STUDY OF THE REGULARITY OF THE KIRCHHOFF PLATE WITH INTERMEDIATE DAMPING

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ABSTRACT: In this work, we study the regularity of the Kirchhoff plate equation with intermediate damping. The intermediate damping is given by $A^\theta u_t$, where u is the displacement of the plate, $A^\theta = (-\Delta)^\theta$ is a strictly positive self-adjoint operator on a complex Hilbert space for any real value of the parameter θ, here we will consider $\theta \in [0, 2]$. In 2020 Vila et al. [2], published the study of the polynomial decay of this model considering the parameter $\theta \in [0, 1)$. More recently, in 2021, Tebou [23] published a study of asymptotic behavior and regularity considering intermediary damping, but considering $\theta \in [0, 2]$ and also considering a new parameter that considers the plate equations intermediate between the Euler-Bernoulli and Kirchoff plates. Our research, like the two previous ones, also uses the theory of semigroups for the existence, asymptotic behavior, and regularity of the semigroup $S(t)$ associated with the model, we use the good properties of the operator $-\Delta$ to perform a spectral analysis of the model and demonstrate our results in a direct and friendly way: We show that the semigroup $S(t)$ associated with the model is exponentially stable for $\theta \in [1, 2]$, we address the study of the analytic of $S(t)$ for $\theta \in [\frac{3}{2}, 2]$ and that $S(t)$ is not analytic when

$\theta \in [0, \frac{3}{2})$. In the last part of our investigation we show that for $1 < \theta < \frac{3}{2}$, $S(t)$ has Gevrey Sharp classes given by $s > \frac{1}{2\theta - 2}$, it is also shown that when the parameter $\theta \in [0,1]$ the semigroup $S(t)$ does not admit Gevrey classes. For the study of existence, stability and regularity, semigroup theory is used together with frequency domain techniques, multipliers, and spectral analysis of the fractional operator A^θ for $\theta \in [0, 2]$ and inequality interpolation.

KEYWORDS: Asymptotic behaviour, Stability, Regularity, Gevrey Sharp-class, Analyticity, Kirchhoff Plates.

1 I INTRODUCTION

This research work studies the asymptotic behavior and the regularity of the solutions of the following Kirchhoff plate equation: Consider Ω a bounded open set of R^n with smooth boundary,

$$u_{tt} - \gamma\Delta u_{tt} + \beta\Delta^2 u + \alpha(-\Delta)^\theta u_t = 0 \text{ in } \Omega \times (0, \infty), \tag{1}$$

satisfying the boundary conditions

$$u = 0, \quad \Delta u = 0 \quad \text{on} \quad \Sigma = \partial\Omega \times (0, \infty), \tag{2}$$

and the initial data

$$u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x) \quad x \in \Omega. \tag{3}$$

Here, the rotational inertia coefficient γ, the elasticity coefficient β and the damping coefficient α, are positive and the exponent θ is considered in the interval $[0, 2]$. The term $(-\Delta)^\theta u_t$ in equation (1) sets up an intermediate dissipation which includes the frictional damping ($\theta = 0$), structural damping ($\theta = 1$) and the strong or viscous damping ($\theta = 2$).

In 2020 see [2] the study of the asymptotic behavior of the semigroup associated to the system (1)-(3) with the fractional damping term $(-\Delta)^\theta u_t$ for values of the parameter θ varying in the interval $[0, 1)$ and show that the semigroup decays polynomially in time, with the rate $O(t^{\frac{-1}{2-\theta}})$ and that these rates depend on the values of the parameter θ are optimal. More recently in 2021, see [23] studied the plate model with fixed or articulated boundary conditions. Rotation and damping forces involve the spectral fractional Laplacian with powers $\theta \in [0, 1]$ and $\delta \in [0, 2]$, respectively for model:

$$y_{tt} + (-\Delta)^\theta y_{tt} + \alpha\Delta^2 y + b(-\Delta)^\delta y_y = 0 \text{ in } \Omega \times (0, \infty),$$

for boundary conditions

$$(\textbf{Hinged plate}) \; y = 0, \; \Delta y = 0 \text{ on } \Sigma = \partial\Omega \times (0, \infty),$$

or, else

$$(\textbf{Clamped plate}) \; y = 0, \; \partial_\nu y = 0 \text{ on } \Sigma = \partial\Omega \times (0, \infty),$$

where a and b are positive constants, θ and δ are constants with $(\theta, \delta) \in [0, 1] \times [0, 2]$. Using the frequency domain approach and the appropriate interpolation inequalities, they show that the semigroup associated with the model is: (a) analytic for all $\theta \in [0, 1]$ and $\delta \in [(2 + \theta)/2, 2]$, (b) is not analytic for all $\theta \in [0, 1]$ and $\delta \in (\theta, (2 + \theta)/2)$, (c) of the Gevrey class $\alpha > (2\ \theta)/2(1 - \delta)$ for all $\theta \in [0, 1]$ and $\theta < \delta < (2 + \theta)/2$. I also prove that for every admissible value of θ, the semigroup is exponentially stable for $\delta \geq \theta$, and only polynomially stable, with rate $O(t^{-\frac{2-\theta}{2(\theta-\delta)}})$, for $\delta < \theta$, when $\theta > 0$. In particular, in the case of the hinged plate, they show that the polynomial decay rate is optimal. It is well-known that the semigroup of the system (1)–(3) is exponentially stable when the damping in equation (1) is structural. There exist many works about the stability of the solutions of plate models with some dissipative mechanisms. A variety of plate models can be found in the books [14] and [18].

First, investigations of the controllability and stabilization of Kirchhoff Plates can be read in the references [15, 16] and [13]. Other relevant investigations on asymptotic behavior and regularity of Kirchoff plates can be read at [5, 17, 22]. In the last 5 years various studies have emerged on the asymptotic behavior and regularity of coupled plate models, thermoelastic plates with various types of damping and boundary conditions, some of these investigations can be read at [1, 4, 9, 10, 11, 19, 20].

Our proposal here is to approach the Kirchoff plate system, making the parameter vary for [0, 2] and using the good properties of the operator $(-\Delta)^\theta$ of being self-adjoint and positive definite for $\theta \in R$ to do spectral analysis and study the asymptotic behavior and regularity of the system in a more didactic way than the works of [2] and [23].

This research work is organized as follows: in section 2, we present the semigroup of the system (4)-(5) in abstract form using the operator $A := -\Delta$ and we show that our model is well defined using semigroup theory. In section 3, we state and demonstrate the main Result of this work: We start with a subsection dedicated to spectral analysis, followed by subsections dedicated to exponential decay, analyticity, and Gevrey Sharp's classes. We would like to point out that the spectral analysis subsection made it possible to prove the lack of analyticity when $\theta \in [0, \frac{3}{2})$, to show that for $\theta \in (1, \frac{3}{2})$ the determined Gevrey classes are Sharp and also helped to show that the semigroup $S(t)$ does not admit Gevrey classes when $\theta \in [0, 1]$.

2 | WELL-POSEDNESS OF THE SYSTEM

In this section, we will use the semigroup theory to assure the existence and uniqueness of strong solutions for the system (4)–(5). It is well known that the operator $A = -\Delta$ defined in the space $\mathfrak{D}(A) = H^2(\Omega) \cap H_0^1(\Omega)$ is a positive self-adjoint operator in the Hilbert space $L^2(\Omega)$. Even more, this operator has compact inverse. Using this notation for the operator $-\Delta$ the system (1)-(3) can be written in the following abstract setting

$$u_{tt} + \gamma A u_{tt} + \beta A^2 u + \alpha A^\theta u_t = 0, \tag{4}$$

satisfying the initial conditions

$$u(0) = u_0, \; u_t(0) = u_1. \tag{5}$$

It is important recalling that as A, is a positive self-adjoint operator with compact inverse in the Hilbert space $\mathfrak{D}(A^0) := L^2(\Omega)$. Then the operator A^θ is self-adjoint, positive for $\theta \in \mathbb{R}$ and the embedding

$$\mathfrak{D}(A^{\theta_1}) \hookrightarrow \mathfrak{D}(A^{\theta_2}),$$

is compact for $\theta_1 > \theta_2$. Here, the norm in the space $\mathfrak{D}(A^\theta)$ is given by $\|u\|_{\mathfrak{D}(A^\theta)} := \|A^\theta u\|$, where $\|\cdot\|$ denotes the norm of the Hilbert space $\mathfrak{D}(A^0)$. More details about fractional operators can be found in [6]. Note that,

$$\begin{aligned} (I+\gamma A)\colon \mathfrak{D}(A^{\frac{1}{2}}) &\longrightarrow \mathfrak{D}(A^{-\frac{1}{2}}), \\ v &\longmapsto (I+\gamma A)v \end{aligned}$$

is an isometrical bijection when the norm of $H_0^1(\Omega) = \mathfrak{D}(A^{\frac{1}{2}})$ is given by

$$\|v\|^2_{\mathfrak{D}(A^{\frac{1}{2}})} = \|v\|^2_{\mathfrak{D}(A^0)} + \gamma\|A^{\frac{1}{2}}v\|^2_{\mathfrak{D}(A^0)}.$$

Indeed, one has: $\|v\|_{\mathfrak{D}(A^{\frac{1}{2}})} = \|(I+\gamma A)v\|_{\mathfrak{D}(A^{-\frac{1}{2}})} \qquad \forall v \in \mathfrak{D}(A^{\frac{1}{2}})$.

Taking the duality product between equation(4) and u_t and using advantage of the self-adjointness of the powers of the operator A and using the identity $v = u_t$, for every solution of the system (4)-(5) the total energy $\mathfrak{E}\colon \mathbb{R}^+ \to \mathbb{R}^+$ is given in the t by

$$\mathfrak{E}(t) = \frac{1}{2}\left[\|v\|^2 + \gamma\|A^{\frac{1}{2}}v\|^2 + \beta\|Au\|^2\right] \tag{6}$$

and satisfies

$$\frac{d}{dt}\mathfrak{E}(t) = -\alpha\|A^{\frac{\theta}{2}}v\|^2 \quad \text{for} \quad 0 \leq \theta \leq 2. \tag{7}$$

Now, if we consider the vector $U(t) = (u, u_t) = (u, v)$, then the system (4)-(5) can be written in an abstract framework as

$$\frac{d}{dt}U(t) = \mathbb{B}U(t), \quad U(0) = U_0, \tag{8}$$

where $U_0 = (u_0, u_1)$, $U = (u, v)$ and the operator B: D(B) ⊂ X → X is given by

$$\mathbb{B}U := \Big(v, -(I+\gamma A)^{-1}\{\beta A^2 u + \alpha A^\theta v\}\Big), \tag{9}$$

for $U = (u, v)$. This operator will be defined in a suitable subspace of the phase space

$$\mathbb{X} := \mathfrak{D}(A) \times \mathfrak{D}(A^{\frac{1}{2}}),$$

where the inner product is defined by

$$\langle U_1, U_2\rangle_{\mathbb{X}} := \langle v_1, v_2\rangle + \gamma\langle A^{\frac{1}{2}}v_1, A^{\frac{1}{2}}v_2\rangle + \beta\langle Au_1, Au_2\rangle,$$

for $U_i = (u_i, v_i)$, $i = 1, 2$. Here, $\langle \cdot \, , \cdot \rangle$ on the right side of this equation denotes the inner product in the space $\mathfrak{D}(A^0)$. With theses considerations, the domain of the operator B is defined by

$$\mathfrak{D}(\mathbb{B}) := \{U \in \mathbb{X}\colon v \in \mathfrak{D}(A), \ u \in \mathfrak{D}(A^{\frac{3}{2}})\}. \tag{10}$$

To show that the operator B is the generator of a C_0-semigroup, we invoke a result from Liu-Zheng' [18].

Theorem 1 (see Theorem 1.2.4 in [18]) *Let* B *be a linear operator with domain* $\mathfrak{D}$(B) *dense in a Hilbert space* X. *If* B *is dissipative and* 0 ∈ ρ(B)*, the resolvent set of* B*, then* B *is the generator of a* C_0*-semigroup of contractions on* X.

Proof: Let us see that the operator B given in (9) satisfies the conditions of this theorem. Clearly, we see that $\mathfrak{D}$(B) is dense in X. Taking the inner product of BU with U, we have

$$\text{Re}\langle \mathbb{B}U, U\rangle_{\mathbb{X}} = -\alpha\|A^{\frac{\theta}{2}}v\|^2 \leq 0, \quad \forall\, U \in \mathfrak{D}(\mathbb{B}) \qquad \text{and} \qquad 0 \leq \theta \leq 2, \tag{11}$$

then, the operator B is dissipative. To complete the conditions of the above theorem, it remains to show that 0 ∈ ρ(B). Therefore we must show that $(0I - \text{B})^{-1}$ exists and is bounded in X. We will first prove that $(0I - \text{B})^{-1}$ exists, then it must be proved that B is bijective. Here we are going to affirm that B is surjective, then for all $F = (f^1, f^2)^T$ ∈ X the stationary problem BU = F has a solution for $U = (u, v)^T$ ∈ $\mathfrak{D}$(B). From definition of the operator B in (9), this system can be written as

$$v = f^1 \in \mathfrak{D}(A) \qquad \text{and} \qquad \beta A^2 u = -\alpha A^\theta f^1 - (I+\gamma A)f^2 \quad \in \mathfrak{D}(A^{\frac{1}{2}}). \tag{12}$$

From these equations, this problem can be placed in a variational formulation: $u \in \mathfrak{D}(A)$ and we write the second equation of (12) in a variational form, using the sesquilinear form b:

$$b(u, \eta) = \langle h, \eta\rangle \tag{13}$$

where $h = -\alpha A^{\theta} f^1 - (I + \gamma A) f^2 \in \mathfrak{D}(A^{-1})$ and the sesquilinear form b is given by

$$b(u, \eta) = \beta\langle A^2 u, \eta\rangle. \tag{14}$$

as $\mathfrak{D}(A) \hookrightarrow \mathfrak{D}(A^{\frac{1}{2}}) \hookrightarrow \mathfrak{D}(A^0) \hookrightarrow \mathfrak{D}(A^{-\frac{1}{2}}) \hookrightarrow \mathfrak{D}(A^{-1})$, this sesquilinear form is coercive in the space $\mathfrak{D}(A)$. As $h \in \mathfrak{D}(A^{-1})$ from Lax-Milgram's Lemma the variational form has unique solution $u \in \mathfrak{D}(A)$ and it satisfies (12), such that (13) is verify for $\eta \in \mathfrak{D}(A)$. From (14) and for all $\eta \in \mathfrak{D}(A)$, we have

$$\beta\langle A^2 u, \eta\rangle = \langle -\alpha A^{\theta} f^1 - (I + \gamma A) f^2 t, \eta\rangle, \forall \eta \in \mathfrak{D}(A) \tag{15}$$

then

$$\beta A^2 u = -\alpha A^{\theta} f^1 - (I + \gamma A) f^2 \text{ in } \mathfrak{D}(A^{-1}). \tag{16}$$

(16) is solution in the weak sense. From $F = (f^1, f^2)^T \in X$ then $(I + \gamma A) f^2 \in D(A^{-\frac{1}{2}})$ and $0 \leq \theta \leq 2$, we have

$\mathfrak{D}(A^2) \hookrightarrow \mathfrak{D}(A^{\theta}) \hookrightarrow \mathfrak{D}(A^0) \hookrightarrow \mathfrak{D}(A^{\frac{-1}{2}})$, from (16), we have

$$u = \frac{1}{\beta}\Big[-\alpha A^{\theta-2} f^1 - (A^{-2} + \gamma A^{-1}) f^2 \Big] \quad \text{in} \quad \mathfrak{D}(A^{\frac{3}{2}}). \tag{17}$$

From first equation of (12), we have

$$v = f^1 \in \mathfrak{D}(A) \tag{18}$$

therefore, $U = (u, v) \in \mathfrak{D}(B)$.

The injectivity of B follows from the uniqueness given by the Lemma of Lax- Milgram's. It remains to show that B^{-1} is a bounded operator. From equations (17) and (18), we get

$$\begin{aligned} \|U\|^2_{\mathbb{X}} &= \|f^1\|^2 + \gamma\|A^{\frac{1}{2}} f^1\|^2 + \beta\left\| A\frac{1}{\beta}\Big[-\alpha A^{\theta-2} f^1 - (A^{-2} + \gamma A^{-1}) f^2 \Big]\right\|^2 \\ &\leq C\|Af^1\|^2 + C\|Af^1\|^2 + C\left\|A^{\theta-1} f^1\right\|^2 + C\left\|A^{-1} f^2\right\|^2 + C\left\|A^0 f^2\right\|^2 \end{aligned} \tag{19}$$

Using continuous embedding $\mathfrak{D}(A) \hookrightarrow \mathfrak{D}(A^{\theta-1}) \hookrightarrow \mathfrak{D}(A^{-1})$, $1 \geq \theta - 1 \geq -1$ and $\frac{1}{2} \geq -1$, $\mathfrak{D}(A^{\frac{1}{2}}) \hookrightarrow \mathfrak{D}(A^{-1})$ then

$$\|U\|_{\mathbb{X}}^2 \le C\left\|Af^1\right\|^2 + C\left\|A^{\frac{1}{2}}f^2\right\|^2 + C\left\|f^2\right\|^2 \qquad (20)$$

therefore, of definition of the norm $\|F\|_{X}$, we get

$$\|U\|_{\mathbb{X}}^2 = \|\mathbb{B}^{-1}F\|_{\mathbb{X}}^2 \le C\|F\|_{\mathbb{X}}^2.$$

Therefore B^{-1} is bounded. So we come to the end of the proof of this theorem.

As a consequence of the previous Theorem 1, we obtain

Theorem 2 *Given* $U_0 \in H$ *there exists a unique weak solution U to the problem* (8) satisfying

$$U \in C([0, +\infty), X).$$

Futhermore, if $U_0 \in \mathfrak{D}(Bk)$, $k \in N$, *then the solution U of* (8) *satisfies*

$$U \in \bigcap_{j=0}^{k} C^{k-j}([0,+\infty), \mathfrak{D}(\mathbb{B}^j)).$$

Theorem 3 (Lions' Interpolation) *Let* $\alpha < \beta < \gamma$. *The there exists a constant* $L = L(\alpha, \beta, \gamma)$ *such that*

$$\|A^\beta u\| \le L\|A^\alpha u\|^{\frac{\gamma-\beta}{\gamma-\alpha}} \cdot \|A^\gamma u\|^{\frac{\beta-\alpha}{\gamma-\alpha}}$$

for every $u \in \mathfrak{D}(A^\gamma)$

Proof: See Theorem 5.34 [6]

Theorem 4 (Hilla-Yosida) *A linear (unbounded) operator* B *is the infinitesimal generator of a* C_0 – *semigroup of contractions* $S(t)$, $t \ge 0$, *if and only if*

I. B *is closed and* $\mathfrak{D}(B) = X$,

II. the resolvent set $\rho(B)$ *of* B *contains* R^+ *and for every* $\lambda > 0$,

$$\|(\lambda I - \mathbb{B})^{-1}\|_{\mathcal{L}(\mathbb{X})} \le \frac{1}{\lambda}.$$

Proof: See [8].

3 | STABILITY AND REGULARITY

3.1 Spectral Analysis of the fractional operator A^θ for $\theta \in [0, 2]$

Since A is a positive self-adjoint operator with compact resolvent, its spectrum is constituted by positive eigenvalues ϕ_n, $n \in N$, with $\phi_n \to \infty$. Let us denote by (e_n) the corresponding eigenvectors, that is

$$Ae_n = \phi_n e_n,\ n \in \mathrm{N} \text{ and } A^r e_n = \phi_n^r e_n,\ r \in \mathrm{R}.$$

We consider $F_n = (0, -\tilde{e}_n) \in X$ where $\tilde{e}_n = \frac{e_n}{\|A^{\frac{1}{2}} e_n\|}$, then the solution $U = (u, v)$ of the system $(i\lambda I - \mathrm{B})U = F_n$ satisfies the following conditions:

$$v = i\lambda u \text{ and } i\lambda v + (I + \gamma A)^{-1}\{\beta A^2 u + \alpha A^\theta v\} = -\tilde{e}_n.$$

By substituting the first identity and applying the operator $(I + \gamma \mathrm{A})$ in the second equation, we obtain

$$\lambda^2 (I + \gamma A)u - \beta A^2 u - i\alpha\lambda \mathrm{A}^\theta u = (I + \gamma A)\tilde{e}_n.$$

Now, we are going to look for by solutions of the form $u = \eta\ \tilde{e}_n$ for some complex number η. Therefore, the coefficient η must satisfy the equation

$$\lambda^2(I+\gamma A)\left(\eta\frac{e_n}{\|A^{\frac{1}{2}}e_n\|}\right) - \beta A^2\left(\eta\frac{e_n}{\|A^{\frac{1}{2}}e_n\|}\right) - i\alpha\lambda A^\theta\left(\eta\frac{e_n}{\|A^{\frac{1}{2}}e_n\|}\right) = (I+\gamma A)\left(\frac{e_n}{\|A^{\frac{1}{2}}e_n\|}\right),$$

equivalent

$$\left[\frac{1}{\|A^{\frac{1}{2}}e_n\|}\right]\left[\lambda^2(I+\gamma A)(\eta e_n) - \beta A^2(\eta e_n) - i\alpha\lambda A^\theta(\eta e_n)\right] = \left[\frac{1}{\|A^{\frac{1}{2}}e_n\|}\right](I+\gamma A)(e_n).$$

Then

$$\lambda^2(1+\gamma\phi_n)\eta - \beta\phi_n^2\eta - i\alpha\lambda\phi_n^\theta\eta = (1+\gamma\phi_n). \tag{21}$$

Solving this equation we have

$$\eta = \frac{1+\gamma\phi_n}{\lambda^2(1+\gamma\phi_n) - \beta\phi_n^2 - i\alpha\lambda\phi_n^\theta}. \tag{22}$$

In this point, taking

$$\lambda^2 = \lambda_n^2 := \frac{\beta\phi_n^2}{1+\gamma\phi_n} \tag{23}$$

using (23) in (22), we obtain

$$\eta = \eta_n = i\frac{1+\gamma\phi_n}{\alpha\lambda_n\phi_n^\theta}. \tag{24}$$

If we introduce the notation $x_n \approx y_n$ meaning $\lim_{n\to\infty}\frac{|x_n|}{|y_n|}$ is a positive real number, then from (23) and (24) we can assert that $|\lambda_n| \approx |\phi_n|^{\frac{1}{2}}$ and $|\eta_n| \approx |\lambda_n|^{1-2\theta}$.

Therefore, if $U_n = (u_n, v_n)$ is the solution of the system $(i\lambda_n - A)U_n = F_n$, we obtain

$$\|A^{\frac{1}{2}}v_n\| = \|\lambda_n A^{\frac{1}{2}}u_n\| = |\lambda_n||\eta_n|\|A^{\frac{1}{2}}\tilde{e}_n\| \geq K|\lambda_n|^{2-2\theta},$$

for some $K > 0$ and n large enough. From this estimative, we conclude that

$$\|U_n\|_{\mathbb{X}} \geq \gamma\|A^{\frac{1}{2}}v_n\| \geq \gamma K|\lambda_n|^{2-2\theta} \qquad \text{for} \qquad 0 \leq \theta \leq 2. \tag{25}$$

3.2 Exponential decay of the semigroup $S(t) = e^{t\mathbb{B}}$ for $\theta \in [1, 2]$

In this section, we will study the asymptotic behavior of the semigroup of the system (4)-(5). We will use the following spectral characterization of exponential stability of semigroups due to Gearhart[7](Theorem 1.3.2 book of Liu-Zheng [18]).

Theorem 5 (see [18]) *Let* $S(t) = e^{t\mathbb{B}}$ *be a* C_0*-semigroup of contractions on a Hilbert space* X. *Then* $S(t)$ *is exponentially stable if and only if*

$$\rho(\mathbb{B}) \supseteq \{i\lambda/\lambda \in \mathbb{R}\} \equiv i\mathbb{R} \tag{26}$$

and

$$\limsup_{|\lambda|\to\infty} \|(i\lambda I - \mathbb{B})^{-1}\|_{\mathcal{L}(\mathbb{X})} < \infty \tag{27}$$

holds.

Remark 6 *Note that to show the condition* (27) *it is enough to show that: Let* $\delta > 0$. *There exists a constant* $C_\delta > 0$ *such that the solutions of the system* (4)- (5) *for* $|\lambda| > \delta$, *satisfy the inequality*

$$\|U\|_{\mathbb{X}} \leq C_\delta\|F\|_{\mathbb{X}} \text{ for } 1 \leq \theta \leq 2. \tag{28}$$

In view this Theorem 5, we will try to obtain some estimates for the solution $U = (u, v) \in \mathfrak{D}(\mathbb{B})$ of the system

$$(i\lambda I - \mathbb{B})U = F \tag{29}$$

where $F = (f, g) \in \mathbb{X}$. This system, written in components, reads

$$i\lambda u - v = f \text{ in } \mathfrak{D}(A) \tag{30}$$

$$i\lambda(I + \gamma A)v + \beta A^2 u + \alpha A^\theta v = (I + \gamma A)g \text{ in } \mathfrak{D}(A^{\frac{1}{2}}) \tag{31}$$

From (11), we have the first estimate

$$\alpha\|A^{\frac{\theta}{2}}v\|^2 = \mathrm{Re}\langle(i\lambda I - B)U, U\rangle_X = \mathrm{Re}\langle F, U\rangle_X \leq \|F\|_X\|U\|_X. \quad (32)$$

Next, we show some lemmas that will lead us to the proof of the main theorem of this section.

Lemma 7 *Let* $\delta > 0$. *There exists* $C_\delta > 0$ *such that the solutions of the system* (4)-(5) *for* $|\lambda| > \delta$, *satisfy*

$$\begin{array}{rrcllr} (i) & \|Au\|^2 & \leq & C_\delta\|F\|_{\mathbb{X}}\|U\|_{\mathbb{X}} & \text{for} \quad 1 \leq \theta \leq 2. & (33) \\ (ii) & \|v\|^2 + \gamma\|A^{\frac{1}{2}}v\|^2 & \leq & C_\delta\|F\|_{\mathbb{X}}\|U\|_{\mathbb{X}} & \text{for} \quad 1 \leq \theta \leq 2. & (34) \end{array}$$

Proof: (*i*) Taking the duality product between equation (31) and u, taking advan- tage of the self-adjointness of the powers of the operator A, we obtain

$$\begin{aligned} \beta\|Au\|^2 &= \langle(I+\gamma A)v, i\lambda u\rangle - \alpha\langle A^\theta v, u\rangle + \langle(I+\gamma A)g, u\rangle \\ &= \|v\|^2 + \gamma\|A^{\frac{1}{2}}v\|^2 + \langle v, f\rangle + \gamma\langle A^{\frac{1}{2}}v, A^{\frac{1}{2}}f\rangle - i\alpha\lambda\|A^{\frac{\theta}{2}}u\|^2 \\ &\quad -\alpha\langle Af, A^{\theta-1}u\rangle + \langle g, u\rangle + \gamma\langle A^{\frac{1}{2}}g, A^{\frac{1}{2}}u\rangle. \qquad (35) \end{aligned}$$

Taking real part, applying Cauchy-Schwarz inequality and norms $\|F\|_X$ and $\|U\|_X$, and estimative (32), finish proof of item (*i*) this lemma.

Proof: (*ii*) It is enough to observe that, if $1 \leq \theta \leq 2$ then $\frac{1}{2} \leq \frac{\theta}{2}$. Applying continuous immersions and estimative (32), we finish the proof of item (*ii*) this lemma. Q

Theorem 8 *The semigroup* $S(t) = e^{tB}$ *is exponentially stable when the parameter* θ *assumes values in the interval* $[1, 2]$.

Proof: Adding the estimates (33) and (34) of Lemma 7, we have

$$\|U\|_X^2 \leq C_\delta\|F\|_X\|U\|_X \text{ for } 1 \leq \theta \leq 2. \quad (36)$$

Therefore, the condition (27) for $\theta \in [1, 2]$ is verified. Next, we show the condition (26) of Theorem 5.

Lemma 9 *Let* $\varrho(B)$ *be the resolvent set of operator* B. *Then*

$$i\mathbb{R} \subseteq \varrho(B). \quad (37)$$

Proof: Since B is the infinitesimal generator of a C_0– semigroup of contractions $S(t)$, $t \geq 0$, from Theorem 4, B is a closed operator, and as $\mathfrak{D}(B)$ has compact embedding into the energy space X, the spectrum $\sigma(B)$ contains only eigenvalues. Therefore, to prove $i\mathbb{R} \subseteq \rho(B)$ by using an argument by contradiction, so we suppose that $i\mathbb{R} \not\subset \rho(B)$. As $0 \in \rho(B)$ and $\rho(B)$ is open, we consider the highest positive number λ_0 such that the $]-i\lambda_0, i\lambda_0[\subset \rho(B)$ then $i\lambda_0$ or $-i\lambda_0$ is an element of the spectrum $\sigma(B)$. We Suppose $i\lambda_0 \in \sigma(B)$ (if $-i\lambda_0 \in \sigma(B)$ the proceeding is similar) such that the non-trivial eigenfunction is $U = (u, v) \in \mathfrak{D}(B)$, then $\mathbb{B}U = i\lambda_0 U$, on the other hand, from the stationary equation (29) for $i\lambda_0$ and $F = 0$, from estimates (36) with $F = 0$, we get $U = 0$, which is a contradiction, since U is the eigenfunction associated with the imaginary eigenvalue $i\lambda_0$. Hence, $i\mathbb{R} \subseteq \rho(\mathbb{B})$. This completes the proof of this lemma.

Therefore the semigroup $S(t) = e^{t\mathbb{B}}$ is exponentially stable for $\theta \in [1, 2]$, thus we finish the proof of this theorem.

3.3 Regularity

The study of the regularity of the solutions of the model of this research is centralized to study the analyticity and the existence of Gevrey classes; in the first subsection, we will demonstrate that the semigroup $S(t)$ is analytic when the parameter θ assumes values in the interval $[\frac{3}{2}, 2]$ we also show the lack of analyticity when θ $[0, \frac{3}{2})$, in the second part we show that when the parameter θ assumes values in the range $(1, \frac{3}{2})$, $S(t)$ supports Gevrey Sharp classes and when θ $[0, 1]$, $S(t)$ does not support Gevrey classes. Here we would like to point out that the semigroup being analytic is more regular than only admitting Gevrey classes. Analyticity or Gevrey Classes imply differentiability of the semigroup.

3.4 Study of Analyticity and Lack Analyticity of the semi- group $S(t) = e^{t\mathbb{B}}$

In this subsection, we will show that the semigroup $S(t)$, is analytic for the parameter $\frac{3}{2} \leq \phi \leq 2$ and is not analytic when the parameter assumes values in the interval $[0, \frac{3}{2})$.

The following theorem characterizes the analyticity of the semigroups $S(t)$.

Theorem 10 (see [18]) *Let $S(t) = e^{t\mathbb{B}}$ be C_0-semigroup of contractions on a Hilbert space* $\mathbb{X}$. *Suppose that*

$$\rho(\mathbb{B}) \supseteq \{i\lambda/\lambda \in \mathbb{R}\} \equiv i\mathbb{R}. \tag{38}$$

Then $S(t)$ is analytic if and only if

$$\limsup_{|\lambda|\to\infty} \|\lambda(i\lambda I - \mathbb{B})^{-1}\|_{\mathcal{L}(\mathbb{X})} < \infty, \tag{39}$$

holds.

3.4.1 Analyticity for $\theta \in [\frac{3}{2}, 2]$

Theorem 11 *The semigroup $S(t) = e^{t\mathbb{B}}$ is analytic for $\frac{3}{2} \leq \theta \leq 2$.*

Proof: From Lemma(9), (38) is verified. Therefore, it remains to prove (39), for that it is enough to show that, let $\delta > 0$. There exists a constant $C_\delta > 0$ such that the solutions of the system (4)-(5) for $|\lambda| > \delta$, satisfy the inequality

$$|\lambda|\frac{\|U\|_{\mathbb{X}}}{\|F\|_{\mathbb{X}}}\leq C_\delta \quad\Longleftrightarrow\quad |\lambda|\|U\|^2_{\mathbb{X}}\leq C_\delta\|F\|_{\mathbb{X}}\|U\|_{\mathbb{X}}<\infty,\quad \text{for}\quad \frac{3}{2}\leq\theta\leq 2. \qquad (40)$$

Lemma 12 *Let* $\delta>0$*. There exists* $C_\delta>0$ *such that the solutions of the system* (4)- (5) *for* $|\lambda|>\delta$*, satisfy*

$$(i)\quad |\lambda|\|Au\|^2 \leq C_\delta\|F\|_{\mathbb{X}}\|U\|_{\mathbb{X}} \quad \text{for}\quad \frac{3}{2}\leq\theta\leq 2, \qquad (41)$$

$$(ii)\quad |\lambda|\left[\|v\|^2+\gamma\|A^{\frac{1}{2}}v\|^2\right] \leq C_\delta\|F\|_{\mathbb{X}}\|U\|_{\mathbb{X}} \quad \text{for}\quad \frac{3}{2}\leq\theta\leq 2. \qquad (42)$$

Proof: (*i*) Using equation (30) in (31), we have

$$i\lambda(I+\gamma A)v+\beta A^2u+i\lambda\alpha A^\theta u-\alpha A^\theta f=(I+\gamma A)g. \qquad (43)$$

Applying the duality product the equation (43) with $A^{2-\theta}u$ and taking into account that the fractional powers of the operator A are self-adjoint and using (30), we obtain

$$\begin{aligned} i\lambda\alpha\|Au\|^2 &= \langle (I+\gamma A)v, A^{2-\theta}(i\lambda u)\rangle-\beta\|A^{\frac{4-\theta}{2}}u\|^2+\alpha\langle Af,Au\rangle+\langle (I+\gamma A)g,A^{2-\theta}u\rangle\\ &= \|A^{\frac{2-\theta}{2}}v\|^2+\gamma\|A^{\frac{3-\theta}{2}}v\|^2+\langle A^{1-\theta}v,Af\rangle+\gamma\langle A^{2-\theta}v,Af\rangle-\beta\|A^{\frac{4-\theta}{2}}u\|^2\\ &\quad +\alpha\langle Af,Au\rangle+\langle A^{\frac{1}{2}}g,A^{\frac{3}{2}-\theta}u\rangle+\gamma\langle A^{\frac{1}{2}}g,A^{\frac{5}{2}-\theta}u\rangle. \qquad (44)\end{aligned}$$

As for $\frac{3}{2}\leq\theta\leq 2$ we have $1-\theta<2-\theta\leq\frac{1}{2}$ and $\frac{3}{2}-\theta<\frac{5}{2}-\theta\leq 1$, taking imaginary part in (44) and applying Cauchy-Schwarz inequality and continuous immersion, we finish the proof of item (*i*) this lemma.

Proof: (*ii*) Now, applying the duality product the equation (31) with v and taking into account that the fractional powers of the operator A are self-adjoint and using (30), we obtain

$$\begin{aligned} i\lambda\left[\|v\|^2+\gamma\|A^{\frac{1}{2}}v\|^2\right] &= -\beta\langle A^2u, i\lambda u-f\rangle-\alpha\|A^{\frac{\theta}{2}}v\|^2+\langle (I+\gamma A)g,v\rangle\\ &= i\beta\lambda\|Au\|^2+\beta\langle Au,Af\rangle-\alpha\|A^{\frac{\theta}{2}}v\|^2\\ &\quad +\langle A^{\frac{1}{2}}g,A^{-\frac{1}{2}}v\rangle+\gamma\langle A^{\frac{1}{2}}g,A^{\frac{1}{2}}v\rangle, \qquad (45)\end{aligned}$$

finally, taking imaginary part, now applying Cauchy-Schwarz and item (*i*) this lemma, we finish the proof of item (*ii*) this lemma.

Finally, from estimates of Lemma 12, finish to proof of theorem.

3.4.2 Lack Analyticity for $\theta\in[0,3)$

Multiplying by $|\lambda_n|$ both sides gives inequality (25), we have

$$|\lambda_n|\|U_n\|_{\mathbb{X}}\geq\gamma\|A^{\frac{1}{2}}v_n\|\geq\gamma K|\lambda_n|^{3-2\theta} \quad \text{for } 0\leq\theta\leq 2.$$

As $3-2\theta>0 \Longleftrightarrow \theta<\frac{3}{2}$, then if $\theta\in[0,\frac{3}{2})$, we have $|\lambda_n|\|U_n\|_{\mathbb{X}}\to\infty$ when $|\lambda_n|\to\infty$. Therefore $S(t)$ is not analytic when $\theta\in[0,\frac{3}{2})$.

3.5 Gevrey Sharp-class and Lack of Gevrey Class

In this subsection, we will show that the semigroup $S(t)$, has Gevrey sharp-classes depending on the parameter θ, for parameter values in the range $(1, \frac{3}{2})$ and has no Gevrey classes when the parameter θ assumes values in the range [0, 1].

Before exposing our results, it is useful to recall the next definition and result presented in [3, 10] (adapted from [21], Theorem 4, p. 153]).

Definition 13 *Let $t_0 \geq 0$ be a real number. A strongly continuous semigroup $S(t)$, defined on a Banach space $\mathbb{X}$, is of Gevrey class $s > 1$ for $t > t_0$, if $S(t)$ is infinitely differentiable for $t > t_0$, and for every compact set $K \subset (t_0, \infty)$ and each $\mu > 0$, there exists a constant $C = C(\mu, K) > 0$ such that*

$$||S^{(n)}(t)||_{\mathcal{L}(\mathbb{X})} \leq C\mu^n(n!)^s, \text{ for all t K, n = 0, 1, 2...} \tag{46}$$

Theorem 14 ([21]) *Let $S(t)$ be a strongly continuous and bounded semigroup on a Hilbert space $\mathbb{X}$. Suppose that the infinitesimal generator $\mathbb{B}$ of the semigroup $S(t)$ satisfies the following estimate, for some $0 < \Psi < 1$:*

$$\lim_{|\lambda|\to\infty} \sup |\lambda|^{\Psi}||(i\lambda I - \mathbb{B})^{-1}||_{\mathcal{L}(\mathbb{X})} < \infty. \tag{47}$$

Then $S(t)$ is of Gevrey class s for $t > 0$, for every $s > \frac{1}{\Psi}$.

3.5.1 Gevrey Sharp-Class for $\theta \in (1, \frac{3}{2})$

Lemma 15 *Let $\delta > 0$. There exists $C_\delta > 0$ such that the solutions of the system (4)-(5) for $|\lambda| > \delta$, satisfy*

$$\begin{aligned}
(i) \quad \|A^{\frac{1+\theta}{2}}u\|^2 &\leq C_\delta\|F\|_{\mathbb{X}}\|U\|_{\mathbb{X}} \quad \text{for} \quad 1 \leq \theta \leq \frac{3}{2}, && (48)\\
(ii) \quad |\lambda|\|A^{\frac{2\theta-1}{4}}v\|^2 &\leq C_\delta\|F\|_{\mathbb{X}}\|U\|_{\mathbb{X}} \quad \text{for} \quad 1 \leq \theta \leq \frac{3}{2}, && (49)\\
(iii) \quad |\lambda|^{2\theta-2}\|A^{\frac{1}{2}}v\|^2 &\leq C_\delta\|F\|_{\mathbb{X}}\|U\|_{\mathbb{X}} \quad \text{for} \quad 1 \leq \theta \leq \frac{3}{2}. && (50)
\end{aligned}$$

Proof: (i) Applying the duality product the equation (31) with $A^{\theta-1}u$ and taking into account that the fractional powers of the operator A are self-adjoint and using (30), we obtain

$$\begin{aligned}
\beta\|A^{\frac{1+\theta}{2}}u\|^2 &= \langle (I+\gamma A)v, A^{\theta-1}i\lambda u\rangle - \alpha\langle A^\theta(i\lambda u - f), A^{\theta-1}u\rangle + \langle (I+\gamma A)g, A^{\theta-1}u\rangle\\
&= \|A^{\frac{\theta-1}{2}}v\|^2 + \gamma\|A^{\frac{\theta}{2}}v\|^2 - i\lambda\alpha\|A^{\frac{2\theta-1}{2}}u\|^2 + \alpha\langle Af, A^{2\theta-2}u\rangle\\
&\quad + \langle A^{\frac{1}{2}}g, A^{\theta-\frac{3}{2}}u\rangle + \gamma\langle A^{\frac{1}{2}}g, A^{\theta-\frac{1}{2}}u\rangle + \langle v, A^{\theta-1}f\rangle + \gamma\langle A^{\frac{1}{2}}v, A^{\theta-\frac{1}{2}}f\rangle. \quad (51)
\end{aligned}$$

Taking real part and for $1 \leq \theta \leq \frac{3}{2}$ we have $\frac{\theta-1}{2} < \frac{\theta}{2}$, $2\theta - 2 \leq 1$, applying continuous embedding, we obtain

$$\|A^{\frac{1+\theta}{2}}u\|^2 \leq C_\delta\{\|A^{\frac{\theta}{2}}v\|^2 + \|Af\|\|Au\| + \|A^{\frac{1}{2}}g\|\|Au\| + \|A^{\frac{1}{2}}v\|\|A^{\theta-\frac{1}{2}}f\|\}.$$

From (32) and items *i* and (*ii*) of Lemma 7, we finish the proof of item (*i*) this lemma.

Proof: (*ii*) Applying the duality product the equation (31) with $A^{\theta-\frac{3}{2}}v$ and taking into account that the fractional powers of the operator *A* are self-adjoint and using (30), we obtain

$$i\lambda\left[\|A^{\frac{2\theta-3}{4}}v\|^2+\gamma\|A^{\frac{2\theta-1}{4}}v\|^2\right] = -\beta\langle A^{\frac{1+\theta}{2}}u, A^{\frac{\theta}{2}}v\rangle-\alpha\|A^{\frac{4\theta-3}{4}}v\|^2 + \langle A^{\frac{1}{2}}g, A^{\theta-2}v\rangle+\gamma\langle A^{\frac{1}{2}}g, A^{\theta-1}v\rangle.$$

Taking imaginary part and applying Cauchy-Schwarz, Young inequalities and as for $1\leq\theta\leq\frac{3}{2}$ we have $\theta-1\leq\frac{1}{2}$ applying continuous immersions, we obtain

$$|\lambda|\left[\|A^{\frac{2\theta-3}{4}}v\|^2+\gamma\|A^{\frac{2\theta-1}{4}}v\|^2\right]\leq C_\delta\{\|A^{\frac{1+\theta}{2}}u\|^2 + \|A^{\frac{\theta}{2}}v\|^2+\|F\|_{\mathbb{X}}\|U\|_{\mathbb{X}}\}\quad\text{for}\quad 1\leq\theta\leq\frac{3}{2}. \quad (52)$$

Finally of estimative (32) and item (*i*) this lemma, we finish the proof of item (ii) this lemma.

Proof: (*iii*) As for $1\leq\theta\leq\frac{3}{2}$, we have $\frac{1}{2}\in[\frac{2\theta-1}{4},\frac{\theta}{2}]$. We are going to use an interpolation inequality Theorem 3. Since

$$\frac{1}{2}=\phi\left(\frac{2\theta-1}{4}\right)+(1-\phi)\frac{\theta}{2},\quad \text{for } \phi=2\theta-2.$$

using inequalities (32) and (48) we get that

$$\begin{aligned}\|A^{\frac{1}{2}}v\|^2 &\leq C\|A^{\frac{2\theta-1}{4}}v\|^{2\phi}\|A^{\frac{\theta}{2}}v\|^{2(1-\phi)}\\ &\leq C_\delta|\lambda|^{2-2\theta}\{\|F\|_{\mathbb{X}}\|U\|_{\mathbb{X}}\}^{2\theta-2}\{\|F\|_{\mathbb{X}}\|U\|_{\mathbb{X}}\}^{3-2\theta}.\end{aligned}$$

Therefore, we conclude the proof this item (*iii*).

Lemma 16 *Let* $\delta>0$. *There exists* $C_\delta>0$ *such that the solutions of the system* (1)-(3) *for* $|\lambda|>\delta$, *satisfy*

$$\begin{aligned}(i)\quad |\lambda|\|A^{\frac{2\theta+1}{4}}u\| &\leq C_\delta\|F\|_{\mathbb{X}}\|U\|_{\mathbb{X}}\quad\text{for}\quad 1\leq\theta\leq\frac{3}{2} \quad (53)\\ (ii)\quad |\lambda|^{2\theta-2}\|Au\|^2 &\leq C_\delta\|F\|_{\mathbb{X}}\|U\|_{\mathbb{X}}\quad\text{for}\quad 1\leq\theta\leq\frac{3}{2}. \quad (54)\end{aligned}$$

Proof: (*i*) Applying the duality product to the second equation of (30) with $A^{\frac{2\theta+1}{2}}$ and taking into account that the fractional powers of the operator *A* are self-adjoint and using (31), we obtain

$$\begin{aligned}i\lambda\|A^{\frac{2\theta+1}{4}}u\|^2 &= \langle A^{\frac{2\theta-3}{2}}v, A^2u\rangle+\langle Af, A^{\theta-\frac{1}{2}}u\rangle\\ &= \frac{1}{\beta}\langle A^{\frac{2\theta-3}{2}}v, -i\lambda(I+\gamma A)v-\alpha A^\theta v+(I+\gamma A)g\rangle+\langle Af, A^{\frac{2\theta-1}{2}}u\rangle\\ &= \frac{i\lambda}{\beta}\|A^{\frac{2\theta-3}{4}}v\|^2+\frac{i\gamma\lambda}{\beta}\|A^{\frac{2\theta-1}{4}}v\|^2-\frac{\alpha}{\beta}\|A^{\frac{4\theta-3}{4}}v\|^2+\frac{1}{\beta}\langle A^{\theta-2}v, A^{\frac{1}{2}}g\rangle\\ &\quad+\frac{\gamma}{\beta}\langle A^{\theta-1}v, A^{\frac{1}{2}}g\rangle+\langle Af, A^{\frac{2\theta-1}{2}}u\rangle. \quad (55)\end{aligned}$$

Taking imaginary part and for $1\leq\theta\leq\frac{3}{4}$ we have $\frac{2\theta-3}{4}<\frac{2\theta-1}{4}\leq\frac{1}{2}$, $\theta-1\leq\frac{1}{2}$ and $\frac{2\theta-1}{2}$

≤ 1 applying continuous embedding and item (*ii*) of Lemma 15, finish the proof of item (*i*) this lemma.

Proof (*ii*) As for $1 \leq \theta \leq \frac{3}{2}$, we have $1 \in [\frac{2\theta+1}{4}, \frac{1+\theta}{2}]$. We are going to use an interpolation inequality Theorem 3. Since

$$1 = \phi\left(\frac{2\theta+1}{4}\right) + (1-\phi)\left(\frac{1+\theta}{2}\right), \quad \text{for} \quad \phi = 2\theta - 2,$$

using inequalities (53) and (54) we get that

$$\begin{aligned} \|Au\|^2 &\leq C\|A^{\frac{2\theta+1}{4}}v\|^{2\phi}\|A^{\frac{1+\theta}{2}}v\|^{2(1-\phi)} \\ &\leq C_\delta|\lambda|^{2-2\theta}\{\|F\|_{\mathbb{X}}\|U\|_{\mathbb{X}}\}^{2\theta-2}\{\|F\|_{\mathbb{X}}\|U\|_{\mathbb{X}}\}^{3-2\theta}. \end{aligned}$$

Therefore, we conclude the proof this item (*ii*).

Our main result in this subsection is as follows:

Theorem 17 *Let* $S(t) = e^{t\mathbb{B}}$ *strongly continuos-semigroups of contractions on the Hilbert space* H*, the semigroups* $S(t)$ *is of Grevrey class s, for every* $s > \frac{1}{2\theta-2}$ *for* $\theta \in (1, \frac{3}{2})$, *such that we have the resolvent estimative:*

$$\limsup_{|\lambda|\to\infty} |\lambda|^{2\theta-2}\|(i\lambda I - \mathbb{B})^{-1}\|_{\mathcal{L}(\mathbb{X})} < \infty, \qquad \text{for} \quad 1 < \theta < \frac{3}{2}. \tag{56}$$

Proof: Note that the estimate

$$|\lambda|^{2-2\theta}\|(i\lambda I - \mathbb{B})^{-1}F\|_{\mathbb{X}} = |\lambda|^{2-2\theta}\|U\|_{\mathbb{X}} \leq C_\delta\|F\|_{\mathbb{X}} \quad \text{for} \quad 1 < \theta < \frac{3}{2} \tag{57}$$

implies the inequality (56). Adding the estimates (50) of Lemma 15 and (54) of Lemma 16 and as $|\lambda|^{2\theta-2}\|v\|^2 \leq C|\lambda|\|A^{\frac{1}{2}}v\|^2$, finish proof this theorem.

[Gevrey Sharp-class] The Gevrey class determined above is Sharp, meaning by "Gevrey Sharp", the statement of the following theorem:

Theorem 18 *The Gevrey class determined by the function* $\Psi(\theta) = 2\theta - 2$ *for* $\theta \in (1, \frac{3}{2})$ is sharp, in the sense:

If $\Phi = \Psi + \delta_1 = 2\theta = 2 + \delta_1$ for all $\delta_1 > 0$

such that $0 < \Phi < 1$ and $1 < \theta < \frac{3}{2}$, (58)

then

$$s > \frac{1}{\Phi} \text{ for } 1 < \theta < \frac{3}{2},$$

is not a Gevrey class of the semigroup $S(t) = e^{t\mathbb{B}}$.

Proof: We will use the results (56) of Theorem 17 and the estimative (25), for $\delta_1 > 0$, we have

$$|\lambda_n|^{\Phi} \| U_n \|_X = |^{\lambda n|2\theta-2+\delta 1} \| U_n \|_H \geq K|\lambda_n|^{\delta 1} \to \infty, \text{ when } |\lambda_n| \to \infty$$

Therefore Φ not verify the conditions (56) of the Theorem 17 concerning class Gevrey. Then the Gevrey classe $s > \frac{1}{2\theta-2}$ for $\theta \in (1, \frac{3}{2})$ the semigroup $S(t)$ is Sharp.

3.5.2 Lack Gevrey class for θ ∈ [0, 1]

Finally, if there is a Gevrey class for $S(t)$ when $\theta \in [0, 1]$, there must exist a $\Phi \in (0, 1)$ such that the identity

$$\lim_{|\lambda_n|\to\infty} \sup |\lambda_n|^{\Phi} ||(i\lambda I - \mathbb{B})^{-1}||_{\mathcal{L}(\mathbb{X})} < \infty. \quad (59)$$

is verified. Although, multiplying both sides of the inequality (25) by $|\lambda_n|^{\Phi}$ for $\Phi \in (0, 1)$ and considering $\theta \in [0, 1]$, we have

$$|\lambda_n|^{\Phi} \| U_n \|_H \geq K|\lambda|^{2-2\theta+\Phi}.$$

Therefore, to verify (59) we must have $2 - 2\theta + \Phi \leq 0 \Longleftrightarrow \Phi \leq 0$, this is absurd, because to have Gevrey classes $\Phi \in (0, 1)$. Consequently the semigroup $S(t)$ does not admit a Gevrey class for $\theta \in [0, 1]$. We emphasize that this was already expected from the results of [2].

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