REGULARITY OF THE TIMOSHENKO'S SYSTEM WITH THERMOELASTICITY OF TYPE III AND FRACTIONAL DAMPING

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ABSTRACT: The article presents the study of the regularity of two thermoelastic beam systems defined by the Timoshenko beam model coupled with the heat conduction of Green-Naghdiy theory of type III, both mathematical models are differentiated by their coupling terms that arise as a consequence of the constitutive laws initially considered. The systems presented in this work have 3 fractional dampings: $\mu_1(-\Delta)^\tau \phi_t$, $\mu_2(-\Delta)^\sigma \psi_t$ and $K(-\Delta)^\xi \theta_t$, where ϕ, ψ and θ are transverse displacement, rotation angle and empirical temperature of the bean respectively and the parameters $(\tau, \sigma, \xi) \in [0, 1]^3$. It is noted that for values 0 and 1 of the parameter τ, the so-called frictional or viscous damping will be faced, respectively. The main contribution of this article is to show that the corresponding semigroup $S_i(t) = e^{B_i t}$, with $i = 1, 2$, is of Gevrey class $s > \frac{1}{\min\{\frac{\tau}{1+\tau}, \frac{\sigma}{1+\sigma}, \frac{\xi}{1+\xi}\}}$, with $(\tau, \sigma, \xi) \in R_{CG} := (0, 1]^3$. It is also showed that $S_1(t) = e^{B_1 t}$ is analytic in the region $R_{A_1} := \{(\tau, \sigma, \xi) \in [\frac{1}{2}, 1]^3\}$ and $S_2(t) = e^{B_2 t}$ is analytic in the region $R_{A_2} := \{(\tau, \sigma, \xi) \in [\frac{1}{2}, 1]^3 / \tau = \xi\}$.
KEYWORDS: Gevrey class, Analyticity, Fractional damping, Semigroup theory.

1 | INTRODUCTION

In this article, it is investigated the regularity of the semigroup associated with the thermoelastic beam system where the transversal vibrations are given by Timoshenko's model (See Timoshenko [24]) and the balance of the energy is described by the Green-Naghdi theory, known as thermo- elasticity of type III(See Green and Naghdi [7]). The equation of motion and energy balance for this thermoelastic Timoshenko system is given by

$$\rho A \phi_{tt} - S_x = 0 \quad = \quad 0, \quad x \in (0, L),\ t \in \mathbb{R}^+,$$
$$\rho I \psi_{tt} - M_x + S \quad = \quad 0, \quad x \in (0, L),\ t \in \mathbb{R}^+,$$

$$\rho_3 u_t + q_x + \gamma\psi_{xt} = 0, \quad x \in (0, L), \ t \in \mathbb{R}^+.$$

The constant ρ denotes the density, the cross-sectional area, and I the area moment of inertia. By S one denotes the shear force, M is the bending moment and q is the heat flux. The function ϕ is the transverse displacement, ψ is the rotation angle of a filament of the beam and u is the temperature difference. Here, t is the time variable and x is the coordinate space along the beam.

In this investigation, two systems that differ only in their coupling terms are analyzed, which are obtained by starting from similar constitutive laws.

The first best-known mathematical model in the literature, and studied for example in [5], [14] and [15], starts from the following constitutive laws:

$$S = kAG(\phi_x + \psi), \qquad M = EI\psi_x + \beta u \qquad \text{and} \qquad q = -\delta\alpha_x - K\alpha_{xt}, \qquad (1)$$

where α is the so-called thermal displacement whose time derivative is the empirical temperature u, i.e., $\alpha_t = u$, E and G are elastic constants, k the shear coefficient for measuring the stiffness of materials ($k < 1$), δ and K denote the thermal conductivity, β the coefficient of linear thermal expansion and γ a coupling constant.

To simplify the notation let us denote by $\rho_1 = \rho A$, $\rho_2 = \rho I$, $\kappa = kAG$ and $b = EI$. Under these conditions, the system can be written as

$$\rho_1\phi_{tt} - \kappa(\phi_x + \psi)_x = 0, \quad x \in (0, L), \ t \in \mathbb{R}^+, \qquad (2)$$
$$\rho_2\psi_{tt} - b\psi_{xx} + \kappa(\phi_x + \psi) + \beta u_x = 0, \quad x \in (0, L), \ t \in \mathbb{R}^+, \qquad (3)$$
$$\rho_3 u_{tt} - \delta u_{xx} + \gamma\psi_{ttx} - K u_{txx} = 0, \quad x \in (0, L), \ t \in \mathbb{R}^+. \qquad (4)$$

In order to exhibit the dissipative natural of system (2)–(4), it is convenient to introduce a new variable (see [25]):

$$\theta(t, x) = \int_0^t u(s, x)ds + \frac{1}{\delta}\chi(x), \qquad (5)$$

where $\chi \in H_0^1(0, L)$ solves the following Cauchy problem

$$\begin{cases} \chi_{xx} = \rho_3 u_1 - K u_{0xx} + \gamma\psi_{1x} & \text{in} \quad (0, L), \\ \chi(x) = 0, & x = 0, L. \end{cases} \qquad (6)$$

Then, considering 3 fractional damping $\mu_1(-\Delta)^\tau \phi_t$, $\mu_2(-\Delta)^\sigma \psi_t$ and $K(-\Delta)^\xi \theta_t$ and using (5) and (6) the starting system (2)–(4) is transformed to

$$\rho_1\phi_{tt} - \kappa(\phi_x + \psi)_x + \mu_1(-\Delta)^\tau\phi_t = 0, \quad x \in (0, L), \ t \in \mathbb{R}^+, \qquad (7)$$
$$\rho_2\psi_{tt} - b\Delta\psi + \kappa(\phi_x + \psi) + \beta\theta_{tx} + \mu_2(-\Delta)^\sigma\psi_t = 0, \quad x \in (0, L), \ t \in \mathbb{R}^+, \qquad (8)$$
$$\rho_3\theta_{tt} - \delta\Delta\theta + \gamma\psi_{tx} + K(-\Delta)^\xi\theta_t = 0, \quad x \in (0, L), \ t \in \mathbb{R}^+, \qquad (9)$$

where the parameters τ, σ and ξ take values in the range [0, 1] and μ_1, μ_2 are positive real constants.

The second mathematical model addressed in this research that was proposed in 2017 by Santos et al. [17], which consists of adjusting the couplings of the first system

couplings to satisfy type III heat conduction that acts mainly on the shear force, and as a result, the transverse shear force $S = \kappa(\phi_x + \psi)$ is obtained from the transverse distribution of shear stresses acting on each cross-section of the beam. For this purpose, it is necessary to start with the used constitutive laws

$$S = \kappa(\phi_x + \psi) - \mu u, \qquad M = b\psi_x \qquad \text{and} \qquad q = -\delta\alpha_x - K\alpha_{xt}, \qquad (10)$$

where α is the so-called thermal displacement, the time derivative of which is the empirical tem- perature u, i.e., $\alpha_t = u$. From these constitutive laws, by making appropriate changes, one arrives at the system to follow (for more details see [17]):

$$
\begin{aligned}
\rho_1\phi_{tt} - \kappa(\phi_x + \psi)_x + \mu\theta_{tx} + \mu_1(-\Delta)^\tau\phi_t &= 0, & x \in (0, L),\ t \in \mathbb{R}^+, & \quad (11)\\
\rho_2\psi_{tt} - b\Delta\psi + \kappa(\phi_x + \psi) - \mu\theta_t + \mu_2(-\Delta)^\sigma\psi_t &= 0, & x \in (0, L),\ t \in \mathbb{R}^+, & \quad (12)\\
\rho_3\theta_{tt} - \delta\Delta\theta + \mu(\phi_x + \psi)_t + \gamma(-\Delta)^\xi\theta_t &= 0, & x \in (0, L),\ t \in \mathbb{R}^+. & \quad (13)
\end{aligned}
$$

To make the calculations in the results of the test more practical, the two systems are abstractly rewritten by using the operator: $A : D(A) \subset L^2(0, L) \to L^2(0, L)$, where

$$A = -\Delta = -(\,\cdot\,)_{xx}, \quad D(A) = H^2(0, L) \cap H_0^1(0, L). \qquad (14)$$

It is known that the operator given in (14) is self-adjoint, positive, and has inverse compact on a complex Hilbert space $D(A^0) = L^2(0, L)$. Therefore, the operator A^v is self-adjoint positive for all $v \in \mathbb{R}$, bounded by $v \leq 0$, and the embedding

$$D(A^{\nu_1}) \hookrightarrow D(A^{\nu_2}),$$

is continuous for $v_1 > v_2$. Here, the norm in $D(A^v)$ is given by $\|u\|_{D(A^v)} := \|A^v u\|$, $u \in D(A^v)$, where $\|.\|$ denotes the norm in the Hilbert space $L^2(0, L) = D(A^0)$. Some of these spaces are: $D(A^{1/2}) = H_0^1(0, L)$, $D(A^0) = L^2(0, L)$ and $D(A^{-1/2}) = H^{-1}(0, L)$.

Studying the regularity (Gevrey class and/or Analyticity) of systems is relevant to science. In applied sciences, obtaining information about the solutions of the mathematical model under study such as regularity (smoothness) has the same importance as knowing the asymptotic behavior of the model's solution. From a mathematical point of view, when one talks about the regularity of solutions, concepts like differentiable, Gevrey class, and analyticity come to mind. It is already known that semigroups $S(t) = e^{Bt}$ of the Gevrey class have better regular properties than a differentiable semigroup, but are less regular than an analytic semigroup. The Gevrey rate $s > 1$ 'measures' the degree of divergence of its power series. It should be noted that the Gevrey class or analyticity of the model in particular implies three very important properties: The first one is the property of the smoothing effect on initial data, i. e., no matter how irregular the initial data is, the solutions of the models become very smooth in positive finite-time. The second property is that systems are exponentially stable. Finally, these systems benefit from the property of linear stability, which means that the type of the semigroup is equal to the spectral limit of its infinitesimal operator.

During the last decades, various authors have studied some physical phenomena

for the Tim- oshenko beam system formulated in different mathematical models. Most of them focused on studying the asymptotic behavior, always trying to obtain the best decay rate and using dissipa- tions in some of the system equations. In the following paragraphs, some of these investigations are mentioned.

In 1987, Kim and Renardy [9] studied the asymptotic behavior of the Timoshenko beam consid- ering two boundary dissipations. They demonstrated the exponential decay of energy associated with the model using the multiplier technique and also determined numerical estimates of the eigenvalues of the operator associated with this system. Later in 2008, Messaoudin and Said- Houari [14] also studied the asymptotic behavior of the Type III thermoelastic Timoshenko system with mixed boundary conditions (Dirichlet- Dirichlet-Neuman), they demonstrated using the en- ergy method that if the velocities of the waves associated with the hyperbolic part of the system are equal, then the system decays exponentially. The complement of the study of the asymptotic behavior of this same system due to different mixed conditions (Neumann-Dirichlet-Neumann) was studied again in 2013 by Messaoudin and Fareh [15], in this new research they show that if the speeds of the waves associated with the hyperbolic part of the system are different, then the system decays polynomially. In the following year, the work of Fatori et al. [5] who also studied this same system, but with two types of boundary conditions (Dirichlet-Dirichlet-Dirichlet) and (Dirichlet- Neumann-Dirichlet), using semigroup technique they showed that the corresponding semigroup is exponentially stable If and only if the velocities associated with the hyperbolic part of the system are equal, in the absence of exponential decay they show that the corresponding semigroup is polynomially stable and the determined rate is optimal.

More recently, in 2017, Santos et al.[17] studied the type III thermoelastic Timoshenko beam system given by:

$$\begin{aligned}
\rho_1\phi_{tt} - \kappa(\phi_x + \psi)_x + \mu\theta_{tx} &= 0, & x \in (0,L),\ t \in \mathbb{R}^+, \\
\rho_2\psi_{tt} - b\psi_{xx} + \kappa(\phi_x + \psi) - \mu\theta_t &= 0, & x \in (0,L),\ t \in \mathbb{R}^+, \\
\rho_3\theta_{tt} - \delta\theta_{xx} + \mu(\phi_x + \psi)_t - \gamma\theta_{txx} &= 0, & x \in (0,L),\ t \in \mathbf{R}^+,
\end{aligned}$$

and the boundary conditions (Dirichlet-Dirichlet-Dirichlet) given by

$$\phi(0,t) = \phi(L,0) = \psi(0,t) = \psi(L,0) = \theta(0,t) = \theta(L,t) = 0, \qquad \forall t > 0,$$

or with the boundary conditions(Dirichlet-Neumann-Neumann) given by

$$\phi(0,t) = \phi(L,0) = \psi_x(0,t) = \psi_x(L,0) = \theta_x(0,t) = \theta_x(L,t) = 0, \qquad \forall t > 0,$$

note that this system differs slightly from the terms of coupling. In this research, the authors also applied the semigroup technique and showed that the associated semigroup can be exponen- tially or polynomially stable depending on the relationships between the wave propagation velocity coefficients, specifically in the case of polynomial decay it was proved that the rate found is optimal.

About investigations the regularity of semigroups associated with various mathematical mod- els, one could mention the paper/article of Fatori et al.[4], whose work the authors studied the differentiability and analyticity, in addition to studying the asymptotic behavior via semigroups. For analyticity, they used a theorem that can be found in the Liu-Zheng book[12] or the work of Hao et al.[8]. Other more recently published works have explored the regularity of solutions using the Gevrey class introduced in Taylor's thesis [19] (1989). In the same direction, one could mention [1, 2, 6].

Among recent research that has studied the asymptotic behavior and/or regularity of models with fractional damping, one could mention the works of Sare et al.[18], in that paper, the authors investigated coupled systems thermoelastic type, where they address two cases with Fourier's heat law and the other with Cattaneo considering in both cases the rotational inertial term. Further- more, they, study the exponential stability, possible regions of loss of exponential stability and polynomial stability, and, more recently, the work of Keyantuo et al.[20](2020) to be published. In this latter work, the authors studied the thermoelastic plate model with a fractional Laplacian between the Euler-Bernoulli and Kirchhoff model with two types of boundary conditions. In ad- dition to studying the asymptotic and analytical behavior, the authors show that the underlying semigroups are of Gevrey class s for every $s > \frac{2-\theta}{2-4\theta}$ for both the clamped and hinged bound-ary conditions when the parameter θ lies in the interval (0, 1⁄2). Moreover, one could cite the investigations [3, 10, 11, 16, 22, 23].

One motivation for deciding to study the regularity (determination of the Gevrey classes and analyticity) of Timoshenko's systems was in the direction of complementing the work published in 2005 by Raposo et al. [21], in this work the authors study the asymptotic behavior (exponential decay of the Timoshenko system), the studied system considers two weak (frictional) dampings given by the speed of the transversal and rotational movements. The investigated model is given by

$$\rho_1 u_{tt} - k(u_x - \psi)_x + u_t = 0 \quad \text{in} \quad (0, L) \times (0, \infty),$$
$$\rho_2 \psi_{tt} - b\psi_{xx} + k(u_x - \psi) + \psi_t = 0, \quad \text{in} \quad (0, L) \times (0, \infty),$$
$$u(0, t) = u(L, t) = \psi(0, t) = \psi(L, t) = 0 \qquad t > 0.$$

The authors use the frequency domain technique (spectral characterization) to study the asymp- totic behavior via semigroups.

This article is organized as follows. In section 2, using the theory of semigroup, the well- posedness of both systems is studied. Section 3 is dedicated to the study of the regularity of both models, this subsection is subdivided into two parts: in subsection 3.1 Gevrey class of the semigroup associated with the first system $S(t) = e^{B_1 t}$ in the region R_{GC} = (0, 1]³ is determined and it is shown that $S(t) = e^{B_1 t}$ is analytic in the region R_{A1}. Finally, in subsection 3.2 we show that the Gevrey class of $S(t) = e^{B_2 t}$ is $s > \frac{1}{\min\{\frac{\tau}{1+\tau}, \frac{\sigma}{1+\sigma}, \frac{\xi}{1+\xi}\}}$ for the parameters (τ, σ, ξ) within the region R_{GC} and this subsection is finished by proving that $S(t) = e^{B_2 t}$ is analytic when the 3 parameters τ , σ, and ξ take values in the closed

interval $[\frac{1}{2}, 1]$ such that τ = ξ. The investigation ends with an observation regarding the asymptotic behavior of $S(t) = e^{B_i t}$ in which it is illustrated that the necessary estimates given in Propositions(5) and (6) imply that the semigroup associated with the system (15)–(17) and (21)–(22) and the system (18)-(22) respectivamently are exponentially stable for (τ, σ, ξ) ∈ [0, 1]³.

2 I WELL-POSEDNESS: SEMIGROUP APPROACH

Using the *A* operator defined in (14) the system (7)–(9) will be given by

$$\rho_1 \phi_{tt} - \kappa(\phi_x + \psi)_x + \mu_1 A^\tau \phi_t = 0, \quad x \in (0, L), \ t \in \mathbb{R}^+, \tag{15}$$
$$\rho_2 \psi_{tt} + bA\psi + \kappa(\phi_x + \psi) + \beta\theta_{tx} + \mu_2 A^\sigma \psi_t = 0, \quad x \in (0, L), \ t \in \mathbb{R}^+, \tag{16}$$
$$\rho_3 \theta_{tt} + \delta A\theta + \gamma\psi_{tx} + KA^\xi \theta_t = 0, \quad x \in (0, L), \ t \in \mathbb{R}^+. \tag{17}$$

And again using the *A* operator on (11)–(13), leads to

$$\rho_1 \phi_{tt} - \kappa(\phi_x + \psi)_x + \mu\theta_{tx} + \mu_1 A^\tau \phi_t = 0, \quad x \in (0, L), \ t \in \mathbb{R}^+, \tag{18}$$
$$\rho_2 \psi_{tt} + bA\psi + \kappa(\phi_x + \psi) - \mu\theta_t + \mu_2 A^\sigma \psi_t = 0, \quad x \in (0, L), \ t \in \mathbb{R}^+, \tag{19}$$
$$\rho_3 \theta_{tt} + \delta A\theta + \mu(\phi_x + \psi)_t + \gamma A^\xi \theta_t = 0, \quad x \in (0, L), \ t \in \mathbb{R}^+. \tag{20}$$

Additionally, the appropriate initial conditions for both systems are considered

$$\varphi(0, \cdot) = \phi_0, \quad \varphi_t(0, \cdot) = \phi_1, \quad \psi(0, \cdot) = \psi_0, \quad \psi_t(0, \cdot) = \psi_1, \quad \theta(0, \cdot) = \theta_0 \quad \text{and} \quad \theta_t(0, \cdot) = \theta_1, \tag{21}$$

and boundary conditions for both systems of type Dirichlet–Dirichlet–Dirichlet

$$\phi(t, 0) = \phi(t, L) = 0, \quad \psi(t, 0) = \psi(t, L) = 0, \quad \theta(t, 0) = \theta(L, 0) = 0 \quad t > 0. \tag{22}$$

Taking *ϕ* = *ϕ_t*, Ψ = *ψ_t*, Θ = *θ_t*, considering *U* = (*ϕ*, *ϕ*, *ψ*, Ψ, *θ*, Θ) and *U_0* = (*ϕ_0*, *ϕ_1*, *ψ_0*, *ψ_1*, *θ_0*, *θ_1*), both systems, can be written in the following abstract framework

$$\frac{d}{dt} U(t) = \mathcal{B}_i U(t), \quad U(0) = U_0, \tag{23}$$

where for the operator B₁ for the first system is defined by

$$\mathcal{B}_1 U := \begin{bmatrix} \Phi \\ \frac{1}{\rho_1}[\kappa(\phi_x + \psi)_x - \mu_1 A^\tau \Phi] \\ \Psi \\ -\frac{1}{\rho_2}[bA\psi + \kappa(\phi_x + \psi) + \beta\Theta_x + \mu_2 A^\sigma \Psi] \\ \Theta \\ -\frac{1}{\rho_3}[\delta A\theta + \gamma\Psi_x + KA^\xi \Theta] \end{bmatrix}, \tag{24}$$

and the operator B₂ for the second system is defined by

$$\mathcal{B}_2 U := \begin{bmatrix} \Phi \\ \frac{1}{\rho_1}[\kappa(\phi_x + \psi)_x - \mu\Theta_x - \mu_1 A^\tau \Phi] \\ \Psi \\ \frac{1}{\rho_2}[-bA\psi - \kappa(\phi_x + \psi) + \mu\Theta - \mu_2 A^\sigma \Psi] \\ \Theta \\ \frac{1}{\rho_3}[-\delta A\theta - \mu(\Phi_x + \Psi) - \gamma A^\xi \Theta] \end{bmatrix}, \tag{25}$$

for $U = (\phi, \phi, \psi, \Psi, \theta, \Theta)$. Both operators are defined in the energy space

$$\mathcal{H}_1 = \mathcal{H}_2 = \mathcal{H} := [D(A^{\frac{1}{2}}) \times D(A^0)]^3.$$

H_1 is a Hilbert space with the inner product given by

$$\langle U, U^* \rangle_{\mathcal{H}_1} := \rho_1 \beta \gamma \langle \Phi, \Phi^* \rangle + \rho_2 \kappa \gamma \langle \Psi, \Psi^* \rangle + \beta \kappa \gamma \langle \phi_x + \psi, \phi_x^* + \psi^* \rangle + b\kappa\gamma\langle\psi_x, \psi_x^*\rangle$$
$$+ \beta\delta\kappa\langle\theta_x, \theta_x^*\rangle + \beta\kappa K\langle\Theta, \Theta^*\rangle,$$

for $U = (\phi, \phi, \psi, \Psi, \theta, \Theta)$, $U^* = (\phi^\cdot, \phi^\cdot, \psi^\cdot, \Psi^\cdot, \theta^\cdot, \Theta^\cdot) \in H$, and induced norm

$$\|U\|^2_{\mathcal{H}_1} := \rho_1\beta\gamma\|\Phi\|^2 + \rho_2\kappa\gamma\|\Psi\|^2 + \beta\kappa\gamma\|\phi_x + \psi\|^2 + b\kappa\gamma\|A^{\frac{1}{2}}\psi\|^2 + \beta\delta\kappa\|A^{\frac{1}{2}}\theta\|^2 + \beta\kappa K\|\Theta\|^2.$$

In these conditions, he domain of B_1 is defined as

$$\mathcal{D}(\mathcal{B}_1) := \{ U \in \mathcal{H} / \mathcal{B}(U) \in \mathcal{H} + \text{ boundary conditions (22) } \}$$
$$:= D(A) \times D(A^{\frac{1}{2}}) \cap D(A^\tau) \times D(A) \times D(A^{\frac{1}{2}}) \cap D(A^\sigma) \times D(A) \times D(A^{\frac{1}{2}}) \cap D(A^\xi). \tag{26}$$

And H_2 is a Hilbert space with the inner product given by

$$\langle U, U^* \rangle_{\mathcal{H}_2} := \rho_1\langle \Phi, \Phi^* \rangle + \rho_2\langle \Psi, \Psi^* \rangle + \kappa\langle \phi_x + \psi, \phi_x^* + \psi^* \rangle + b\langle\psi_x, \psi_x^*\rangle + \delta\langle\theta_x, \theta_x^*\rangle + \rho_3\langle\Theta, \Theta^*\rangle,$$

for $U = (\phi, \Phi, \psi, \Psi, \theta, \Theta), U^* = (\phi^*, \Phi^*, \psi^*, \Psi^*, \theta^*, \Theta^*) \in \mathcal{H}$, and induced norm

$$\|U\|^2_{\mathcal{H}_2} := \rho_1\|\Phi\|^2 + \rho_2\|\Psi\|^2 + \kappa\|\phi_x + \psi\|^2 + b\|A^{\frac{1}{2}}\psi\|^2 + \delta\|A^{\frac{1}{2}}\theta\|^2 + \rho_3\|\Theta\|^2.$$

In these conditions, the domain of B_2 is defined as

$$\mathcal{D}(\mathcal{B}_2) := D(A) \times D(A^{\frac{1}{2}}) \cap D(A^\tau) \times D(A) \times D(A^{\frac{1}{2}}) \cap D(A^\sigma) \times D(A) \times D(A^{\frac{1}{2}}) \cap D(A^\xi). \tag{27}$$

To show that both operators B_i, $i = 1, 2$ are generators of a C_0 semigroup we invoke a result from Liu-Zheng' book[12].

Theorem 1 (see Theorem 1.2.4 in [12]) *Let B_i be a linear operator with domain* $D(B_i)$ *dense in a Hilbert space* H. *If B_i is dissipative and $0 \in \rho(B_i)$, the resolvent set of B_i, then B_i is the generator of a C_0- semigroup of contractions on* H.

Next, we apply the Theorem 1 only for the first system, the proof of the second system is completely similar. Let us see that the operator B_1 defined in (24) satisfies the conditions of this theorem. Clearly, $D(B_1)$ is dense in H. Effecting the internal product of $B_1 U$ with U, lead to

$$\text{Re}\langle \mathcal{B}_1 U, U \rangle = -\beta\gamma\mu_1\|A^{\frac{\tau}{2}}\Phi\|^2 - \kappa\gamma\mu_2\|A^{\frac{\sigma}{2}}\Psi\|^2 - \frac{\beta\kappa K^2}{\rho_3}\|A^{\frac{\xi}{2}}\Theta\|^2, \quad \forall\, U \in \mathcal{D}(\mathcal{B}_1), \tag{28}$$

that is, the operator B_1 is dissipative.

To complete the conditions of the above theorem, it remains to show that $0 \in \rho(\mathrm{B}_1)$. Let $F = (f^1, f^2, f^3, f^4, f^5, f^6) \in \mathrm{H}$, let us see that the stationary problem $\mathrm{B}_1 U = F$ has a solution $U = (\phi, \phi, \psi, \Psi, \theta, \Theta)$. From the definition of the operator B_1 given in (24), this system can be written as

$$
\begin{aligned}
\Phi &= f^1, & \kappa(\phi_x + \psi)_x &= \rho_1 f^2 + \mu_1 A^\tau f^1, & (29) \\
\Psi &= f^3, & bA\psi + \kappa(\phi_x + \psi) &= -\rho_2 f^4 - \mu_2 A^\sigma f^3 - \beta f_x^5, & (30) \\
\Theta &= f^5, & \delta A\theta &= -\gamma f_x^3 - KA^\xi f^5 - \rho_3 f^6. & (31)
\end{aligned}
$$

Therefore, it is not difficult to see that there exists only one solution ϕ, ψ and θ of the system

$$
\begin{aligned}
\kappa(\phi_x + \psi)_x = \rho_1 f^2 + \mu_1 A^\tau f^1 & \in & D(A^0), \\
bA\psi + \kappa(\phi_x + \psi) = -\rho_2 f^4 - \mu_2 A^\sigma f^3 - \beta f_x^5 & \in & D(A^0), \\
\delta A\theta = -\gamma f_x^3 - KA^\xi f^5 - \rho_3 f^6 & \in & D(A^0). \quad (32)
\end{aligned}
$$

Therefore:

$$
\|U\|_{\mathcal{H}} \le C \|F\|_{\mathcal{H}},
$$

which in particular implies that $\left\| \mathrm{B}_1^{-1} F \right\|_{\mathrm{H}} \le C \| F \|_{\mathrm{H}}$, so we have that 0 belongs to the resolvent set $\rho(\mathrm{B}_1)$. Consequently, from Theorem 1 we have B_1 as the generator of a contractions semigroup.

As a consequence of the above Theorem(1), it follows

Theorem 2 *Given $U_0 \in \mathrm{H}$ there exists a unique weak solution U to the problem* (23) *satisfying*

$$
U \in C([0, +\infty), \mathcal{H}).
$$

Futhermore, if $U_0 \in D(\mathrm{B}_i^k)$, $k \in \mathrm{N}$, then the solution U of (23) *satisfies*

$$
U \in \bigcap_{j=0}^k C^{k-j}([0, +\infty), D(\mathcal{B}_i^j)).
$$

3 | REGULARIZATION RESULTS

This section is divided into two subsections, in each of them, the Gevrey class and analytics of the corresponding systems are studied. Using the characterization results presented in [20] (adapted from [19], Theorem 4, p. 153]) it is shown that the corresponding semigroups $S(t) = e^{\mathrm{B}it}$ are of Gevrey class $s > \frac{1}{r}$ for $r = \min \left\{ \frac{\tau}{1+\tau}, \frac{\sigma}{1+\sigma}, \frac{\xi}{1+\xi} \right\} \in (0, 1)$, with $(\tau, \sigma, \xi) \in R_{CG} := (0, 1]^3$. And for the study of analyticity the main tool used is the characterization of analytical semigroups due to Liu and Zheng[12] (See the book by Liu-Zheng - Theorem 1.3.3. It has proved that $S_1(t) = e^{\mathrm{B}1t}$ is analytic in the region $R_{A1} := [1/2, 1]^3$ and $S_2(t) = e^{\mathrm{B}2t}$ is analytic in the region $R_{A2} := \{(\tau, \sigma, \xi) \in [1/2, 1]^3 / \tau = \xi\}$.

In what follows: C, C_δ and C_ε will denote positive constants that assume different

values in different places.

3.1 Regularity of the first system

Next, two lemma are presented where two estimates are tested which are fundamental for the determination of Gevrey class and the analytics of the associated semigroup $S(t) = e^{B_1 t}$.

Lemma 3 *Let $S(t) = e^{B_1 t}$ be a C_0-semigroup on contractions on Hilbert space $H=H_1$, the solutions of the system* (15)–(17) *and* (21)–(22) *satisfy the inequality*

$$\limsup_{|\lambda| \to \infty} \|(i\lambda I - \mathcal{B}_1)^{-1}\|_{\mathcal{L}(\mathcal{H})} < \infty. \tag{33}$$

Proof: To show the (33) inequality, it suffices to show that, given $\delta > 0$ there exists a constant $C_\delta > 0$ such that the solutions of the system (15)–(17) and (21)–(22) for $|\lambda| > \delta$ satisfy the inequality

$$\frac{\|U\|_{\mathcal{H}}}{\|F\|_{\mathcal{H}}} \leq C_\delta \iff \|U\|_{\mathcal{H}} = \|(i\lambda I - \mathcal{B}_1)^{-1} F\|_{\mathcal{H}} \leq C_\delta \|F\|_{\mathcal{H}}. \tag{34}$$

If $\lambda \in \mathbb{R}$ and $F = (f^1, f^2, f^3, f^4, f^5, f^6) \in H$ then the solution $U = (\phi, \Phi, \psi, \Psi, \theta, \Theta) \in D(B_1)$ the resolvent equation $(i\lambda I - B_1)U = F$ can be written in the form

$$i\lambda\phi - \Phi = f^1 \quad \text{in} \quad D(A^{\frac{1}{2}}), \tag{35}$$

$$i\lambda\Phi - \frac{\kappa}{\rho_1}(\phi_x + \psi)_x + \frac{\mu_1}{\rho_1}A^\tau\Phi = f^2 \quad \text{in} \quad D(A^0), \tag{36}$$

$$i\lambda\psi - \Psi = f^3 \quad \text{in} \quad D(A^{\frac{1}{2}}), \tag{37}$$

$$i\lambda\Psi + \frac{b}{\rho_2}A\psi + \frac{\kappa}{\rho_2}(\phi_x + \psi) + \frac{\beta}{\rho_2}\Theta_x + \frac{\mu_2}{\rho_2}A^\sigma\Psi = f^4 \quad \text{in} \quad D(A^0), \tag{38}$$

$$i\lambda\theta - \Theta = f^5 \quad \text{in} \quad D(A^{\frac{1}{2}}), \tag{39}$$

$$i\lambda\Theta + \frac{\delta}{\rho_3}A\theta + \frac{\gamma}{\rho_3}\Psi_x + \frac{K}{\rho_3}A^\xi\Theta = f^6 \quad \text{in} \quad D(A^0). \tag{40}$$

Using the fact that the operator B_1 is dissipative, result in

$$\beta\gamma\mu_1\|A^{\frac{\tau}{2}}\Phi\|^2 + \kappa\gamma\mu_2\|A^{\frac{\sigma}{2}}\Psi\|^2 + \frac{\beta\kappa K^2}{\rho_3}\|A^{\frac{\xi}{2}}\Theta\|^2$$
$$= \text{Re}\langle(i\lambda - \mathcal{B}_1)U, U\rangle = \text{Re}\langle F, U\rangle \leq \|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}}. \tag{41}$$

On the other hand, performing the duality product of (36) for $\beta\gamma\rho_1\phi$, and remembering that the operators A^v for all $v \in \mathbb{R}$ are self-adjoint, resulting in

$$\beta\kappa\gamma\langle(\phi_x + \psi), \phi_x\rangle = \beta\gamma\rho_1\|\Phi\|^2 + \beta\gamma\rho_1\langle\Phi, f^1\rangle$$
$$- i\lambda\mu_1\beta\gamma\|A^{\frac{\tau}{2}}\phi\|^2 + \mu_1\beta\gamma\langle A^{\frac{\tau}{2}}f^1, A^{\frac{\tau}{2}}\phi\rangle + \beta\gamma\rho_1\langle f^2, \phi\rangle,$$

now performing the duality product of (38) for $\beta\gamma\rho_2\psi$, and remembering that the operators A^v for all $v \in \mathbb{R}$ are self-adjoint, leading to

$$\beta\kappa\gamma\langle((\phi_x+\psi),\psi\rangle + b\beta\gamma\|A^{\frac{1}{2}}\psi\|^2 = \beta\gamma\rho_2\|\Psi\|^2 + \beta\gamma\rho_2\langle\Psi,f^3\rangle$$
$$+ \beta^2\gamma\langle\Theta,\psi_x\rangle - i\lambda\beta\gamma\mu_2\|A^{\frac{\sigma}{2}}\psi\|^2 + \beta\gamma\mu_2\langle A^{\frac{\sigma}{2}}f^3, A^{\frac{\sigma}{2}}\psi\rangle + \beta\gamma\rho_2\langle f^4,\psi\rangle.$$

Adding the last 2 equations, result in

$$\beta\kappa\gamma\|\phi_x+\psi\|^2 + b\beta\gamma\|A^{\frac{1}{2}}\psi\|^2 = \beta\gamma\rho_1\|\Phi\|^2 + \beta\gamma\rho_2\|\Psi\|^2 - i\lambda\beta\gamma\{\mu_1\|A^{\frac{\tau}{2}}u\|^2$$
$$+ \mu_2\|A^{\frac{\sigma}{2}}\psi\|^2\} + \beta\gamma\rho_1\langle\Phi,f^1\rangle + \beta\gamma\rho_2\langle\Psi,f^3\rangle$$
$$+ \beta\gamma\mu_1\langle A^{\frac{\tau}{2}}f^1, A^{\frac{\tau}{2}}\phi\rangle + \beta\gamma\mu_2\langle A^{\frac{\sigma}{2}}f^3, A^{\frac{\sigma}{2}}\phi\rangle$$
$$+ \beta\gamma\rho_1\langle f^2,\phi\rangle + \beta\gamma\rho_2\langle f^4,\psi\rangle + \beta^2\gamma\langle\Theta,\psi_x\rangle. \quad (42)$$

Taking real part, using norm $\|F\|_{\mathsf{H}}$ and $\|U\|_{\mathsf{H}}$ and applying Cauchy-Schwarz and Young Inequa- lities, for $\varepsilon > 0$ exists $C_\varepsilon > 0$ which does not depend on λ such that

$$\beta\kappa\gamma\|\phi_x+\psi\|^2 + b\beta\gamma\|A^{\frac{1}{2}}\psi\|^2 \le \beta\gamma\rho_1\|\Phi\|^2 + \beta\gamma\rho_2\|\Psi\|^2 + \varepsilon\|A^{\frac{1}{2}}\psi\|^2 + C_\varepsilon\|\Theta\|^2 + C_\delta\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}}.$$

From estimative (41) and the fact $0 \le \frac{\tau}{2}$, $0 \le \frac{\sigma}{2}$ and $0 \le \frac{\xi}{2}$, the continuous embedding $D(A^{\theta_2}) \to D(A^{\theta_1})$, $\theta_2 > \theta_1$, lead to

$$\beta\kappa\gamma\|\phi_x+\psi\|^2 + b\beta\gamma\|A^{\frac{1}{2}}\psi\|^2 \quad \le \quad C_\delta\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}}. \quad (43)$$

On the other hand, performing the duality product of (40) for $\beta\kappa\rho_3\theta$, and remembering that the operators A^v for all $v \in \mathbb{R}$ are self-adjoint and from (39), result in

$$\beta\kappa\delta\|A^{\frac{1}{2}}\theta\|^2 = \delta\beta\kappa\|\Theta\|^2 + \beta\kappa\rho_3\langle\Theta,f^5\rangle + \beta\kappa\gamma\langle\Psi,\theta_x\rangle$$
$$- i\kappa\beta K\lambda\|A^{\frac{\xi}{2}}\theta\|^2 + \kappa\beta K\langle A^{\frac{\xi}{2}}f^5, A^{\frac{\xi}{2}}\theta\rangle + \beta\kappa\rho_3\langle f^6,\theta\rangle$$

Taking real part and by using the inequalities Cauchy-Schwarz and Young and norms $\|F\|_{\mathsf{H}}$ and $\|U\|_{\mathsf{H}}$, from $|\lambda| > 1$, for $\varepsilon > 0$ exists $C_\varepsilon > 0$ which does not depend on λ such that

$$\beta\kappa\delta\|A^{\frac{1}{2}}\theta\|^2 \quad \le \quad C\{\|\Theta\|^2 + \|\Theta\|\|f^5\| + \|A^{\frac{\xi}{2}}f^5\|\|A^{\frac{\xi}{2}}\theta\| + \|f^6\|\|\theta\|\} + C_\varepsilon\|\Psi\|^2 + \varepsilon\|A^{\frac{1}{2}}\theta\|^2.$$

Then, from estimative (41) and the fact $0 \le \frac{\sigma}{2}$ and $0 \le \frac{\xi}{2} \le \frac{1}{2}$ the continuous embedding $D(A^{\theta_2}) \to D(A^{\theta_1})$, $\theta_2 > \theta_1$, lead to

$$\beta\kappa\delta\|A^{\frac{1}{2}}\theta\|^2 \le C_\delta\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}}. \quad (44)$$

Finally, from estimates (41), (43) and (44), the proof of this lemma is completed.

Lemma 4 *Let $\delta > 0$. There exists a constant $C_\delta > 0$ such that the solutions of the system* (15)–(17) *and* (21)–(22) *for $|\lambda| \ge \delta$ satisfy the inequalities*

$$(i)\ |\lambda|\beta\gamma[\kappa\|\phi_x+\psi\|^2 + b\|A^{\frac{1}{2}}\psi\|^2] \le |\lambda|\beta\gamma[\rho_1\|\Phi\|^2 + \rho_2\|\Psi\|^2] + C_\delta\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}}. \quad (45)$$

$$(ii)\ \beta\kappa\delta|\lambda|\|A^{\frac{1}{4}}\theta\|^2 \quad \le \quad \beta\kappa\rho_3|\lambda|\|\Theta\|^2 + C_\delta\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}}. \quad (46)$$

Proof: **Item** (*i*)**:** Performing the duality product of (36) for $\beta\gamma\rho_1\lambda\phi$, and remembering that the operators A^v for all $v \in \mathbb{R}$ are self-adjunct, result in

$$\beta\kappa\gamma\lambda\langle(\phi_x+\psi),\phi_x\rangle \;=\; \beta\gamma\rho_1\lambda\|\Phi\|^2 + \beta\gamma\rho_1\langle\lambda\Phi,f^1\rangle - i\mu_1\beta\gamma\|A^{\frac{\tau}{2}}\Phi\|^2$$
$$-i\mu_1\beta\gamma\langle A^\tau\Phi,f^1\rangle + i\beta\gamma\rho_1\langle f^2,\Phi\rangle + i\beta\gamma\rho_1\langle f^2,f^1\rangle.$$

Now performing the duality product of (38) for $\beta\gamma\rho_2\lambda\psi$, and remembering that the operators A^v for all $v \in \mathbb{R}$ are self-adjunct leads to

$$\lambda\beta\gamma[\kappa\langle(\phi_x+\psi),\psi\rangle + b\|A^{\frac{1}{2}}\psi\|^2] = \beta\gamma\rho_2\lambda\|\Psi\|^2 + \beta\gamma\rho_2\langle\lambda\Psi,f^3\rangle + i\beta^2\gamma\langle\Theta,\psi_x\rangle$$
$$+i\beta^2\gamma\langle\Theta,f_x^3\rangle - i\mu_2\beta\gamma\|A^{\frac{\sigma}{2}}\Psi\|^2 - i\mu_2\beta\gamma\langle A^\sigma\Psi,f^3\rangle$$
$$+i\beta\gamma\rho_2\langle f^4,\Psi\rangle + i\beta\gamma\rho_2\langle f^4,f^3\rangle.$$

Adding the last 2 equations, result in

$$\lambda\beta\gamma[\kappa\|\phi_x+\psi\|^2 + b\|A^{\frac{1}{2}}\psi\|^2] = \lambda\beta\gamma[\rho_1\|\Phi\|^2 + \rho_2\|\Psi\|^2] - i\beta\gamma\{\mu_1\|A^{\frac{\tau}{2}}\Phi\|^2$$
$$+ \mu_2\|A^{\frac{\sigma}{2}}\Psi\|^2\} + \beta\gamma\rho_1\langle\lambda\Phi,f^1\rangle + \beta\gamma\rho_2\langle\lambda\Psi,f^3\rangle - i\beta\gamma\{\mu_1\langle A^\tau\Phi,f^1\rangle$$
$$+ \mu_2\langle A^\sigma\Psi,f^3\rangle\} + i\beta\gamma\rho_1\langle f^2,\Phi\rangle + i\beta\gamma\rho_2\langle f^4,\Psi\rangle + i\beta\gamma\rho_1\langle f^2,f^1\rangle$$
$$+ i\beta\gamma\rho_2\langle f^4,f^3\rangle + i\beta^2\gamma\langle\Theta,\psi_x\rangle + i\beta^2\gamma\langle\Theta,f_x^3\rangle. \quad (47)$$

On the other hand, from (36) and (38), result in

$$\beta\gamma\{\rho_1\langle\lambda\Phi,f^1\rangle + \rho_2\langle\lambda\Psi,f^3\rangle\} = i\beta\gamma\{\kappa\langle(\phi_x+\psi),f_x^1\rangle$$
$$+ \mu_1\langle A^\tau\Phi,f^1\rangle - \rho_1\langle f^2,f^1\rangle + b\langle A^{\frac{1}{2}}\psi,A^{\frac{1}{2}}f^3\rangle + \kappa\langle(\phi_x+\psi),f^3\rangle$$
$$- \beta\langle\Theta,f_x^3\rangle + \mu_2\langle A^\sigma\Psi,f^3\rangle - \rho_2\langle f^4,f^3\rangle\}. \quad (48)$$

Using the identity (48) in the (47) equation and simplifying, lead to

$$\lambda\beta\gamma[\kappa\|\phi_x+\psi\|^2 + b\|A^{\frac{1}{2}}\psi\|^2] = \lambda\beta\gamma[\rho_1\|\Phi\|^2 + \rho_2\|\Psi\|^2]$$
$$- i\{\mu_1\|A^{\frac{\tau}{2}}\Phi\|^2 + \mu_2\|A^{\frac{\sigma}{2}}\Psi\|^2\} + i\beta\gamma\kappa\langle\phi_x,f_x^1\rangle + i\beta\gamma\kappa\langle\psi,f_x^1\rangle$$
$$+ ib\beta\gamma\langle A^{\frac{1}{2}}\psi,A^{\frac{1}{2}}f^3\rangle i\beta\gamma\kappa\langle\phi_x,f^3\rangle + i\beta\gamma\kappa\langle\psi,f^3\rangle$$
$$+ i\beta\gamma\rho_1\langle f^2,\Phi\rangle + i\beta\gamma\rho_2\langle f^4,\Psi\rangle + i\beta^2\gamma\langle\Theta,\psi_x\rangle. \quad (49)$$

As for $\varepsilon > 0$ exists $C_\varepsilon >$ such that $|i\beta^2\gamma\,\Theta,\,\psi| \le \varepsilon\|A^{\frac{1}{2}}\psi\|^2 + C_\varepsilon\|\Theta\|^2$, taking the real part, and applying the Cauchy-Schwarz and Young inequalities, estimative (34) of Lemma(3) and using the definitions of the F and U norm in (49) the proof of item (i) this lemma is completed.

On the other hand, performing the duality product of (40) for $\beta\kappa\rho_3\lambda\theta$, and remembering that the operators A^θ for all $\theta \in \mathbb{R}$ are self-adjunct, result in

$$\beta\kappa\delta\lambda\|A^{\frac{1}{2}}\theta\|^2 \;=\; \beta\kappa\rho_3\langle\lambda\Theta,i\lambda\theta\rangle \;-\; \beta\kappa\gamma\langle\psi_x,\lambda\theta\rangle \;+\; \beta\kappa K\langle A^\xi\Theta,\lambda\theta\rangle \;+\; \beta\kappa\rho_3\langle f^6,\lambda\theta\rangle. \quad (50)$$

As

$$\rho_3\langle\lambda\Theta,i\lambda\theta\rangle \;=\; \rho_3\lambda\|\Theta\|^2 + \rho_3\langle\lambda\Theta,f^5\rangle$$
$$=\; \rho_3\lambda\|\Theta\|^2 + i\delta\langle A^{\frac{1}{2}}\theta,A^{\frac{1}{2}}f^5\rangle - i\gamma\langle\Psi,f_x^5\rangle + iK\langle A^\xi\Theta,f^5\rangle - i\rho_3\langle f^6,f^5\rangle. \quad (51)$$
$$-\gamma\langle\psi_x,\lambda\theta\rangle \;=\; -i\gamma\langle\psi_x,\Theta\rangle - i\gamma\langle\psi_x,f^5\rangle. \quad (52)$$
$$K\langle A^\xi\Theta,\lambda\theta\rangle \;=\; -iK\|A^{\frac{\xi}{2}}\Theta\|^2 - iK\langle A^\xi\Theta,f^5\rangle. \quad (53)$$
$$\rho_3\langle f^6,\lambda\theta\rangle \;=\; i\rho_3\langle f^6,\Theta\rangle + i\rho_3\langle f^6,f^5\rangle. \quad (54)$$

Using (51)–(54) in (50), lead to

$$\beta\kappa\delta\lambda\|A^{\frac{1}{2}}\theta\|^2 = \beta\kappa\rho_3\lambda\|\Theta\|^2 + i\beta\kappa\delta\langle A^{\frac{1}{2}}\theta, A^{\frac{1}{2}}f^5\rangle - i\beta\kappa\gamma\langle\Psi, f_x^5\rangle - i\beta\kappa\gamma\langle\psi_x, \Theta\rangle$$
$$- i\beta\kappa\gamma\langle\psi_x, f^5\rangle - i\beta\kappa K\|A^{\frac{\xi}{2}}\Theta\|^2 + i\beta\kappa\rho_3\langle f^6, \Theta\rangle. \quad (55)$$

Taking real part of the equation (55) and using the inequalities Cauchy-Schwarz and Young and norms $\|F\|_H$ and $\|U\|_H$, from $|\lambda| > 1$, for $\varepsilon > 0$ exists $C_\varepsilon > 0$ such that

$$\beta\kappa\delta|\lambda|\|A^{\frac{1}{2}}\theta\|^2 \leq \beta\kappa\rho_3|\lambda|\|\Theta\|^2 + C\{\|A^{\frac{1}{2}}\theta\|\|A^{\frac{1}{2}}f^5\| + \|\Psi\|\|f_x^5\|$$
$$+ \|\psi_x\|^2 + \|\Theta\|^2 + \|\psi_x\|\|f^5\| + \|f^6\|\|\theta\|\}.$$

Then, from Lemma(3) and as $\|\psi_x\|^2 = \|A^{\frac{1}{2}}\psi\|^2$, the proof of item (*ii*) of this lemma is finished. Q

3.1.1 Gevrey class of the first system

Definition 5 *Let $t_0 \geq 0$ be a real number. A strongly continuous semigroup S(t), defined on a Banach space* H*, is of Gevrey class s > 1 for t > t_0, if S(t) is infinitely differentiable for t > t_0, and for every compact set $K \subset (t_0,\infty)$ and each $\mu > 0$, there exists a constant C = C(μ, K) > 0 such that*

$$\|S^{(n)}(t)\|_{\mathcal{L}(\mathcal{H})} \leq C\mu^n(n!)^s, \quad for\ all \quad t \in K, n = 0, 1, 2... \quad (56)$$

Theorem 6 ([19]) *Let S(t) be a strongly continuous and bounded semigroup on a Hilbert space . Suppose that the infinitesimal generator of the semigroup S(t) satisfies the following estimate, for some 0 < r < 1:*

$$\lim_{|\lambda|\to\infty} \sup |\lambda|^r\|(i\lambda I - \mathcal{B})^{-1}\|_{\mathcal{L}(\mathcal{H})} < \infty. \quad (57)$$

Then S(t) is of Gevrey class s for t > 0, for every $s > \frac{1}{r}$.

The main result of this subsection is as follows:

Theorem 7 *The semigroup S(t) = $e^{B_1 t}$ associated to the system* (15)–(17) *and* (21)–(22) *is of Gevrey class s for every $s > \frac{1}{r}$ for r = $\min\{\frac{\tau}{1+\tau}, \frac{\sigma}{1+\sigma}, \frac{\xi}{1+\xi}\} \in (0, 1)$, with ($\tau$, σ, ξ) $\in R_{CG}:=(0, 1]^3$.*

Proof: From the resolvent equation $F = (i\lambda I - B_1)U$ for $\lambda \in R$, $U = (i\lambda I - B_1)^{-1}F$. Furthermore, show (57) of theorem(6) it is enough to show, that for $\varepsilon > 0$ exists $C_\delta > 0$, such that:

$$\frac{|\lambda|^r\|(i\lambda - \mathcal{B}_1)^{-1}F\|_{\mathcal{H}}}{\|F\|_{\mathcal{H}}} \leq C_\delta \iff |\lambda|^r\|U\|_{\mathcal{H}} \leq C_\delta\|F\|_{\mathcal{H}}, \quad F \in \mathcal{H} \quad \text{and} \quad C_\delta > 0.$$

Equivalent to

$$|\lambda|^r\|U\|_{\mathcal{H}}^2 \leq C_\delta\{\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}}\} \quad \text{for} \quad r \in (0, 1). \quad (58)$$

where $r = \min\left\{\frac{\tau}{1+\tau}, \frac{\sigma}{1+\sigma}, \frac{\xi}{1+\xi}\right\}$, $0 < \tau \leq 1$, $0 < \sigma \leq 1$ and $0 < \xi \leq 1$.
Next, $|\lambda|^{\frac{\tau}{1+\tau}}\|\Phi\| + |\lambda|^{\frac{\sigma}{1+\sigma}}\|\Psi\| + |\lambda|^{\frac{\xi}{1+\xi}}\|\Theta\|$ will be estimated.
Let's start by estimating the term $|\lambda|^{\frac{\tau}{1+\tau}}\|\Phi\|$**:** It is assume that $|\lambda| > 1$, some ideas

could be borrowed from [13]. Set $\phi = \phi_1 + \phi_2$, where $\phi_1 \in D(A)$ and $\phi_2 \in D(A^0)$, with

$$i\lambda\Phi_1 + A\Phi_1 = f^2, \qquad\qquad i\lambda\Phi_2 = \frac{\kappa}{\rho_1}A\phi + \frac{\kappa}{\rho_1}\psi_x - \frac{\mu_1}{\rho_1}A^\tau\Phi + A\Phi_1. \qquad (59)$$

Firstly, applying in the product duality the first equation in (59) by ϕ_1, then by $A\phi_1$ and recalling that the operator A is self-adjoint, resulting in

$$i\lambda\|\Phi_1\|^2 + \|A^{\frac{1}{2}}\Phi_1\|^2 = \langle f^2, \Phi_1 \rangle, \qquad (60)$$

taking first the imaginary part of (60) and in the sequence, the real part in addition to applyingCauchy-Schwarz inequality, results in

$$|\lambda|\|\Phi_1\| \leq C\|F\|_{\mathcal{H}} \qquad \text{and} \qquad \|A^{\frac{1}{2}}\Phi_1\| \leq C\|F\|_{\mathcal{H}}. \qquad (61)$$

Applying the $A^{-\frac{1}{2}}$ operator on the second equation of (59), result in

$$i\lambda A^{-\frac{1}{2}}\Phi_2 = \frac{\kappa}{\rho_1}A^{\frac{1}{2}}\phi + \frac{\kappa}{\rho_1}A^{-\frac{1}{2}}\psi_x - \frac{\mu_1}{\rho_1}A^{\tau-\frac{1}{2}}\Phi + A^{\frac{1}{2}}\Phi_1,$$

then, as $\|A^{-\frac{1}{2}}\psi_x\|^2 = \langle -A^{-\frac{1}{2}}\psi_{xx}, A^{-\frac{1}{2}}\psi \rangle = \langle A^{\frac{1}{2}}\psi, A^{-\frac{1}{2}}\psi \rangle = \|\psi\|^2$ and $\tau - \frac{1}{2} \leq \frac{\tau}{2}$ taking into account the continuous embedding $D(A^{\theta 2}) \to D(A^{\theta 1})$, $\theta_2 > \theta_1$ and using (60), result in

$$|\lambda|\|A^{-\frac{1}{2}}\Phi_2\| \leq C\{\|A^{\frac{1}{2}}\phi\| + \|A^{\frac{\tau}{2}}\Phi\| + \|\psi\|\} + \|A^{\frac{1}{2}}\Phi_1\|, \qquad (62)$$

using (34), (41) and second estimative the equation (61), lead to

$$\|A^{-\frac{1}{2}}\Phi_2\|^2 \leq C|\lambda|^{-2}\{\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}} + \|F\|_{\mathcal{H}}^2\}. \qquad (63)$$

On the other hand, from $\phi_2 = \phi - \phi_1$, (41) and as $\frac{\tau}{2} \leq \frac{1}{2}$ the inequality of (60), result in

$$\|A^{\frac{\tau}{2}}\Phi_2\|^2 \quad \leq \quad C\{\|A^{\frac{\tau}{2}}\Phi\|^2 + \|A^{\frac{\tau}{2}}\Phi_1\|^2\} \leq C\{\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}} + \|F\|_{\mathcal{H}}^2\}. \qquad (64)$$

By Lions' interpolations inequality $0 \in [-\frac{1}{2}, \frac{\tau}{2}]$, result in

$$\|\Phi_2\|^2 \leq C(\|A^{-\frac{1}{2}}\Phi_2\|^2)^{\frac{\tau}{1+\tau}}(\|A^{\frac{\tau}{2}}\Phi_2\|^2)^{\frac{1}{1+\tau}}. \qquad (65)$$

Then, using (63) and (64) in (65), for $|\lambda| > 1$, result in

$$\|\Phi_2\|^2 \leq C|\lambda|^{\frac{-2\tau}{(1+\tau)}}\{\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}} + \|F\|_{\mathcal{H}}^2\}. \qquad (66)$$

Therefore, as $\|\phi\|^2 \leq \|\phi_1\|^2 + \|\phi_2\|^2$ from first inequality of (61), (66) and $|\lambda|^{-2} \leq |\lambda|^{\frac{-2\tau}{1+\tau}}$, result in

$$|\lambda|^{\frac{\tau}{1+\tau}}\|\Phi\| \leq C_\delta\|F\|_{\mathcal{H}} \quad \Longleftrightarrow \quad |\lambda|\|\Phi\|^2 \leq C_\delta|\lambda|^{\frac{1-\tau}{1+\tau}}\|F\|_{\mathcal{H}}^2 \qquad \text{for} \qquad 0 \leq \tau \leq 1. \qquad (67)$$

On the other hand, let's now estimate the missing term $|\lambda|^{\frac{\sigma}{1+\sigma}}\|\Psi\|$: It is assumed that $|\lambda| > 1$. Set $\Psi = \Psi_1 + \Psi_2$, where $\Psi_1 \in D(A)$ and $\Psi_2 \in D(A^0)$, with

$$i\lambda\Psi_1 + A\Psi_1 = f^4 \qquad \text{and} \qquad i\lambda\Psi_2 = -\frac{b}{\rho_2}A\psi - \frac{\kappa}{\rho_2}\phi_x - \frac{\kappa}{\rho_2}\psi - \frac{\beta}{\rho_2}\Theta_x - \frac{\mu_2}{\rho_2}A^\sigma\Psi + A\Psi_1. \qquad (68)$$

Firstly, applying the product duality the first equation in (68) by Ψ_1, and as the operators A^v forall $v \in \mathbb{R}$ are self-adjoint, result in

$$i\lambda\|\Psi_1\|^2 + \|A^{\frac{1}{2}}\Psi_1\|^2 = \langle f^4, \Psi_1\rangle. \tag{69}$$

Taking first the imaginary part of (69) and in the sequence, the real part in addition to applyingCauchy-Schwarz inequality, results in

$$|\lambda|\|\Psi_1\| \leq C\|F\|_{\mathcal{H}} \qquad \|A^{\frac{1}{2}}\Psi_1\| \leq C\|F\|_{\mathcal{H}}. \tag{70}$$

As follows from the second equation in (68) that

$$i\lambda A^{-\frac{1}{2}}\Psi_2 = -\frac{b}{\rho_2}A^{\frac{1}{2}}\psi - \frac{\kappa}{\rho_2}A^{-\frac{1}{2}}\phi_x - \frac{\kappa}{\rho_2}A^{-\frac{1}{2}}\psi - \frac{\beta}{\rho_2}A^{-\frac{1}{2}}\Theta_x - \frac{\mu_2}{\rho_2}A^{\sigma-\frac{1}{2}}\Psi + A^{\frac{1}{2}}\Psi_1,$$

then, as $\|A^{-\frac{1}{2}}\phi_x\|^2 = \langle -A^{-\frac{1}{2}}\phi_{xx}, A^{-\frac{1}{2}}\phi\rangle = \|\phi\|^2$, $\|A^{-\frac{1}{2}}\Theta_x\|^2 = \|\Theta\|^2$ and $\sigma - \frac{1}{2} \leq \frac{\sigma}{2}$taking into account the continuous embedding $D(A^{\theta_2}) \rightarrow D(A^{\theta_1})$, $\theta_2 > \theta_1$, lead to

$$|\lambda|\|A^{-\frac{1}{2}}\Psi_2\| \leq C\{\|A^{\frac{1}{2}}\phi\| + \|\psi\| + \|\Theta\| + \|A^{\frac{\sigma}{2}}\Psi\|\} + \|A^{\frac{1}{2}}\Psi_1\| \tag{71}$$

Using estimative (34) and estimative (68), for |λ| > 1, result in

$$\|A^{-\frac{1}{2}}\Psi_2\|^2 \leq C|\lambda|^{-2}\{\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}} + \|F\|_{\mathcal{H}}^2\}. \tag{72}$$

On the other hand, from $\Psi_2 = \Psi - \Psi_1$, (41) and as $\frac{\sigma}{2} \leq \frac{1}{2}$ the inequality (70), result in

$$\|A^{\frac{\sigma}{2}}\Psi_2\|^2 \quad \leq \quad C\{\|A^{\frac{\sigma}{2}}\Psi\|^2 + \|A^{\frac{\sigma}{2}}\Psi_1\|^2\} \leq C\{\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}} + \|F\|_{\mathcal{H}}^2\}. \tag{73}$$

Now, by Lions' interpolations inequality $0 \in [-\frac{1}{2}, \frac{\sigma}{2}]$, lead to

$$\|\Psi_2\|^2 \leq C(\|A^{-\frac{1}{2}}\Psi_2\|^2)^{\frac{\sigma}{1+\sigma}}(\|A^{\frac{\sigma}{2}}\Psi_2\|^2)^{\frac{1}{1+\sigma}}. \tag{74}$$

Then, using (72) and (73) in (74), for |λ| > 1, result in

$$\|\Psi_2\|^2 \leq C|\lambda|^{\frac{-2\sigma}{(1+\sigma)}}\{\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}} + \|F\|_{\mathcal{H}}\}. \tag{75}$$

Therefore, as $\|\Psi\|^2 \leq \|\Psi\|^2 + \|\Psi\|^2$ from inequality (70), (75) and $|\lambda|^{-2} \leq |\lambda|^{\frac{-2\sigma}{1+\sigma}}$, result in

$$|\lambda|^{\frac{\sigma}{1+\sigma}}\|\Psi\| \leq C_\delta\|F\|_{\mathcal{H}} \quad \Longleftrightarrow \quad |\lambda|\|\Psi\|^2 \leq |\lambda|^{\frac{1-\sigma}{1+\sigma}}\|F\|_{\mathcal{H}}^2 \qquad \text{for} \qquad 0 \leq \sigma \leq 1. \tag{76}$$

Finally, let's now estimate the missing term $|\lambda|^{\frac{\xi}{1+\xi}}\|\Theta\|$: Now we assume |λ| > 1. Set $\Theta = \Theta_1 + \Theta_2$, where $\Theta_1 \in D(A)$ and $\Theta_2 \in D(A^0)$, with

$$i\lambda\Theta_1 + A\Theta_1 = f^6 \qquad \text{and} \qquad i\lambda\Theta_2 = -\frac{\delta}{\rho_3}A\theta - \frac{\gamma}{\rho_3}\Psi_x - \frac{K}{\rho_3}A^\xi\Theta + A\Theta_1. \tag{77}$$

Firstly, applying the product duality the first equation in (77) by Θ_1 and as the operators A^v forall $v \in \mathbb{R}$ are self-adjoint, lead to

$$i\lambda\|\Theta_1\|^2 + \|A^{\frac{1}{2}}\Theta_1\|^2 = \langle f^6, \Theta_1\rangle. \tag{78}$$

Taking first the imaginary part of (78) and in the sequence, the real part and applying Cauchy-Schwarz inequality, result in

$$|\lambda|\|\Theta_1\| \leq C\|F\|_{\mathcal{H}} \qquad \text{and} \qquad \|A^{\frac{1}{2}}\Theta_1\| \leq C\|F\|_{\mathcal{H}}. \tag{79}$$

As follows from the second equation in (77) that

$$i\lambda A^{-\frac{1}{2}}\Theta_2 = -\frac{\delta}{\rho_3}A^{\frac{1}{2}}\theta - \frac{\gamma}{\rho_3}A^{-\frac{1}{2}}\Psi_x - \frac{K}{\rho_3}A^{\xi-\frac{1}{2}}\Theta + A^{\frac{1}{2}}\Theta_1,$$

then, as $\|A^{-\frac{1}{2}}\Psi_x\|^2 = \|\Psi\|^2$ and $-\frac{1}{2} \leq 0$, $\xi - \frac{1}{2} \leq \frac{\xi}{2}$ taking into account the continuous embedding $D(A^{\theta_2}) \rightarrow D(A^{\theta_1})$, $\theta_2 > \theta_1$, result in

$$|\lambda|\|A^{-\frac{1}{2}}\Theta_2\| \leq C\{\|A^{\frac{1}{2}}\theta\| + \|A^{\frac{\xi}{2}}\Theta\| + \|\Psi\|\} + \|A^{\frac{1}{2}}\Theta_1\| \tag{80}$$

Using estimatites (34), (41) and second estimative of equation (79), for $|\lambda| > 1$, lead to

$$\|A^{-\frac{1}{2}}\Theta_2\|^2 \leq C|\lambda|^{-2}\{\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}} + \|F\|_{\mathcal{H}}^2\}. \tag{81}$$

On the other hand, from $\Theta_2 = \Theta - \Theta_1$, (41) and as $\frac{\xi}{2} \leq \frac{1}{2}$ the second inequality of (79), result in

$$\|A^{\frac{\xi}{2}}\Theta_2\|^2 \leq C\{\|A^{\frac{\xi}{2}}\Theta\|^2 + \|A^{\frac{\xi}{2}}\Theta_1\|^2\} \leq C\{\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}} + \|F\|_{\mathcal{H}}^2\}. \tag{82}$$

Now, by Lions' interpolations inequality $0 \in [-\frac{1}{2}, \frac{\xi}{2}]$, result in

$$\|\Theta_2\|^2 \leq C(\|A^{-\frac{1}{2}}\Theta_2\|^2)^{\frac{\xi}{1+\xi}}(\|A^{\frac{\xi}{2}}\Theta_2\|^2)^{\frac{1}{1+\xi}}. \tag{83}$$

Then, using (81) and (82) in (83), for $|\lambda| > 1$, result in

$$\|\Theta_2\|^2 \leq C|\lambda|^{\frac{-2\xi}{(1+\xi)}}\{\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}} + \|F\|_{\mathcal{H}}\}. \tag{84}$$

Therefore, as $\|\Theta\|^2 \leq \|\Theta\|^2 + \|\Theta\|^2$ from first inequality of (79), estimative (84) and $|\lambda|^{-2} \leq |\lambda|^{\frac{-2\xi}{1+\xi}}$, result in

$$|\lambda|^{\frac{\xi}{1+\xi}}\|\Theta\| \leq C_\delta\|F\|_{\mathcal{H}} \iff |\lambda|\|\Theta\|^2 \leq C_\delta|\lambda|^{\frac{1-\xi}{1+\xi}}\|F\|_{\mathcal{H}}^2 \qquad \text{for} \qquad 0 \leq \xi \leq 1. \tag{85}$$

Using the estimates (67) and (76) in the inequality item (i) of Lemma(4), result in

$$\beta\gamma\kappa\|\phi_x + \psi\|^2 + \beta\gamma b\|A^{\frac{1}{2}}\psi\|^2 \leq C_\delta\{|\lambda|^{\frac{-2\tau}{1+\tau}} + |\lambda|^{\frac{-2\sigma}{1+\sigma}}\}\|F\|_{\mathcal{H}}^2 + C_\delta|\lambda|^{-1}\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}}. \tag{86}$$

Furthermore using estimative (85) in estimate (46) item (ii) of the Lemma 4, we have

$$\beta\kappa\delta\|A^{\frac{1}{2}}\theta\|^2 \leq \beta\kappa\rho_3\|\Theta\|^2 + C_\delta|\lambda|^{-1}\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}} \leq C_\delta|\lambda|^{\frac{-2\xi}{1+\xi}}\|F\|^2 + C_\delta|\lambda|^{-1}\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}}. \tag{87}$$

Finally summing the estimates (67),(76), (85),(86) and (87), we have

$$\|U\|_{\mathcal{H}}^2 \leq C_\delta\left\{|\lambda|^{\frac{-2\tau}{1+\tau}} + |\lambda|^{\frac{-2\sigma}{1+\sigma}} + |\lambda|^{\frac{-2\xi}{1+\xi}}\right\}\|F\|_{\mathcal{H}}^2 + C_\delta|\lambda|^{-1}\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}},$$

as for $(\tau, \sigma, \xi) \in [0, 1]^3$, we have $-1 \leq 2\max\{-\frac{\tau}{1+\tau}, -\frac{\sigma}{1+\sigma}, -\frac{\xi}{1+\xi}\} = -2\min\{\frac{\tau}{1+\tau}, \frac{\sigma}{1+\sigma}, \frac{\xi}{1+\xi}\}$, then

$$\|U\|_{\mathcal{H}}^2 \leq C_\delta |\lambda|^{-2\min\{\frac{\tau}{1+\tau}, \frac{\sigma}{1+\sigma}, \frac{\xi}{1+\xi}\}} \left[\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}} + \|F\|_{\mathcal{H}}^2\right]$$

Equivalently

$$|\lambda|^{\min\{\frac{\tau}{1+\tau}, \frac{\sigma}{1+\sigma}, \frac{\xi}{1+\xi}\}}\|U\|_{\mathcal{H}} \leq C_\delta\|F\|_{\mathcal{H}} \qquad \text{for} \qquad (\tau, \sigma, \xi) \in [0, 1]^3.$$

As $r = \min\{\frac{\tau}{1+\tau}, \frac{\sigma}{1+\sigma}, \frac{\xi}{1+\xi}\} \in (0, 1)$ for $(\tau, \sigma, \xi) \in (0, 1]^3$ the proof of this theorem is finished.

3.1.2 Analyticity of $S(t) = e^{B1t}$ for $(\tau, \sigma, \xi) \in [\frac{1}{2}, 1]^3$

Next, we announce the Liu-Zheng Book Theorem that will be applied in the proof of the analyticity of $S(t) = e^{Bit}$ for $i = 1, 2$ in their respective regions.

Theorem 8 (see [12]) *Let $S(t) = e^{Bt}$ be C_0-semigroup of contractions on a Hilbert space H. Suppose that*

$$\rho(\mathcal{B}) \supseteq \{i\lambda/\lambda \in \mathbb{R}\} \equiv i\mathbb{R} \tag{88}$$

Then $S(t)$ is analytic if and only if

$$\limsup_{|\lambda|\to\infty} \|\lambda(i\lambda I - \mathcal{B})^{-1}\|_{\mathcal{L}(\mathcal{H})} < \infty \tag{89}$$

holds.

Before proving the main result of this section, the following two lemmas will be proved.

Lemma 9 *The C_0–semigroup of contractions $S(t) = e^{B1t}$ satisfies the condition*

$$\rho(\mathcal{B}_1) \supseteq \{i\lambda/\lambda \in \mathbb{R}\} \equiv i\mathbb{R}. \tag{90}$$

Proof: Let's prove $i\mathbb{R} \subset \rho(B_1)$ by contradiction, it is supposed that $i\mathbb{R} \not\subset \rho(B_1)$. As $0 \in \rho(B_1)$ and $\rho(B_1)$ is open, the highest positive number λ_0 is considered such that the interval $(-i\lambda_0, i\lambda_0) \subset \rho(B_1)$, then $i\lambda_0$ or $-i\lambda_0$ is an element of the spectrum $\sigma(B_1)$. It is supposed that $i\lambda_0 \in \sigma(B_1)$ (if $-i\lambda_0 \in \sigma(B_1)$ the proceeding is similar). Then, for $0 < v < \lambda_0$ there exist a sequence of real numbers (λ_n), with $0 < v \leq \lambda_n < \lambda_0$, $\lambda_n \to \lambda_0$, and a vector sequence $U_n = (\phi_n, \Phi_n, \psi_n, \Psi_n, \theta_n, \Theta_n) \in D(B_1)$ with unitary norms, such that

$$\|(i\lambda_n - \mathcal{B}_1)U_n\|_{\mathcal{H}} = \|F_n\|_{\mathcal{H}} \to 0,$$

as $n \to \infty$. From (43) for $0 \leq \tau \leq 1$ and $0 \leq \sigma \leq 1$, result in

$$\beta\kappa\gamma\|\phi_{xn} + \psi_n\|^2 + b\beta\gamma\|A^{\frac{1}{2}}\psi_n\|^2 \leq C_\delta\|F_n\|_{\mathcal{H}}\|U_n\|_{\mathcal{H}}.$$

From (44), for $0 \leq \xi \leq 1$, result in

$$\beta\kappa\delta\|A^{\frac{1}{2}}\theta_n\|^2 \leq C_\delta\|F_n\|_{\mathcal{H}}\|U_n\|_{\mathcal{H}}. \tag{91}$$

In addition to the estimative (41), for $0 \leq \tau \leq 1$ and $0 \leq \sigma \leq 1$, result in

$$\beta\gamma\mu_1\|A^{\frac{\tau}{2}}\Phi_n\|^2 + \kappa\gamma\mu_2\|A^{\frac{\sigma}{2}}\Psi_n\|^2 + \frac{\beta\kappa K}{\rho_3}\|A^{\frac{\xi}{2}}\Theta_n\|^2 \leq C_\delta\|F_n\|_{\mathcal{H}}\|U_n\|_{\mathcal{H}}.$$

Consequently, $\|U_n\|^2_{\mathsf{H}} \to 0$. Therefore, lead to $\|U_n\|_{\mathsf{H}} \to 0$ but this is absurd, since $\|U_n\|_{\mathsf{H}} = 1$ for all $n \in \mathbb{N}$. Thus, $i\mathbb{R} \subset \rho(B_1)$. This completes the proof.

Lemma 10 *Let $\delta > 0$. There exists a constant $C_\delta > 0$ such that the solutions of* (18)–(20) *and* (21)–(22) *for* $|\lambda| \geq \delta$ *satisfy the inequality*

$$(i) \quad |\lambda|\|\Phi\|^2 \leq C_\delta\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}} \qquad \text{for} \quad \frac{1}{2} \leq \tau \leq 1. \tag{92}$$

$$(ii) \quad |\lambda|\|\Psi\|^2 \leq C_\delta\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}} \qquad \text{for} \quad \frac{1}{2} \leq \sigma \leq 1. \tag{93}$$

$$(iii) \quad |\lambda|\|\Theta\|^2 \leq C_\delta\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}} \qquad \text{for} \quad \frac{1}{2} \leq \xi \leq 1. \tag{94}$$

Proof: **Item** (*i*): of this lemma will be initially shown, performing the duality product of (36) for $\frac{\rho_1}{\mu_1} A^{-\tau} \lambda\phi$, and recalling that the operator A^v is self-adjoint for all $v \in \mathbb{R}$, results in

$$\begin{aligned}
\lambda\|\Phi\|^2 &= \frac{\kappa}{\mu_1}\langle\lambda(-A\phi + \psi_x), A^{-\tau}\Phi\rangle + \frac{\rho_1}{\mu_1}\langle f^2, \lambda A^{-\tau}\Phi\rangle - i\frac{\rho_1}{\mu_1}\lambda^2\|A^{-\frac{\tau}{2}}\Phi\|^2 \\
&= i\frac{\kappa}{\mu_1}\langle(A\Phi + Af^1), A^{-\tau}\Phi\rangle - i\frac{\kappa}{\mu_1}\langle\Psi_x + f_x^3, A^{-\tau}\Phi\rangle \\
&\quad + \frac{1}{\mu_1}\langle f^2, i\kappa A^{1-\tau}\phi - i\kappa A^{-\tau}\psi_x + i\mu_1\Phi - i\rho_1 A^{-\tau}f_2\rangle - i\frac{\rho_1}{\mu_1}\lambda^2\|A^{-\frac{\tau}{2}}\Phi\|^2 \\
&= i\frac{\kappa}{\mu_1}\|A^{\frac{1-\tau}{2}}\Phi\|^2 + i\frac{\kappa}{\mu_1}\langle A^{\frac{1}{2}}f^1, A^{\frac{1}{2}-\tau}\Phi\rangle + i\frac{\kappa}{\mu_1}\langle\Psi, A^{-\tau}\Phi_x\rangle - i\frac{\kappa}{\mu_1}\langle f_x^3, A^{-\tau}\Phi\rangle \\
&\quad - i\langle f^2, \Phi\rangle - i\frac{\kappa}{\mu_1}\langle f^2, A^{1-\tau}\phi\rangle + i\frac{\kappa}{\mu_1}\langle A^{-\tau}f^2, \psi_x\rangle + i\frac{\rho_1}{\mu_1}\|A^{-\frac{\tau}{2}}f^2\|^2 \\
&\quad - i\frac{\rho_1}{\mu_1}\lambda^2\|A^{-\frac{\theta}{2}}\Phi\|^2.
\end{aligned}$$

Noting that: $\|A^{-\tau}\Phi_x\|^2 = \langle A^{-\tau}\Phi_x, A^{-\tau}\Phi_x\rangle = \langle-A^{-\tau}\Phi_{xx}, A^{-\tau}\Phi\rangle = \|A^{\frac{1-2\tau}{2}}\Phi\|^2$, taking real part and considering that $\frac{1}{2} \leq \tau \leq 1$ using estimative (41) and using Cauchy-Schwarz and Young inequalities, for $\varepsilon > 0$ exists C_ε, such that

$$|\lambda|\|\Phi\|^2 \leq C_\delta\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}} + \varepsilon\|A^{-\tau}\Phi_x\|^2 + C_\varepsilon\|\Psi\|^2.$$

As $0 \leq \frac{\sigma}{2}$, then from estimative (41) $\|\Psi\|^2 \leq C_\delta\|F\|_{\mathsf{H}}\|U\|_{\mathsf{H}}$. From the continuous embedding for $|\lambda| \geq 1$, the proof of item (*i*) of this lemma is finished.

Item (*ii*): Again similarly, performing the duality product of (38) for $\frac{\rho_2}{\mu_2} A^{-\sigma}\lambda\Psi$, using (37), and recalling the self-adjointness of A^v, $v \in \mathbb{R}$, leads to

$$\begin{aligned}
\lambda\|\Psi\|^2 &= -\frac{b}{\mu_2}\langle A\lambda\psi, A^{-\sigma}\Psi\rangle - \frac{\kappa}{\mu_2}\langle\lambda(\phi_x + \psi), A^{-\sigma}\Psi\rangle + \frac{\rho_2}{\mu_2}\langle f^4, \lambda A^{-\sigma}\Psi\rangle - i\frac{\rho_2}{\mu_2}\lambda^2\|A^{-\frac{\sigma}{2}}\Psi\|^2 \\
&= i\frac{b}{\mu_2}\|A^{\frac{1-\sigma}{2}}\Psi\|^2 + i\frac{b}{\mu_2}\langle A^{\frac{1}{2}}f^3, A^{\frac{1}{2}-\sigma}\Psi\rangle + i\frac{\kappa}{\mu_2}\langle A^{-\sigma}\Phi_x, \Psi\rangle + i\frac{\kappa}{\mu_2}\langle f_x^1, A^{-\sigma}\Psi\rangle \\
&\quad + i\frac{\kappa}{\mu_2}\|A^{-\frac{\sigma}{2}}\Psi\|^2 + i\frac{\kappa}{\mu_2}\langle f^3, A^{-\sigma}\Psi\rangle - i\frac{b}{\rho_2}\langle f^4, A^{1-\sigma}\psi\rangle - i\frac{\kappa}{\mu_2}\langle f^4, A^{-\sigma}\phi_x\rangle \\
&\quad - i\frac{\kappa}{\mu_2}\langle f^4, A^{-\sigma}\psi\rangle - i\frac{\beta}{\mu_2}\langle f^4, A^{-\sigma}\Theta\rangle - i\langle f^4, \Psi\rangle + i\frac{\rho_2}{\mu_2}\|A^{-\frac{\sigma}{2}}f^4\|^2 - i\frac{\rho_2}{\mu_2}\lambda^2\|A^{-\frac{\sigma}{2}}\Psi\|^2.
\end{aligned}$$

Noting that: $\|A^{\frac{1-2\sigma}{2}}\Phi\|^2 = \|A^{-\sigma}\Phi_x\|^2$, taking real part, considering $\frac{1}{2} \leq \sigma \leq 1$, using

Cauchy-Schwarz and Young inequalities, for ε > 0 exists C_ε such that

$$|\lambda|\|\Psi\|^2 \;\le\; C_\delta\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}} + C_\varepsilon\|A^{\frac{1-2\sigma}{2}}\Phi\|^2 + \varepsilon\|\Psi\|^2.$$

As $-\frac{1-2\sigma}{2} \le 0 \Longleftrightarrow \frac{1}{2} \le \sigma \le 1$. Considering $|\lambda| \ge 1$, using estimative (34), norms $\big\|F\big\|_{\mathcal{H}}$ and $\big\|U\big\|_{\mathcal{H}}$, the proof of item (*ii*) of this lemma is finished.

Item (*iii*): Finally, let's prove the (*iii*) item of this lemma, performing the duality product of (40) for $\frac{\rho_3}{K}A^{-\xi}\lambda\Theta$, using (37), and recalling the self-adjointness of A^v, $v \in \mathbb{R}$, result in

$$
\begin{aligned}
\lambda\|\Theta\|^2 \;=\;& -i\lambda^2\frac{\rho_3}{K}\|A^{-\frac{\xi}{2}}\Theta\|^2 + i\frac{\delta}{K}\|A^{\frac{1-\xi}{2}}\Theta\|^2 + i\frac{\delta}{K}\langle A^{\frac{1}{2}}f^5, A^{\frac{1}{2}-\xi}\Theta\rangle - i\frac{\delta}{K}\langle f^6, A^{1-\xi}\theta\rangle \\
&+\frac{\gamma}{K}\langle\sqrt{|\lambda|}\Psi, \frac{\lambda}{\sqrt{|\lambda|}}A^{-\xi}\Theta_x\rangle - i\frac{\gamma}{K}\langle f^6, A^{-\xi}\Psi_x\rangle - i\langle f^6, \Theta\rangle + i\frac{\rho_3}{K}\|A^{-\frac{\xi}{2}}f^6\|^2.
\end{aligned}
$$

Taking the real part, applying Cauchy-Schwarz and Young inequalities, for ε > 0 exists $C_\varepsilon > 0$ such that

$$
\begin{aligned}
|\lambda|\|\Theta\|^2 \le C\{&\|A^{\frac{1-\xi}{2}}\Theta\|^2 + \|A^{\frac{1}{2}}f^5\|\|A^{\frac{1}{2}-\xi}\Theta\| \\
&+ \|f^6\|\|A^{1-\xi}\theta\| + \|f^6\|\|A^{-\xi}\Psi_x\|\} + C_\varepsilon|\lambda|\|\Psi\|^2 + \varepsilon|\lambda|\|A^{-\xi}\Theta_x\|^2. \quad (95)
\end{aligned}
$$

Considering $\frac{1}{2} \le \xi \le 1$, $\frac{1-2\xi}{2} \le 0$, and remembering the identities $\|A^{-\xi}\Psi_x\|^2 = \|A^{\frac{1-2\xi}{2}}\Psi\|^2, \|A^{-\xi}\Theta_x\|^2 = \|A^{\frac{1-2\xi}{2}}\Theta\|^2$ in addition to using the item (*ii*) of this lemma the proof of the item (*iii*) of this lemma is finished.

The main result of this subsection is the following theorem

Theorem 11 *The semigroup S(t) = e^{B^1t} associated to the system* (15)–(17) *and* (21)–(22) *is analytic when the three parameters* τ, σ *and* ξ *vary in the interval* $[\frac{1}{2}, 1]$.
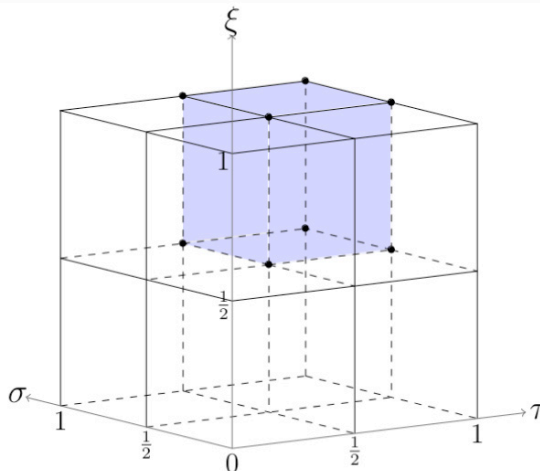

FIG. 01: Region R_{A1} of Analyticity de $S(t) = e^{B^1t}$

Proof: We will prove this theorem will be proved using the Theorem 8, so the two

conditions (88)and (89) must be proved. From Lemma 9, the condition (88) is verified, it remains to verify the condition (89), note that proving this condition is equivalent to: let δ > 0 there exists a constant C_δ > 0 such that the solutions of (15)–(21) for |λ| ≥ δ satisfy the inequality

$$|\lambda|\|U\|_{\mathcal{H}}^2 \le C_\delta \|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}}. \tag{96}$$

It is not difficult to see that this inequality (96) arises from the inequalities of the Lemmas 4 and10, so the proof of this theorem is finished.

3.2 Regularity of the second system

Next, two fundamental Lemmas for the determination of the Gevrey class and the proof of the analyticity of the semigroup $S(t) = e^{\mathcal{B}2t}$ are presented.

Lemma 12 *Let $S(t) = e^{\mathcal{B}2t}$ be a C_0-semigroup on contractions on Hilbert space $H=H_2$, the solutions of the system* (18)–(22) *satisfy the inequality*

$$\limsup_{|\lambda|\to\infty} \|(i\lambda I - \mathcal{B}_2)^{-1}\|_{\mathcal{L}(\mathcal{H})} < \infty. \tag{97}$$

Proof: To show the (33) inequality, it suffices to show that, given δ > 0 there exists a constant C_δ > 0 such that the solutions of the system (18)–(22) for |λ| > δ satisfy the inequality

$$\frac{\|U\|_{\mathcal{H}}}{\|F\|_{\mathcal{H}}} \le C_\delta \Longleftrightarrow \|U\|_{\mathcal{H}}^2 = \|(i\lambda I - \mathcal{B}_2)^{-1}F\|_{\mathcal{H}}^2 \le C_\delta\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}}. \tag{98}$$

If $F = (f^1, f^2, f^3, f^4, f^5, f^6) \in$ H then the solution $U = (\phi, \Phi, \psi, \Psi, \theta, \Theta) \in D(\mathcal{B}_2)$ the resolvent equation $(i\lambda I - B_2)U = F$ can be written in the form

$$
\begin{aligned}
i\lambda\phi - \Phi &= f^1 \quad \text{in} \quad \mathrm{D}(A^{\frac{1}{2}}), & (99)\\
i\lambda\Phi - \frac{\kappa}{\rho_1}(\phi_x + \psi)_x + \frac{\mu}{\rho_1}\Theta_x + \frac{\mu_1}{\rho_1}A^\tau\Phi &= f^2 \quad \text{in} \quad \mathrm{D}(A^0), & (100)\\
i\lambda\psi - \Psi &= f^3 \quad \text{in} \quad \mathrm{D}(A^{\frac{1}{2}}), & (101)\\
i\lambda\Psi + \frac{b}{\rho_2}A\psi + \frac{\kappa}{\rho_2}(\phi_x + \psi) - \frac{\mu}{\rho_2}\Theta + \frac{\mu_2}{\rho_2}A^\sigma\Psi &= f^4 \quad \text{in} \quad \mathrm{D}(A^0), & (102)\\
i\lambda\theta - \Theta &= f^5 \quad \text{in} \quad D(A^{\frac{1}{2}}), & (103)\\
i\lambda\Theta + \frac{\delta}{\rho_3}A\theta + \frac{\mu}{\rho_3}(\Phi_x + \Psi) + \frac{\gamma}{\rho_3}A^\xi\Theta &= f^6 \quad \text{in} \quad D(A^0). & (104)
\end{aligned}
$$

Using the fact that the operator is dissipative B_2, result in

$$\mu_1\|A^{\frac{\tau}{2}}\Phi\|^2 + \mu_2\|A^{\frac{\sigma}{2}}\Psi\|^2 + \gamma\|A^{\frac{\xi}{2}}\Theta\|^2 = \mathrm{Re}\langle(i\lambda - \mathcal{B}_2)U, U\rangle = \mathrm{Re}\langle F, U\rangle \le \|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}}. \tag{105}$$

On the other hand, performing the duality product of (100) for $\rho_1\phi$, and as the operators A^v for all $v \in$ R are self-adjoint, lead to

$$\kappa\langle(\phi_x + \psi), \phi_x\rangle = \rho_1\|\Phi\|^2 + \rho_1\langle\Phi, f^1\rangle - i\lambda\mu_1\|A^{\frac{\tau}{2}}\phi\|^2 + \mu_1\langle A^{\frac{\tau}{2}}f^1, A^{\frac{\tau}{2}}\phi\rangle + \rho_1\langle f^2, \phi\rangle + \mu\langle\Theta, \phi_x\rangle,$$

now performing the duality product of (99) for $\rho_2\psi$, and as the operators A^v for all v $\in$ R are self-adjoint, result in

$$\kappa\langle(\phi_x+\psi),\psi\rangle + b\|A^{\frac{1}{2}}\psi\|^2 = \rho_2\|\Psi\|^2 + \rho_2\langle\Psi,f^3\rangle$$
$$+ \mu\langle\Theta,\psi\rangle - i\lambda\mu_2\|A^{\frac{\sigma}{2}}\psi\|^2 + \mu_2\langle A^{\frac{\sigma}{2}}f^3, A^{\frac{\sigma}{2}}\psi\rangle + \rho_2\langle f^4,\psi\rangle.$$

Adding the last 2 equations, result in

$$\kappa\|\phi_x+\psi\|^2 + b\|A^{\frac{1}{2}}\psi\|^2 = \rho_1\|\Phi\|^2 + \rho_2\|\Psi\|^2 - i\lambda\{\mu_1\|A^{\frac{\tau}{2}}u\|^2 + \mu_2\|A^{\frac{\sigma}{2}}\psi\|^2\}$$
$$+ \rho_1\langle\Phi,f^1\rangle + \rho_2\langle\Psi,f^3\rangle + \mu_1\langle A^{\frac{\tau}{2}}f^1, A^{\frac{\tau}{2}}\phi\rangle + \mu_2\langle A^{\frac{\sigma}{2}}f^3, A^{\frac{\sigma}{2}}\phi\rangle$$
$$+ \rho_1\langle f^2,\phi\rangle + \rho_2\langle f^4,\psi\rangle + \mu\langle\Theta,\phi_x+\psi\rangle. \quad (106)$$

Taking real part, applying Cauchy-Schwarz and Young inequalities, for ε > 0 exists C_ε > 0 such that

$$\kappa\|\phi_x+\psi\|^2 + b\|A^{\frac{1}{2}}\psi\|^2 \le \rho_1\|\Phi\|^2 + \rho_2\|\Psi\|^2 + \varepsilon\|\phi_x+\psi\|^2 + C_\varepsilon\|\Theta\|^2 + C_\delta\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}}.$$

From estimative (102) and the fact $0 \le \frac{\tau}{2}$, $0 \le \frac{\sigma}{2}$ and $0 \le \frac{\xi}{2}$ the continuous embedding $D(A^{\theta 2}) \to D(A^{\theta 1})$, $\theta_2 > \theta_1$, result in

$$\kappa\|\phi_x+\psi\|^2 + b\|A^{\frac{1}{2}}\psi\|^2 \le C_\delta\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}}. \quad (107)$$

On the other hand, performing the duality product of (101) for $\rho_3\theta$, and remembering that the operators A^v for all v $\in$ R are self-adjoint and from (100), lead to

$$\delta\|A^{\frac{1}{2}}\theta\|^2 = \rho_3\|\Theta\|^2 + \rho_3\langle\Theta,f^5\rangle + \mu\langle\Phi,\theta_x\rangle - \mu\langle\Psi,\theta\rangle - i\lambda\gamma\|A^{\frac{\xi}{2}}\theta\|^2 + \gamma\langle A^{\frac{\xi}{2}}f^5, A^{\frac{\xi}{2}}\theta\rangle + \rho_3\langle f^6,\theta\rangle.$$

Taking the real part, using the inequalities Cauchy-Schwarz and Young, for ε > 0 exists C_ε > 0 which does not depend on λ such that

$$\delta\|A^{\frac{1}{2}}\theta\|^2 \le C\{\|\Theta\|^2 + \|\Theta\|\|f^5\| + \|A^{\frac{\xi}{2}}f^5\|\|A^{\frac{\xi}{2}}\theta\| + \|f^6\|\|\theta\|\}$$
$$+ C_\varepsilon\{\|\Phi\|^2 + \|\Psi\|^2\} + \varepsilon\{\|\theta\|^2 + \|A^{\frac{1}{2}}\theta\|^2\}.$$

Then, from estimative (102) and the fact $0 \le \frac{\tau}{2}$, $0 \le \frac{\sigma}{2}$ and $0 \le \frac{\xi}{2} \le \frac{1}{2}$ the continuous embedding $D(A^{\theta 2}) \to D(A^{\theta 1})$, $\theta_2 > \theta_1$, result in

$$\beta\kappa\delta\|A^{\frac{1}{2}}\theta\|^2 \le C_\delta\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}}. \quad (108)$$

Finally, from estimates (102), (104) and (105), the proof of this lemma is finished.

Lemma 13 *Let* δ > 0. *There exists a constant* C_δ > 0 *such that the solutions of* (18)− (22) *for* |λ| ≥ δ *satisfy the inequalities*

$$(i) \ |\lambda|[\kappa\|\phi_x+\psi\|^2 + b\|A^{\frac{1}{2}}\psi\|^2] \le |\lambda|[\rho_1\|\Phi\|^2 + \rho_2\|\Psi\|^2 + C\|\Theta\|^2] + C_\delta\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}}. \quad (109)$$
$$(ii) \ \delta|\lambda|\|A^{\frac{1}{2}}\theta\|^2 \le C|\lambda|\{\|\Theta\|^2 + \|\Phi\|^2\} + C_\delta\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}}. \quad (110)$$

Proof: **Item** (*i*)**:** Performing the duality product of (100) for $\rho_1\lambda\phi$, and remembering that the operators A^v for all $v \in$ R are self-adjunct, lead to

$$\kappa\lambda\langle(\phi_x+\psi),\phi_x\rangle = \rho_1\lambda\|\Phi\|^2 + \rho_1\langle\lambda\Phi,f^1\rangle - i\mu_1\|A^{\frac{\tau}{2}}\Phi\|^2$$
$$-i\mu_1\beta\gamma\langle A^\tau\Phi,f^1\rangle + i\rho_1\langle f^2,\Phi\rangle + i\rho_1\langle f^2,f^1\rangle + \mu\langle\Theta,\lambda\phi_x\rangle.$$

Now performing the duality product of (99) for $\rho_2\lambda\psi$, and remembering that the operators A^v for all $v \in R$ are self-adjunct, resulting in

$$\lambda[\kappa\langle(\phi_x+\psi),\psi\rangle + b\|A^{\frac{1}{2}}\psi\|^2] = \rho_2\lambda\|\Psi\|^2 + \rho_2\langle\lambda\Psi,f^3\rangle - i\mu_2\|A^{\frac{\sigma}{2}}\Psi\|^2 - i\mu_2\langle A^\sigma\Psi,f^3\rangle$$
$$+i\rho_2\langle f^4,\Psi\rangle + i\rho_2\langle f^4,f^3\rangle + \mu\langle\Theta,\lambda\psi\rangle.$$

Adding the last 2 equations, result in

$$\lambda[\kappa\|\phi_x+\psi\|^2 + b\|A^{\frac{1}{2}}\psi\|^2] = \lambda[\rho_1\|\Phi\|^2 + \rho_2\|\Psi\|^2] - i\{\mu_1\|A^{\frac{\tau}{2}}\Phi\|^2$$
$$+ \mu_2\|A^{\frac{\sigma}{2}}\Psi\|^2\} + \rho_1\langle\lambda\Phi,f^1\rangle + \rho_2\langle\lambda\Psi,f^3\rangle - i\{\mu_1\langle A^\tau\Phi,f^1\rangle$$
$$+ \mu_2\langle A^\sigma\Psi,f^3\rangle\} + i\rho_1\langle f^2,\Phi\rangle + i\rho_2\langle f^4,\Psi\rangle + i\rho_1\langle f^2,f^1\rangle$$
$$+ i\rho_2\langle f^4,f^3\rangle + \mu\langle\sqrt{|\lambda|}\Theta,\frac{\lambda}{\sqrt{|\lambda|}}(\phi_x+\psi)\rangle. \quad (111)$$

On the other hand, from (100) and (102), result in

$$\rho_1\langle\lambda\Phi,f^1\rangle + \rho_2\langle\lambda\Psi,f^3\rangle = i\{\kappa\langle(\phi_x+\psi),f_x^1\rangle - \mu\langle\Theta,f_x^1\rangle$$
$$+ \mu_1\langle A^\tau\Phi,f^1\rangle - \rho_1\langle f^2,f^1\rangle + b\langle A^{\frac{1}{2}}\psi,A^{\frac{1}{2}}f^3\rangle + \kappa\langle(\phi_x+\psi),f^3\rangle$$
$$- \mu\langle\Theta,f^3\rangle + \mu_2\langle A^\sigma\Psi,f^3\rangle - \rho_2\langle f^4,f^3\rangle\}. \quad (112)$$

Using the identity (112) in the (111) equation and simplifying, leads to

$$\lambda[\kappa\|\phi_x+\psi\|^2 + b\|A^{\frac{1}{2}}\psi\|^2] = \lambda[\rho_1\|\Phi\|^2 + \rho_2\|\Psi\|^2] - i\{\mu_1\|A^{\frac{\tau}{2}}\Phi\|^2 + \mu_2\|A^{\frac{\sigma}{2}}\Psi\|^2\} + i\kappa\langle\phi_x,f_x^1\rangle \ (113)$$
$$+ i\kappa\langle\psi,f_x^1\rangle - i\mu\langle\Theta,f_x^1\rangle + ib\langle A^{\frac{1}{2}}\psi,A^{\frac{1}{2}}f^3\rangle + i\kappa\langle\phi_x,f^3\rangle + i\kappa\langle\psi,f^3\rangle$$
$$- i\mu\langle\Theta,f^3\rangle + i\rho_1\langle f^2,\Phi\rangle + i\rho_2\langle f^4,\Psi\rangle + \mu\langle\sqrt{|\lambda|}\Theta,\frac{\lambda}{\sqrt{|\lambda|}}(\phi_x+\psi)\rangle.$$

As for $\varepsilon > 0$, exists $C_\varepsilon > 0$, such that

$$\left|\mu\langle\sqrt{|\lambda|}\Theta,\frac{\lambda}{\sqrt{|\lambda|}}(\phi_x+\psi)\rangle\right| \leq C_\varepsilon|\lambda|\|\Theta\|^2 + \varepsilon|\lambda|\|\phi_x+\psi\|^2,$$

taking the real part of the equation (113), applying the Cauchy-Schwarz and Young inequalities and estimative (98) of Lemma 12, the proof of item (*i*) this lemma is completed.

On the other hand, performing the duality product of (104) for $\rho_3\lambda\theta$, and remembering that the operators A^v for all $v \in R$ are self-adjunct, result in

$$\delta\lambda\|A^{\frac{1}{2}}\theta\|^2 = \rho_3\lambda\|\Theta\|^2 + i\mu\langle\Phi,\Theta_x\rangle - i\gamma\|A^{\frac{\xi}{2}}\Theta\|^2 + i\rho_3\langle f^6,\Theta\rangle - i\mu\langle\Psi,\Theta\rangle. \quad (114)$$

As

$$i\mu\langle\Phi,\Theta_x\rangle = i\mu\langle\Phi,i\lambda\theta_x - f_x^5\rangle = \mu\langle\sqrt{|\lambda|}\Phi,\frac{\lambda}{\sqrt{|\lambda|}}\theta_x\rangle - i\mu\langle\Phi,f_x^5\rangle. \quad (115)$$

Then, using (115) in (114) and taking real part, lead to

$$\delta|\lambda|\|A^{\frac{1}{2}}\theta\|^2 = \rho_3\lambda\|\Theta\|^2 + \mu\langle\sqrt{|\lambda|}\Phi, \frac{\lambda}{\sqrt{|\lambda|}}\theta_x\rangle - i\mu\langle\Phi, f_x^5\rangle + i\rho_3\langle f^6, \Theta\rangle - i\mu\langle\Psi, \Theta\rangle, \quad (116)$$

applying Cauchy-Schwarz and Young inequalities, for ε > 0 exists $C_\varepsilon > 0$, such that

$$\delta|\lambda|\|A^{\frac{1}{2}}\theta\|^2 \leq \rho_3|\lambda|\|\Theta\|^2 + \varepsilon|\lambda|\|A^{\frac{1}{2}}\theta\|^2 + C_\varepsilon|\lambda|\|\Phi\|^2 + C\{\|\Phi\|\|A^{\frac{1}{2}}f^5\| + \|f^6\|\|\Theta\| + \|\Phi\|^2 + \|\Theta\|^2\}.$$

Then, from Lemma 12, the proof of item (*ii*) of this lemma is finished.

3.2.1 Gevrey class of the second system

Theorem 14 *The semigroup S(t) = e^{B2t} associated to system (18)–(22) is of Gevrey class* v *for every s >* $\frac{1}{r}$ *for r* = min$\{\frac{\tau}{1+\tau}, \frac{\sigma}{1+\sigma}, \frac{\xi}{1+\xi}\} \in (0, 1)$, *with (τ, σ, ξ)* ∈∈ R_{CG}:= (0, 1]³.

Proof: From the resolvent equation $F = (i\lambda IB_2)U$ for λ∈R, result in $U = (i\lambda I-B_2)^{-1}F$. Furthermore to show (57) this is theorem(6) it is enough to show:

$$|\lambda|^r\|U\|_{\mathcal{H}}^2 \leq C_\delta\{\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}}\} \quad \text{for} \quad r \in (0, 1). \quad (117)$$

where $r = \min\left\{\frac{\tau}{1+\tau}, \frac{\sigma}{1+\sigma}, \frac{\xi}{1+\xi}\right\}$, 0 < τ ≤ 1, 0 < σ ≤ 1 and 0 < ξ ≤ 1.

Next, $|\lambda|^{\frac{\tau}{1+\tau}}\|\Phi\|$, $|\lambda|^{\frac{\sigma}{1+\sigma}}\|\Psi\|$ and $|\lambda|^{\frac{\xi}{1+\xi}}\|\Theta\|$will be estimated.

Let's start by estimating the term $|\lambda|^{\frac{\tau}{1+\tau}}\|\Phi\|$: It is assumed that |λ| > 1. Set $\phi = \phi_1 + \phi_2$, where $\phi_1 \in D(A)$ and $\phi_2 \in D(A^0)$, with

$$i\lambda\Phi_1 + A\Phi_1 = f^2, \qquad i\lambda\Phi_2 = -\frac{\kappa}{\rho_1}A\phi + \frac{\kappa}{\rho_1}\psi_x - \frac{\mu}{\rho_1}\Theta_x - \frac{\mu_1}{\rho_1}A^\tau\Phi + A\Phi_1. \quad (118)$$

Firstly, applying the product duality the first equation in (118) by ϕ_1, then by $A\phi_1$, and recalling that the operator A is self-adjoint, resulting in

$$i\lambda\|\Phi_1\|^2 + \|A^{\frac{1}{2}}\Phi_1\|^2 = \langle f^2, \Phi_1\rangle. \quad (119)$$

Taking first the imaginary part of (119) and in the sequence, the real part and applying Cauchy-Schwarz inequality, results in

$$|\lambda|\|\Phi_1\| \leq C\|F\|_{\mathcal{H}} \qquad \|A^{\frac{1}{2}}\Phi_1\| \leq C\|F\|_{\mathcal{H}}. \quad (120)$$

Applying the $A^{-\frac{1}{2}}$ operator on the second equation of (118), lead to

$$i\lambda A^{-\frac{1}{2}}\Phi_2 = -\frac{\kappa}{\rho_1}A^{\frac{1}{2}}\phi + \frac{\kappa}{\rho_1}A^{-\frac{1}{2}}\psi_x - \frac{\mu}{\rho_1}A^{-\frac{1}{2}}\Theta_x - \frac{\mu_1}{\rho_1}A^{\tau-\frac{1}{2}}\Phi + A^{\frac{1}{2}}\Phi_1,$$

then, as $\|A^{-\frac{1}{2}}\psi_x\|^2 = \|\psi\|^2$ and $\tau-\frac{1}{2} \leq \frac{\tau}{2}$ taking into account the continuous embedding $D(A^{\theta2}) \rightarrow D(A^{\theta1})$, $\theta_2 > \theta_1$, result in

$$|\lambda|^2\|A^{-\frac{1}{2}}\Phi_2\|^2 \leq C\{\|A^{\frac{1}{2}}\phi\|^2 + \|A^{\frac{\tau}{2}}\Phi\|^2 + \|\psi\|^2 + \|\Theta\|^2\} + \|A^{\frac{1}{2}}\Phi_1\|^2 \quad (121)$$

Using (98) and second estimative of equation (120), result in

$$\|A^{-\frac{1}{2}}\Phi_2\|^2 \leq C|\lambda|^{-2}\{\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}} + \|F\|_{\mathcal{H}}^2\}. \quad (122)$$

On the other hand, from $\phi_2 = \phi - \phi_1$, (105) and as $\frac{\tau}{2} \leq \frac{1}{2}$ the inequality (120), result in

$$\|A^{\frac{\tau}{2}}\Phi_2\|^2 \leq C\{\|A^{\frac{\tau}{2}}\Phi\|^2 + \|A^{\frac{\tau}{2}}\Phi_1\|^2\} \leq C\{\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}} + \|F\|_{\mathcal{H}}^2\}. \tag{123}$$

Now, by Lions' interpolations inequality $0 \in [-\frac{1}{2}, \frac{\tau}{2}]$, result in

$$\|\Phi_2\|^2 \leq C(\|A^{-\frac{1}{2}}\Phi_2\|^2)^{\frac{\tau}{1+\tau}}(\|A^{\frac{\tau}{2}}\Phi_2\|^2)^{\frac{1}{1+\tau}}. \tag{124}$$

Then, using (122) and (123) in (124), result in

$$\|\Phi_2\|^2 \leq C|\lambda|^{\frac{-2\tau}{(1+\tau)}}\{\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}} + \|F\|_{\mathcal{H}}^2\}. \tag{125}$$

Therefore, as $\|\phi\|^2 \leq \|\phi\|^2 + \|\phi\|^2$ from inequality (120), (125) and $|\lambda|^{-2} \leq |\lambda|^{\frac{-2\tau}{1+\tau}}$, we have

$$|\lambda|^{\frac{\tau}{1+\tau}}\|\Phi\| \leq C_\delta\|F\|_{\mathcal{H}} \iff |\lambda|\|\Phi\|^2 \leq C_\delta|\lambda|^{\frac{-2\tau}{1+\tau}}\{\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}} + \|F\|_{\mathcal{H}}^2\} \quad \text{for} \quad 0 \leq \tau \leq 1. \tag{126}$$

On the other hand, let's now estimate the missing term $|\lambda|^{\frac{\sigma}{1+\sigma}}\|\Psi\|$**:** Set $\Psi = \Psi_1 + \Psi_2$, where $\Psi_1 \in D(A)$ and $\Psi_2 \in D(A^0)$, with

$$i\lambda\Psi_1 + A\Psi_1 = f^4 \qquad i\lambda\Psi_2 = -\frac{b}{\rho_2}A\psi - \frac{\kappa}{\rho_2}\phi_x - \frac{\kappa}{\rho_2}\psi + \frac{\mu}{\rho_2}\Theta - \frac{\mu_2}{\rho_2}A^\sigma\Psi + A\Psi_1. \tag{127}$$

Firstly, applying the product duality on the first equation in (127) by Ψ_1, the by $A\Psi_1$ and recalling that the operator A is self-adjoint, resulting in

$$i\lambda\|\Psi_1\|^2 + \|A^{\frac{1}{2}}\Psi_1\|^2 = \langle f^4, \Psi_1\rangle. \tag{128}$$

Taking first the imaginary part of (128) and in the sequence, the real part and applying Cauchy-Schwarz inequality, lead to

$$|\lambda|\|\Psi_1\| \leq C\|F\|_{\mathcal{H}} \quad \text{and} \quad \|A^{\frac{1}{2}}\Psi_1\| \leq C\|F\|_{\mathcal{H}}. \tag{129}$$

It arises from the second equation in (127) that

$$i\lambda A^{-\frac{1}{2}}\Psi_2 = -\frac{b}{\rho_2}A^{\frac{1}{2}}\psi - \frac{\kappa}{\rho_2}A^{-\frac{1}{2}}\phi_x - \frac{\kappa}{\rho_2}A^{-\frac{1}{2}}\psi + \frac{\mu}{\rho_2}A^{-\frac{1}{2}}\Theta - \frac{\mu_2}{\rho_2}A^{\sigma-\frac{1}{2}}\Psi + A^{\frac{1}{2}}\Psi_1,$$

then, as $\|A^{-\frac{1}{2}}\phi_x\|^2 = \|\phi\|^2$, $\|A^{-\frac{1}{2}}\Theta_x\|^2 = \|\Theta\|^2$ and $\sigma - \frac{1}{2} \leq \frac{\sigma}{2}$ taking into account the continuous embedding $D(A^{\theta_2}) \to D(A^{\theta_1})$, $\theta_2 > \theta_1$, result in

$$|\lambda|^2\|A^{-\frac{1}{2}}\Psi_2\|^2 \leq C\{\|\phi\|^2 + \|A^{\frac{1}{2}}\psi\|^2 + \|A^{-\frac{1}{2}}\Theta\|^2 + \|A^{\frac{\sigma}{2}}\Psi\|^2\} + \|A^{\frac{1}{2}}\Psi_1\|^2 \tag{130}$$

Using estimative (98) of Lemma 12 and estimative (129), yields

$$\|A^{-\frac{1}{2}}\Psi_2\|^2 \leq C|\lambda|^{-2}\{\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}} + \|F\|_{\mathcal{H}}\}. \tag{131}$$

On the other hand, from $\Psi_2 = \Psi - \Psi_1$, (105) and as $0 \leq \frac{\sigma}{2} \leq \frac{1}{2}$, the inequality (129), result in

$$\|A^{\frac{\sigma}{2}}\Psi_2\|^2 \leq C\{\|A^{\frac{\sigma}{2}}\Psi\|^2 + \|A^{\frac{\sigma}{2}}\Psi_1\|^2\} \leq C\{\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}} + \|F\|_{\mathcal{H}}^2\}. \tag{132}$$

Now using interpolation inequality $0 \in [-\frac{1}{2}, \frac{\sigma}{2}]$. Since

$$0 = \eta\left(-\frac{1}{2}\right) + (1-\eta)\frac{\sigma}{2}, \quad \text{for} \quad \eta = \frac{\sigma}{1+\sigma},$$

using (131) and (132) we get that

$$\|\Psi_2\|^2 \;\leq\; C_\delta(\|A^{-\frac{1}{2}}\Psi_2\|^2)^\eta(\|A^{\frac{\sigma}{2}}\Psi_2\|^2)^{1-\eta} \leq |\lambda|^{\frac{-2\sigma}{1+\sigma}}\{\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}} + \|F\|_{\mathcal{H}}^2\}. \tag{133}$$

Also, as $\|\Psi\|2 \leq C\{\|\Psi\|2 + \|\Psi\|2\}$ from inequality (129), estimative (133) and as $|\lambda|^{-2} \leq |\lambda|^{\frac{-2\sigma}{1+\sigma}}$, result in

$$|\lambda|^{\frac{\sigma}{1+\sigma}}\|\Psi\| \leq C_\delta\|F\|_{\mathcal{H}} \iff |\lambda|\|\Psi\|^2 \leq C_\delta|\lambda|^{\frac{-2\sigma}{1+\sigma}}\{\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}} + \|F\|_{\mathcal{H}}^2\} \quad \text{for} \quad 0 \leq \sigma \leq 1. \tag{134}$$

Finally, let's now estimate the missing term $|\lambda|\frac{\xi}{1+\xi}\|\Theta\|$. Set $\Theta = \Theta_1 + \Theta_2$, where $\Theta_1 \in D(A)$ and $\Theta_2 \in D(A^0)$, with

$$i\lambda\Theta_1 + A\Theta_1 \;=\; f^6 \qquad i\lambda\Theta_2 \;=\; -\frac{\delta}{\rho_3}A\theta - \frac{\mu}{\rho_3}\Phi_x - \frac{\mu}{\rho_3}\Psi - \frac{\gamma}{\rho_3}A^\xi\Theta + A\Theta_1. \tag{135}$$

Firstly, applying the product duality the first equation in (135) by Θ_1, and as the operators A^v for all $v \in \mathbb{R}$ are self-adjoint, result in

$$i\lambda\|\Theta_1\|^2 + \|A^{\frac{1}{2}}\Theta_1\|^2 = \langle f^6, \Theta_1\rangle. \tag{136}$$

Taking first the imaginary part of (136) and in the sequence, the real part and applying Cauchy-Schwarz inequality, lead to

$$|\lambda|\|\Theta_1\| \leq C\|F\|_{\mathcal{H}} \qquad \text{and} \qquad \|A^{\frac{1}{2}}\Theta_1\| \leq C\|F\|_{\mathcal{H}}. \tag{137}$$

It arises from the second equation in (135) that

$$i\lambda A^{-\frac{1}{2}}\Theta_2 = -\frac{\delta}{\rho_3}A^{\frac{1}{2}}\theta - \frac{\mu}{\rho_3}A^{-\frac{1}{2}}\Phi_x - \frac{\mu}{\rho_3}A^{-\frac{1}{2}}\Psi - \frac{\gamma}{\rho_3}A^{\xi-\frac{1}{2}}\Theta + A^{\frac{1}{2}}\Theta_1,$$

then, as $\|A^{-\frac{1}{2}}\phi_x\|^2 = \|\phi\|^2$, $-\frac{1}{2} \leq 0$ and $\xi - \frac{1}{2} \leq \frac{\xi}{2}$ taking into account the continuous embedding $D(A^{\theta_2}) \to D(A^{\theta_1})$, $\theta_2 > \theta_1$ and using estimative (98) and second estimative of equation (137), lead to

$$\|A^{-\frac{1}{2}}\Theta_2\|^2 \leq C|\lambda|^{-2}\{\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}} + \|F\|_{\mathcal{H}}\}. \tag{138}$$

On the other hand, from $\Theta_2 = \Theta - \Theta_1$, (105) and as $\frac{\xi}{2} \leq \frac{1}{2}$ the inequality (137), result in

$$\|A^{\frac{\xi}{2}}\Theta_2\|^2 \;\leq\; C\{\|A^{\frac{\xi}{2}}\Theta\|^2 + \|A^{\frac{\xi}{2}}\Theta_1\|^2\} \leq C\{\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}} + \|F\|_{\mathcal{H}}^2\}. \tag{139}$$

Now we are going to use an interpolation inequality $0 \in [-\frac{1}{2}, \frac{\xi}{2}]$. Since

$$0 = \eta\left(-\frac{1}{2}\right) + (1-\eta)\frac{\xi}{2}, \quad \text{for} \quad \eta = \frac{\xi}{1+\xi},$$

using (138) and (139) we get that

$$\|\Theta_2\|^2 \le C_\delta(\|A^{-\frac{1}{2}}\Theta_2\|^2)^\eta(\|A^{\frac{\xi}{2}}\Theta_2\|^2)^{1-\eta} \le |\lambda|^{\frac{-2\xi}{1+\xi}}\{\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}} + \|F\|_{\mathcal{H}}^2\} \qquad (140)$$

Also as, $\|\Theta\|^2 \le C\{\|\Theta_1\|^2 + \|\Theta_2\|^2\}$ from first inequality of (137), (140) and as $|\lambda|^{-2} \le |\lambda|^{\frac{-2\xi}{1+\xi}}$, result in

$$|\lambda|^{\frac{\xi}{1+\xi}}\|\Theta\| \le C_\delta\|F\|_{\mathcal{H}} \iff |\lambda|\|\Theta\|^2 \le C_\delta|\lambda|^{\frac{-2\xi}{1+\xi}}\{\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}} + \|F\|_{\mathcal{H}}^2\} \quad \text{for} \quad 0 \le \xi \le 1. \quad (141)$$

Finally, using the estimates (126), (134) and (141) in the inequality item (i) of Lemma 13, resultin

$$\beta\gamma[\kappa\|\phi_x + \psi\|^2 + b\|A^{\frac{1}{2}}\psi\|^2] \le C\{|\lambda|^{\frac{-2\tau}{1+\tau}} + |\lambda|^{\frac{-2\sigma}{1+\sigma}} + |\lambda|^{\frac{-2\xi}{1+\xi}}\}\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}}$$
$$+ C_\delta|\lambda|^{-1}\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}}. \quad (142)$$

And, using the estimates (126) and (141) in the inequality item (ii) of Lemma 13, result in

$$\delta\|A^{\frac{1}{2}}\theta\|^2 \le C\{|\lambda|^{\frac{-2\tau}{1+\tau}} + |\lambda|^{\frac{-2\xi}{1+\xi}}\}\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}} + C_\delta|\lambda|^{-1}\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}}. \quad (143)$$

As for $(\tau, \sigma, \xi) \in [0, 1]^3$, we have $-1 \le 2\max\{-\frac{\tau}{1+\tau}, -\frac{\sigma}{1+\sigma}, -\frac{\xi}{1+\xi}\} = -2\max\{\frac{\tau}{1+\tau}, \frac{\sigma}{1+\sigma}, \frac{\xi}{1+\xi}\}$, for $|\lambda| > 1$.

Finally, summing the the estimates (126), (134), (141), (142) and (143), we obtain

$$\|U\|_{\mathcal{H}}^2 \le C_\delta|\lambda|^{-2\min\{\frac{\tau}{1+\tau}, \frac{\sigma}{1+\sigma}, \frac{\xi}{1+\xi}\}}[\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}} + \|F\|_{\mathcal{H}}].$$

Equivalently:

$$|\lambda|^{\min\{\frac{\tau}{1+\tau}, \frac{\sigma}{1+\sigma}, \frac{\xi}{1+\xi}\}}\|U\|_{\mathcal{H}} \le C_\delta\|F\|_{\mathcal{H}} \qquad \text{for} \qquad (\tau, \sigma, \xi) \in [0, 1]^3.$$

As $r = \min\{\frac{\tau}{1+\tau}, \frac{\sigma}{1+\sigma}, \frac{\xi}{1+\xi}\} \in (0, 1)$ for $(\tau, \sigma, \xi) \in (0, 1]^3$ the proof of this theorem is finished.

3.2.2 Analyticity of $S(t) = e^{B2t}$ for (τ, σ, ξ) $[\frac{1}{2}, 1]^3$ such that $\tau = \xi$. Fig. 02

Before proving the main result of this section, the following lemma will be proved.

Lemma 15 *Let $\delta > 0$. There exists a constant $C_\delta > 0$ such that the solutions of* (18)−(22) *for $|\lambda| \ge \delta$ satisfy the inequality*

$$(i) \quad |\lambda|[\|\Phi\|^2 + \|\Theta\|^2] \le C_\delta\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}} \qquad \text{for} \qquad \frac{1}{2} \le \tau = \xi \le 1. \qquad (144)$$

$$(ii) \quad |\lambda|\|\Psi\|^2 \le C_\delta\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}} \qquad \text{for} \qquad \frac{1}{2} \le \sigma \le 1. \qquad (145)$$

Proof: **Item** (i)**:** Realizing the duality product of (100) with $\frac{\rho_1}{\gamma} A^{-\tau}\lambda\phi$ and using the property that the operator A^v is self-adjoint for all $v \in \mathbb{R}$ and using the equation (99), result in

$$\frac{\mu_1}{\gamma}\lambda\|\Phi\|^2 = \frac{\kappa}{\gamma}\langle\lambda(-A\phi+\psi_x), A^{-\tau}\Phi\rangle + \frac{\rho_1}{\gamma}\langle f^2, \lambda A^{-\tau}\Phi\rangle - i\frac{\rho_1}{\gamma}\lambda^2\|A^{-\frac{\tau}{2}}\Phi\|^2 - \frac{\mu}{\gamma}\langle\Theta_x, \lambda A^{-\tau}\Phi\rangle$$

$$= i\frac{\kappa}{\gamma}\langle(A\Phi+Af^1), A^{-\tau}\Phi\rangle - i\frac{\kappa}{\gamma}\langle\Psi_x + f_x^3, A^{-\tau}\Phi\rangle - i\frac{\rho_1}{\gamma}\lambda^2\|A^{-\frac{\tau}{2}}\Phi\|^2$$

$$+\frac{1}{\gamma}\langle f^2, i\kappa A^{1-\tau}\phi - i\kappa A^{-\tau}\psi_x + i\mu_1\Phi - i\rho_1 A^{-\tau}f_2\rangle + \frac{\mu}{\gamma}\langle\Theta, \lambda A^{-\tau}\Phi_x\rangle.$$

Then

$$\frac{\mu_1}{\gamma}\lambda\|\Phi\|^2 = i\frac{\kappa}{\gamma}\|A^{\frac{1-\tau}{2}}\Phi\|^2 + i\frac{\kappa}{\gamma}\langle A^{\frac{1}{2}}f^1, A^{\frac{1}{2}-\tau}\Phi\rangle + i\frac{\kappa}{\gamma}\langle\Psi, A^{-\tau}\Phi_x\rangle - i\frac{\kappa}{\mu_1}\langle f_x^3, A^{-\tau}\Phi\rangle$$

$$-i\langle f^2, \Phi\rangle - i\frac{\kappa}{\gamma}\langle f^2, A^{1-\tau}\phi\rangle + i\frac{\kappa}{\gamma}\langle A^{-\tau}f^2, \psi_x\rangle + i\frac{\rho_1}{\gamma}\|A^{-\frac{\tau}{2}}f^2\|^2 \qquad (146)$$

$$-i\frac{\rho_1}{\gamma}\lambda^2\|A^{-\frac{\theta}{2}}\Phi\|^2 + \frac{\mu}{\gamma}\langle\Theta, \lambda A^{-\tau}\Phi_x\rangle.$$

On the other hand, now performing the duality product of (104) for $\frac{\rho_3}{\gamma}A^{-\xi}\lambda\Theta$, using (101), and as the operators A^v for all $v \in \mathbb{R}$ are self-adjoint, result in

$$\lambda\|\Theta\|^2 = -i\lambda^2\frac{\rho_3}{\gamma}\|A^{-\frac{\xi}{2}}\Theta\|^2 + i\frac{\delta}{\gamma}\|A^{\frac{1-\xi}{2}}\Theta\|^2 + i\frac{\delta}{\gamma}\langle A^{\frac{1}{2}}f^5, A^{\frac{1}{2}-\xi}\Theta\rangle - i\frac{\delta}{\gamma}\langle f^6, A^{1-\xi}\theta\rangle - i\frac{\mu}{\gamma}\langle f^6, A^{-\xi}\Phi_x\rangle$$

$$-i\frac{\mu}{\gamma}\langle f^6, A^{-\xi}\Psi\rangle - i\langle f^6, \Theta\rangle + i\frac{\rho_3}{\gamma}\|A^{-\frac{\xi}{2}}f^6\|^2 - \frac{\mu}{\gamma}\langle\lambda A^{-\xi}\Phi_x, \Theta\rangle. \qquad (147)$$

Imposing the condition $\tau = \xi$ and since $\text{Re}\{\in\Theta, \lambda A^{-\tau}\phi_x \in -\lambda A^{-\tau}\phi_x, \Theta\in\} = 0$, adding the equations (146) and (147) and then taking the real part, result in

$$\lambda\left[\frac{\mu_1}{\gamma}\|\Phi\|^2 + \|\Theta\|^2\right] = i\frac{\kappa}{\gamma}\langle A^{\frac{1}{2}}f^1, A^{\frac{1}{2}-\tau}\Phi\rangle + i\frac{\kappa}{\gamma}\langle\Psi, A^{-\tau}\Phi_x\rangle - i\frac{\kappa}{\mu_1}\langle f_x^3, A^{-\tau}\Phi\rangle - i\langle f^2, \Phi\rangle$$

$$-i\frac{\kappa}{\gamma}\langle f^2, A^{1-\tau}\phi\rangle + i\frac{\kappa}{\gamma}\langle A^{-\tau}f^2, \psi_x\rangle + i\frac{\delta}{\gamma}\langle A^{\frac{1}{2}}f^5, A^{\frac{1}{2}-\xi}\Theta\rangle$$

$$-i\frac{\delta}{\gamma}\langle f^6, A^{1-\xi}\theta\rangle - i\frac{\mu}{\gamma}\langle f^6, A^{-\xi}\Phi_x\rangle - i\frac{\mu}{\gamma}\langle f^6, A^{-\xi}\Psi\rangle - i\langle f^6, \Theta\rangle$$

From $|i\frac{\kappa}{\gamma}\langle\Psi, A^{-\tau}\Psi_x\rangle| \leq C\{\|\Psi\|^2 + \|A^{\frac{1-2\tau}{2}}\Phi\|^2\}$ and $\frac{1}{2} \leq \tau = \xi \leq 1$, applying Cauchy-Schwarz and Young inequalities, norms $\|F\|_H$ and $\|U\|_H$, the proof of item (*i*) of this lemma is finished.

Item (*ii*): Similarly, performing the duality product of (102) for $\frac{\rho_2}{\mu_2}A^{-\sigma}\lambda\Psi$, using (101), and recalling the self-adjointness of A^v, $v \in \mathbb{R}$, lead to

$$\lambda\|\Psi\|^2 = -i\frac{\rho_2}{\mu_2}\lambda^2\|A^{-\frac{\sigma}{2}}\Psi\|^2 + i\frac{b}{\mu_2}\|A^{\frac{1-\sigma}{2}}\Psi\|^2 + i\frac{b}{\mu_2}\langle A^{\frac{1}{2}}f^3, A^{\frac{1}{2}-\sigma}\Psi\rangle - i\frac{\kappa}{\mu_2}\langle\Phi, A^{-\sigma}\Psi_x\rangle$$

$$+i\frac{\kappa}{\mu_2}\langle f_x^1, A^{-\sigma}\Psi\rangle + i\frac{\kappa}{\mu_2}\|A^{-\frac{\sigma}{2}}\Psi\|^2 + i\frac{\kappa}{\mu_2}\langle f^3, A^{-\sigma}\Psi\rangle$$

$$+\frac{\mu}{\mu_2}\langle\sqrt{|\lambda|}\Theta, \frac{\lambda}{\sqrt{|\lambda|}}A^{-\sigma}\Psi\rangle - i\frac{b}{\mu_2}\langle f^4, A^{1-\sigma}\Psi\rangle - i\frac{\kappa}{\mu_2}\langle f^4, A^{-\sigma}\phi_x\rangle$$

$$-i\frac{\kappa}{\mu_2}\langle f^4, A^{-\sigma}\psi\rangle + i\frac{\mu}{\mu_2}\langle f^4, A^{-\sigma}\Theta\rangle - i\langle f^4, \Psi\rangle + i\frac{\rho_2}{\mu_2}\|f^4\|^2.$$

Noting that: For $\frac{1}{2} \leq \sigma \leq 1$ lead to $\frac{1-\sigma}{2} \leq \frac{\sigma}{2}$, $\frac{1}{2} - \sigma \leq 0$ and $\frac{1-2\sigma}{2} \leq 0$, on the other hand as $\|A^{-\sigma}\Psi_x\|^2 = \|A^{\frac{1-2\sigma}{2}}\Psi\|^2$, taking real part and considering that $\frac{1}{2} \leq \sigma \leq 1$ and using Cauchy-Schwarz and Young inequalities, norms $\|F\|_H$ and $\|U\|_H$, for $\varepsilon > 0$, exists $C_\varepsilon > 0$

independent of λ such that

$$|\lambda|\|\Psi\|^2 \leq C_\delta\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}} + C\{\|\Phi\|^2 + \|\Psi\|^2\} + C_\varepsilon|\lambda|\|\Theta\|^2 + \varepsilon|\lambda|\|\Psi\|^2.$$

Using estimative (98) and estimate (i) of this lemma, the proof of item (ii) of this lemma is finished.

The main result of this subsection is the following theorem

Theorem 16 *The semigroup* $S(t) = e^{B2t}$ *associated to the system* (18)–(22) *is analytic when the three parameters* τ, σ *and* ξ *vary in the interval* $[\frac{1}{2}, 1]$ *with* τ = ξ:
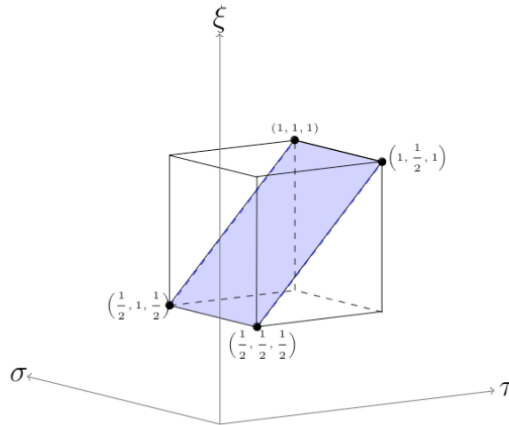

FIG. 02: *Region* R_{A2} *of Analyticity de* $S(t) = e^{B2t}$

Proof: This theorem will be proved again using now the Theorem 8, so the two conditions (88) and (89) must be proved.

Here the condition's test (88) will be omitted because it is completely similar to the test already conducted for the first the first system.

Next, the condition (89) is proved, note that proving this condition is equivalent to show, let δ > 0. There exists a constant C_δ > 0 such that the solutions of (18)–(20) and (21)–(22) for |λ|≥ δ satisfy the inequality

$$|\lambda|\|U\|_{\mathcal{H}}^2 \leq C_\delta\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}}. \tag{148}$$

It is not difficult to see that this inequality (96) follows from the inequalities of the Lemmas 13 and 15, so the proof of this theorem is finished.

Remark 17 (Asymptotic Behavior) *It is emphasized that thanks to the Lemmas 3 and 9 the exponential decay of the first system is obtained when the 3 parameters take values in the closed interval* [0, 1]*, also for the second system using the Lemmas 12 and proving in a completely similar way to the proof of Lemma 9 which* $i\mathbb{R} \subset \rho(B_2)$ *the corresponding semigroup* $S(t) = e^{B2t}$*, will be exponentially stable when the 3 parameters take values in the closed interval* [0, 1]*.*

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