot\nabla)\,\bar{\mathbf{u}},\mathbf{v}\right) + \left((\nu+\nu_r)\,\nabla\bar{\mathbf{u}},\nabla\mathbf{v}\right)dt = \int_0^T \left(\bar{\mathbf{f}},\mathbf{v}\right)dt\,, \tag{44}$$

for all **v** ∈ *V*, with (·, ·) denoting the L^2 (Ω) scalar product. Let **n** be the outward unit surface normal to Γ = ∂Ω. Note that using integration by parts, we can derive

$$(\nabla\cdot\mathbf{w},\mathbf{v}) = \int_\Omega(\nabla\cdot\mathbf{w})\cdot\mathbf{v}d\mathbf{x} = \int_\Gamma(\mathbf{w}\cdot\mathbf{n})\cdot\mathbf{v}ds - \int_\Omega\mathbf{w}\cdot(\nabla\mathbf{v})\,d\mathbf{x} = -\int_\Omega\mathbf{w}\cdot(\nabla\mathbf{v})\,d\mathbf{x} = -\mathbf{w}\cdot(\nabla\mathbf{v})\,,$$

because **v** = 0 on Γ. In this case, we used $\mathbf{w} = (\nu + \nu_r)\,\nabla\bar{\mathbf{u}}$. The pressure term vanishes because

$$(\nabla\bar{p},\mathbf{v}) = \int_\Omega\nabla\bar{p}\cdot\mathbf{v}d\mathbf{x} = \int_\Gamma\bar{p}\,(\mathbf{v}\cdot\mathbf{n})\,ds - \int_\Omega\bar{p}\,(\nabla\cdot\mathbf{v})\,d\mathbf{x} = 0\,,$$

as $\nabla \cdot \mathbf{v} = 0$. Another similar variation of this following formulation. Find $(\mathbf{w}, q) : [0, T] \to \mathbf{X} \times Q$ satisfying $\mathbf{w}(\mathbf{x}, 0) = \bar{\mathbf{u}}_0(\mathbf{x})$ and

$$(\partial_t \mathbf{w}, \mathbf{v}) + a(\mathbf{u}, \mathbf{w}, \mathbf{v}) + b(\mathbf{u}, \mathbf{w}, \mathbf{v}) + (\lambda, \nabla \cdot \mathbf{w}) - (q, \nabla \cdot \mathbf{v}) = (\bar{\mathbf{f}}, \mathbf{v}) \,, \tag{45}$$

for all $(\mathbf{v}, \boldsymbol{\lambda}) \in \mathbf{X} \times Q$ with

$$a(\mathbf{u}, \mathbf{w}, \mathbf{v}) := \alpha(\nabla \cdot \mathbf{u}, \nabla \cdot \mathbf{v}) + \left(\left(2Re^{-1} + \widetilde{C}_S \Delta^2 \left(\sqrt{\overline{S}_{lk}\overline{S}_{lk}}\right)\right) \overline{S}(\mathbf{w}), \overline{S}(v)\right)$$

$$b(\mathbf{u}, \mathbf{w}, \mathbf{v}) := \frac{1}{2}(\mathbf{u}, \nabla \mathbf{w}, \mathbf{v}) - \frac{1}{2}(\mathbf{u}, \nabla \mathbf{v}, \mathbf{w}) \,.$$

with the Smagorinsky parameter $\widetilde{C}_S = \sqrt{2}C_S^2$,, according to [5].

ASYMPTOTIC BEHAVIOR AND REGULARITY

Let us first introduce some standard notations and function spaces which will be used in the following analysis. We denote

$$\mathcal{V} = \left\{\varphi \in \mathcal{D}(\Omega)^3, \nabla \cdot \varphi = 0\right\},$$

$$H = \text{the closure of } \mathcal{V} \text{ in } L^2(\Omega)^3,$$

$$V = \text{the closure of } \mathcal{V} \text{ in } W^{1,3}(\Omega)^3,$$

where $L^2(\Omega)^2$ is the space of functions which are square integrable over Ω with respect to the Lebesgue measure and $W^{1,3}(\Omega)$ is the L^3 Sobolev space. H is a Hilbert space with respect to the inner product. We will use the notation V' for the dual space of V, $\|\cdot\|_V$, for the induced norm and $\langle \cdot, \cdot \rangle$ for the duality product. For spaces of functions which depend on both time and space variables, we will frequently use the twofollowing spaces:

(a) $C([0, T]; X)$ the space of continuous functions $u : [0, T] \to X$, where X is a Banach space with the norm denoted by $|\cdot|_X$. (b) $L^p(0, T; X)$ the space of strongly measurable functions $u :]0, T[\to X$ with a finite norm

$$|u|^p_{L^p(0,T,X)} := \int_0^T |u(t)|^p_X \, dt < \infty.$$

In the case $p = \infty$ the norm is defined by

$$|u|^p_{L^\infty(0,T,X)} := \text{ess } \sup_{t \in]0,T[} |u(t)|_X \,.$$

Finally, we will denote by $|\cdot|_p$ the usual norm in $L^p(\Omega)$.

Lemma 1. *Let X be a Banach space and X_0, X_1 two reflexive, separable Banach spaces. If we assume that*

$$X_0 \hookrightarrow\hookrightarrow X \hookrightarrow X_1$$

the first embedding being compact, then we have the following embedding

$$\left\{v \in L^\alpha(0, T; X_0), \frac{dv}{dt} \in L^\beta(0, T; X_1)\right\} \hookrightarrow\hookrightarrow L^\alpha(0, T; X),$$

where $1 < \alpha, \beta < \infty$. The proof this lemma can be found in [10].

In this context, we consider the weak fomuation for the problem (42). Derived from multiplying the momentum equation by a test function and applying integration by parts, resulting in the issue that will be mentioned in the sequel as:

Problem (S). 5.1.

For $\mathbf{f} \in L^{\frac{3}{2}}(0, T; V')$ and $\mathbf{u}_0 \in H$ given, find $\mathbf{u}$ satisfying

$$\mathbf{u} \in C([0, T]; H) \cap L^3(0, T; V)$$

$$\textit{with } \frac{d\mathbf{u}}{dt} \in L^{\frac{3}{2}}(0, T; V') \tag{46}$$

$$\left\langle \frac{d\mathbf{u}}{dt}, \mathbf{v} \right\rangle + \sum_{i,j=1}^{3} \int_\Omega \mathbb{T}_{ij}(\mathbf{S}(\mathbf{u})) S_{ij}(\mathbf{v}) \, d\mathbf{x} + \int_\Omega (\mathbf{u} \cdot \nabla) \mathbf{u} \cdot \mathbf{v} d\mathbf{x} = \langle \mathbf{f}, \mathbf{v} \rangle, \forall \mathbf{v} \in V$$

and the initial condition

$$\mathbf{u}(0) = \mathbf{u}_0 .$$

Theorem 2. *Let* $\mathbf{u}_0 \in H$ *and* $\mathbf{f} \in L^{\frac{3}{2}}(0, T; V')$. *Then for any* $T > 0$ *the problem (S) has a unique weak solution on* $[0, T]$. *Moreover, if* $\mathbf{u}_0 \in V$ *then the unique weak solution is in* L^∞ $0, T$; $W^{1,3}(\Omega)^3$.

Proof. To prove the existence of a weak solution we used a classical Galerkin method. We omit it, since it is straightforward from the proof done in [10] based on the compactness method. A complete demonstrationcan be found in [11]. We only present here the proof of uniqueness.

Let us suppose that there exist two weak solutions **u** and **v** to problem (S), with the same initial condition $\mathbf{u}_0 \in H$ and let $\mathbf{w} = \mathbf{u} - \mathbf{v}$. After subtracting the weak formulation for **v** from the one for **u** and talking **w** as test function in the resulting equation, we get

$$\frac{1}{2}\frac{d}{dt}|\mathbf{w}|_2^2 + \sum_{i,j=1}^{3}\int_\Omega \left[\mathbb{T}_{ij}(\mathbf{S}(\mathbf{u})) - \mathbb{T}_{ij}(\mathbf{S}(\mathbf{v}))\right] S_{ij}(\mathbf{w})\, d\mathbf{x} = -\int_\Omega (\mathbf{w}\cdot\nabla)\,\mathbf{u}\mathbf{w} d\mathbf{x} . \tag{47}$$

Moreover, from the definition of the tensor T, we have

$$\sum_{i,j=1}^{3}\int_\Omega \left[\mathbb{T}_{ij}(\mathbf{S}(\mathbf{u})) - \mathbb{T}_{ij}(\mathbf{S}(\mathbf{v}))\right] d\mathbf{x} \geq c_1 \sum_{i,j=1}^{3}\int_\Omega |S_{ij}(\mathbf{w})|^2\, d\mathbf{x} , \tag{48}$$

with $c_1 > 0$.

Using Korn's inequality

$$\left(\int_\Omega |\mathbf{S}(\mathbf{u})|^p\, d\mathbf{x}\right)^{\frac{1}{p}} \geq C_p\, |\nabla \mathbf{u}|_p$$

for $u \in W_0^{1,p}$ with $Cp > 0$ $(1 < p < \infty)$ and Ho¨lder's inequality we obtain from Eq. (48)

$$\frac{1}{2}\frac{d}{dt}|\mathbf{w}|_2^2 + c_2\,|\nabla\mathbf{w}|_2^2 \leq \int_\Omega |\mathbf{w}|^2\,|\nabla\mathbf{u}|\, d\mathbf{x} \leq |\nabla\mathbf{u}|_3\,|\mathbf{w}|_3^2 . \tag{49}$$

In three dimensions we have the embedding

$$H^1(\Omega) \subset L^6(\Omega)$$

from which we deduce

$$|\mathbf{w}|_3 \leq |\mathbf{w}|_2^{\frac{1}{2}}\,|\mathbf{w}|_6^{\frac{1}{2}} \leq c_3\,|\mathbf{w}|_2^{\frac{1}{2}}\,|\nabla\mathbf{w}|_2^{\frac{1}{2}}$$

Moreover, it follows from Eq. (49), via Young's inequality, that

$$\frac{d}{dt}|\mathbf{w}|_2^2 + c_4\,|\nabla\mathbf{w}|_2^2 \leq c_5\,|\nabla\mathbf{u}|_3^2\,|\mathbf{w}|_2^2 . \tag{50}$$

Since the function $g(t) = |\nabla\mathbf{u}|_3^2$ is integrable on $]0, T[$ and $\mathbf{w}(0) = 0$, using Gronwall's inequality we get

$$|\mathbf{w}(t)|_2^2 = 0$$

on $[0, T]$ and thus uniqueness of the solution to problem (S).

The uniform in time regularity is related to the asymptotic behavior of the solution that we now consider.

Let $\mathbf{u}_0 \in H$ and suppose now that $\mathbf{f} \in L^2(\Omega)^3$ is time independent.

According Theorem 2 the unique weak solution is continous

$$\mathbf{u} \in C((0, T); H) .$$

Consequently, we can define the family of operators $(S(t))t\geq 0$ by

$$S(t): \quad H \to H \qquad \mathbf{u}_0 \mapsto S(t)_{\mathbf{u}_0 = \mathbf{u}(t)} \tag{51}$$

is the solution to problem (S). It is easy to show that this family form a continuous semigroup for which we have

Proposition 5.1. *The exists a ball*

$$B_\rho = \{\mathbf{u} \in V ; |\nabla\mathbf{u}|_3 \leq \rho\}$$

which absorbs all the balls in the space H.

Proof. Taking in Eq. (46) **u** as test function and using the property

$$\int_\Omega (\mathbf{u}\cdot\nabla)\,\mathbf{u}\cdot\mathbf{u}\, d\mathbf{x}, \forall\mathbf{u} \in V$$

we obtain

$$\frac{d}{dt}\left|\mathbf{u}\right|_2^2+\sum_{i,j=1}^{3}\int_{\Omega}\mathbb{T}_{ij}\left(\mathbf{S}\left(\mathbf{u}\right)\right)S_{ij}\left(\mathbf{u}\right)d\mathbf{u}=\left\langle \mathbf{f},\mathbf{u}\right\rangle .$$

The tensor T_{ij} can be represented through a nonnegative potential $\theta:\mathbf{R}^9\to\mathbf{R}$ given by

$$\theta\left(\mathbf{S}\right)=\int_0^{|\mathbf{S}|^2}\left(\nu+\nu_1\sqrt{\mathbf{y}}\right)d\mathbf{y}. \tag{53}$$

Indeed, we have

$$\mathbb{T}_{ij}\left(\mathbf{S}\right)=\frac{\partial\theta\left(\mathbf{S}\right)}{\partial S_{ij}},\forall i,j=1,2,3.$$

Moreover,

$$\theta\left(0\right)=0 \text{ and } \frac{\partial\theta\left(0\right)}{\partial S_i}=0,\forall i,j=1,2,3.$$

It follows from Eq. (53) that

$$\begin{aligned}\mathbb{T}_{ij}S_{ij}&=2\left(\nu+\nu_1\left|\mathbf{S}\right|\right)\left|\mathbf{S}\right|^2\\&\geq c_1\left(1+\left|\mathbf{S}\right|\right)\left|\mathbf{S}\right|^2\end{aligned} \tag{54}$$

and thus we find

$$\begin{aligned}&\frac{d}{dt}\left|\mathbf{u}\right|_2^2+\left|\mathbf{u}\right|_2^2+\nu c_1C_2^2\left|\nabla\mathbf{u}\right|_2^2\\&+\nu_1c_1C_3^3\leq\left\langle \mathbf{f},\mathbf{u}\right\rangle .\end{aligned} \tag{55}$$

Now, applying Hölder's inequality, followed by Poincar´e's inequality

$$\left|\mathbf{u}\right|_2\leq\lambda_1^{-\frac{1}{2}}\left|\nabla\mathbf{u}\right|_2^1$$

where λ_1 is the first eigenvalue of the Stokes operator, and using the following inequality

$$\lambda_1^{-\frac{1}{2}}\left|\mathbf{f}\right|_2\left|\nabla\mathbf{u}\right|_2\leq\frac{\nu c_1C_2^2}{2}\left|\nabla\mathbf{u}\right|_2^2+\frac{1}{2\nu c_1C_2^2\lambda}\left|\mathbf{f}\right|_2^2$$

we obtain

$$\frac{d}{dt}\left|\mathbf{u}\right|_2^2+\nu c_1C_2^2\left|\nabla\mathbf{u}\right|_2^2+2\nu_1c_1C_3^3$$

$$\left|\nabla\mathbf{u}\right|_3^3\leq\frac{\left|\mathbf{f}\right|_2^2}{\nu c_1C_2^2\lambda_1}$$

respectively,

$$\begin{aligned}&\frac{d}{dt}\left|\mathbf{u}\right|_2^2+\nu\lambda_1c_1C_2^2\left|\nabla\mathbf{u}\right|_2^2+2\nu_1c_1C_3^3\\&\left|\nabla\mathbf{u}\right|_3^3\leq\frac{\left|\mathbf{f}\right|_2^2}{\nu c_1C_2^2\lambda_1}\end{aligned} \tag{56}$$

The classical Gronwall lemma gives

$$\begin{aligned}&\left|\mathbf{u}\right|_2^2\leq\left|\mathbf{u}_0\right|_2^2\exp\left(-\nu\lambda_1c_1C_2^2t\right)+\\&\frac{\left|\mathbf{f}\right|_2^2}{\nu^2c_1^2C_2^4\lambda_1^2}\left(1-\exp\left(-\nu\lambda_1c_1C_2^2t\right)\right)\end{aligned} \tag{57}$$

and thus we have

$$\limsup_{t\to\infty}\left|\mathbf{u}\right|_2\leq\rho_0,\text{ with }\rho_0=\frac{\left|\mathbf{f}\right|_2^2}{\nu c_1C_2^2\lambda_1}.$$

From (57) we infer that the balls of H of radius ρ are absorbing for all $\rho>\rho_0$. Indeed, let $\rho>\rho_0$ and denote by B_0 the ball $B_H(0,\rho)$. Let B be any bounded set in H. Then, there exists $R>0$ such that $B\subset B(0,R)$. Hence we have

$$\begin{aligned}&\left|\mathbf{u}\left(t\right)\right|^2\leq R^2\exp\left(-\nu\lambda_1c_1C_2^2t\right)+\\&\rho_0^2\left(1-\exp\left(-\nu\lambda_1c_1C_2^2t\right)\right).\end{aligned} \tag{58}$$

It is obvious that the condition

$$\begin{aligned}R^2\exp\left(-\nu\lambda_1c_1C_2^2t\right)+\\\rho_0^2\left(1-\exp\left(-\nu\lambda_1c_1C_2^2t\right)\right)<\rho^2\end{aligned}$$

implies

$$\begin{aligned}&S\left(t\right)B\subset B_0,\forall t>t_0\left(B,\rho\right)=t_0\\&=\frac{1}{\nu\lambda_1c_1C_2^2}\log\frac{R^2}{\rho^2-\rho_0^2},\end{aligned} \tag{59}$$

which proves that B_0 is an absorbing set in H.

CONCLUSION

The objective of this study is to reexamine the Smagorinsky model, unveiling, through asymptotic analysis of the LES model, the mathematical formulation of the sub-mesh. The mathematical analysis, elucidated in this work, serves as a pillar for a broader investigation on the energy decay and the regularity of the Navier-Stokes Equations. This investigation, a priori, deepens the Smagorinsky model, with the conviction that the next investi- gations will present an anisotropic model of viscosity for the turbulent one, addressing the regularity within the Navier-Stokes Equations.

This effort has been devoted to effectively presenting a comprehensive mathematical analysis, encouraging deeper exploration, particularly with regard to the persistent conundrum of regularity within the Navier-Stokes equations.

REFERENCES

[1] S. B. Pope, Turbulent Flows, Cambridge University Press., 2000.

[2] V. John, Numerical methods for incompressible flow problems i, Unversity Lecture, Freie Universit¨at Berlin.

[3] J. Hoffman, C. Johnson, A new approach to computational turbulence modeling, Computer Methods in Applied Mechanics and Engineering 195 (23-24) (2006) 2865–2880.

[4] P. Sagaut, Large eddy simulation for incompressible flows: an introduction, Springer Science & Business Media, 2005.

[5] J. H. Ferziger, M. Peri´c, R. L. Street, Computational methods for fluid dynamics, springer, 2019.

[6] A. N. Kolmogorov, The local structure of turbulence in incompressible viscous fluid for very large reynolds numbers, Proceedings of the Royal Society of London. Series A: Mathematical and Physical Sciences 434 (1890) (1991) 9–13.

[7] D. K. Lilly, A proposed modification of the germano subgrid-scale closure method, Physics of Fluids A: Fluid Dynamics 4 (3) (1992) 633–635.

[8] M. Breuer, Large eddy simulation of the subcritical flow past a circular cylinder: numerical and modeling aspects, International journal for numerical methods in fluids 28 (9) (1998) 1281–1302.

[9] D. K. Lilly, A proposed modification of the germano subgrid-scale closure method, Physics of Fluids A: Fluid Dynamics 4 (3) (1992) 633–635.

[10] J.-L. Lions, "quelques m´ethodes de r´esolution des probl`emes aux limites non-lin´eaires", Dunod.

[11] D. Jiroveanu, Analyse th´eorique et num´erique de certains mod`eles de viscosit´e turbulente, Ph.D. thesis, PhD thesis, Universit´e Joseph Fourier (2002).