GROUP OF WEAKLY CONTINUOUS OPERATORS ASSOCIATED TO A SCHRÖDINGER TYPE HOMOGENEOUS MODEL

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ABSTRACT: In this work, we prove the existence and uniqueness of the solution of the Schrödinger type homogeneous model in the periodic distributional space P'. Furthermore, we prove that the solution depends continuously respect to the initial data in P'. Introducing a family of weakly continuous operators, we prove that this family is a group of operators in P'. Then, with this family of operators, we get a fine version of the existence and dependency continuous theorem obtained. Finally, we give some remarks derived from this study.
KEYWORDS: Groups theory, weakly continuous operators, existence of solution, Schrödinger type equation, distributional problem, periodic distributional space.

1 I INTRODUCTION

First, we begin by commenting that [3] has proven the existence of a solution of the Schrodinger type equation in the Hilbert space H^s_{per}. Also in [3] a family of bounded operators is introduced in the Hilbert space H^s_{per} and it is proved that forms a unitary group. Thus motivated by these ideas we will solve the problem (P_2) in the topological dual of P : P', which is not a Banach space.

In this article, we will prove the existence and uniqueness of the solution of (P_2). Furthermore, we will demonstrate that the solution depends continuously with respect to the initial data in P', considering the weak convergence in P'. And we will prove that the introduced family of operators forms a group of weakly continuous linear operators. Thus, with this family we will rewrite our result in a fine version. Our article is organized as follows. In section 2, we indicate the methodology used and cite the references used. In section 3, we put the results obtained from our study. This section is divided into three subsections. Thus, in subsection 3.1 we prove that the problem (P_2) has a unique solution and also demonstrate that the solution depends continuously with respect to the initial data.

In subsection 3.2, we introduce families of weakly continuous linear operators in P' that manage to form a group. In subsection 3.3 we improve Theorem 3.1.

Finally, in section 4 we give the conclusions of this study.

2 | METHODOLOGY

As theoretical framework in this article we use the references [1], [2], [3], [4] and [5] for Fourier Theory in periodic distributional space, periodic Sobolev spaces, topological vector spaces, weakly continuous operators, group of operators and existence of solution of a distributional differential equation.

3 | MAIN RESULTS

The presentation of the results obtained has been organized in subsections and is as follows.

3.1 Solution of the Schrödinger Equation (P_2)

In this subsection we will study the existence of a solution to the problem (P_2) and the continuous dependence of the solution with respect to the initial data in P'.

Theorem 3.1 *Let* $\mu > 0$, $\alpha > 0$ *and the distributional problem*

$$(P_2) \quad \left| \begin{array}{l} u \in C(\mathbb{R}, P') \\ \partial_t u - i\mu\partial_x^2 u + i\alpha u = 0 \ \in P' \\ u(0) = f \in P' \, . \end{array} \right.$$

then (P_2) *has a unique solution* $u \in C^1(\mathbb{R}, P')$. *Furthermore, the solution depends continuously on the initial data. That is, given* f_n, $f \in P'$ *such that* $f_n \xrightarrow{P'} f$ *implies* $u_n(t) \xrightarrow{P'} u(t)$, $\forall t \in \mathbb{R}$, *where* u_n *is solution of* (P_2) *with initial data* f_n *and* u *is solution of* (P_2) *with initial data* f.

Proof.- We have organized the proof as follows.

1. Suppose there exists $u \in C(\mathbb{R}, P')$ satisfying (P_2), then taking the Fourier transform to the equation

$$\partial_t u - i\mu\partial_x^2 u + i\alpha u \;\; = \;\; 0 \, ,$$

we get

$$0 \;\; = \;\; \partial_t \hat{u} - i\mu(ik)^2 \hat{u} + i\alpha\hat{u} \; = \; \partial_t \hat{u} + i\mu k^2 \hat{u} + i\alpha\hat{u},$$

which for each $k \in Z$ is an ODE with initial data $\hat{u}(k, 0) = \hat{f}(k)$.

Thus, we propose an uncoupled system of homogeneous first-order ordinary differential equations

$$(\Omega_k)\left|\begin{array}{l}\hat{u}\in C(I\!R,S'(Z))\\ \partial_t\hat{u}(k,t)+i\mu k^2\hat{u}(k,t)+i\alpha\hat{u}(k,t)=0\\ \hat{u}(k,0)=\hat{f}(k)\text{ with }\hat{f}\in S'(Z)\,,\end{array}\right.$$

$\forall k\in Z$ and we get

$$\hat{u}(k,t)=e^{-i\mu k^2t}e^{-i\alpha t}\hat{f}(k)\,,$$

from where we obtain the explicit expression of u, candidate for solution:

$$u(t) = \sum_{k=-\infty}^{+\infty}\hat{u}(k,t)\phi_k = \sum_{k=-\infty}^{+\infty}e^{-i\mu k^2t}e^{-i\alpha t}\hat{f}(k)\phi_k\,, \tag{3.1}$$

$$= \left[\left(\widehat{f}(k)e^{-i\mu k^2t}e^{-i\alpha t}\right)_{k\in Z}\right]^{\vee}. \tag{3.2}$$

Since $f\in P'$ then $\hat{f}\in S'(Z)$. Thus, we affirm that

$$\left(\widehat{f}(k)e^{-i\mu k^2t}e^{-i\alpha t}\right)_{k\in Z}\in S'(Z)\,,\quad \forall t\in I\!R\,. \tag{3.3}$$

Indeed, let $t\in I\!R$, since $\hat{f}\in S'(Z)$ then satisfies: $\exists C>0$, $\exists N\in I\!N$ such that $|\hat{f}(k)|\leq C|k|^N$, $\forall k\in Z-\{0\}$, using this we get

$$|\widehat{f}(k)e^{-i\mu k^2t}e^{-i\alpha t}|=|\widehat{f}(k)|\underbrace{|e^{-i\mu k^2t}|}_{=1}\underbrace{|e^{-i\alpha t}|}_{=1}=|\widehat{f}(k)|\leq C|k|^N\,.$$

Then,

$$\left(\widehat{f}(k)e^{-i\mu k^2t}e^{-i\alpha t}\right)_{k\in Z}\in S'(Z)\,.$$

If we define

$$u(t):=\left[\left(\widehat{f}(k)e^{-i\mu k^2t}e^{-i\alpha t}\right)_{k\in Z}\right]^{\vee}\,,\qquad \text{for all } t\in I\!R\,, \tag{3.4}$$

we have that $u(t)\in P'$, $\forall t\in I\!R$, since we apply the inverse Fourier transform to$(\hat{f}(k)e^{-i\mu k2t}e^{-i\alpha t})_{k\in Z}\in S'(Z)$.

2. We will prove that u defined in (3.4) is solution of (P_2).

Evaluating (3.2) at $t=0$, we obtain

$$u(0)=\left[\left(\widehat{f}(k)\right)_{k\in Z}\right]^{\vee}=\left[\widehat{f}\right]^{\vee}=f\,.$$

In addition, the following statements are verified.

a) $\partial_t u(t)=i\mu\partial_x^2 u(t)-i\alpha u(t)$ in P', $\forall t\in I\!R$. That is, we will prove that the following equality

$$\underbrace{\lim_{h\to 0}<\frac{u(t+h)-u(t)}{h},\varphi>}_{<\partial_t u(t),\varphi>:=}=i\mu<\partial_x^2u(t),\varphi>-i\alpha<u(t),\varphi>\,,\ \forall\varphi\in P$$

is satisfied, for all $t \in \mathbb{R}$.

Indeed, let $t \in \mathbb{R}$, $\varphi \in P$ and $h \in \mathbb{R} - \{0\}$, we denote

$$I_{h,t} :=< \frac{u(t+h)-u(t)}{h}, \varphi > \ .$$

Thus, we get

$$\begin{aligned}
I_{h,t} &= \frac{1}{h}\{< u(t+h),\varphi > - < u(t),\varphi >\} \\
&= \frac{1}{h}\left\{ \lim_{n\to+\infty} < \sum_{k=-n}^{n} \widehat{f}(k)e^{-i\mu k^2(t+h)}e^{-i\alpha(t+h)}\phi_k,\varphi > \right. \\
&\qquad \left. - \lim_{n\to+\infty} < \sum_{k=-n}^{n} \widehat{f}(k)e^{-i\mu k^2 t}e^{-i\alpha t}\phi_k,\varphi > \right\} \\
&= \frac{1}{h}\left\{ \lim_{n\to+\infty} < \sum_{k=-n}^{n} \widehat{f}(k)e^{-i\mu k^2 t}e^{-i\alpha t}\left(e^{-i\mu k^2 h}e^{-i\alpha h}-1\right)\phi_k,\varphi > \right\} \\
&= \lim_{n\to+\infty} < \sum_{k=-n}^{n} \widehat{f}(k)e^{-i\mu k^2 t}e^{-i\alpha t}\left(\frac{e^{-i\mu k^2 h}e^{-i\alpha h}-1}{h}\right)\phi_k,\varphi > \\
&= \lim_{n\to+\infty} \left\{ \sum_{k=-n}^{n} \widehat{f}(k)e^{-i\mu k^2 t}e^{-i\alpha t}\left(\frac{e^{-i\mu k^2 h}e^{-i\alpha h}-1}{h}\right)\underbrace{< \phi_k,\varphi >}_{=2\pi\widehat{\varphi}(-k)} \right\} \\
&= \lim_{n\to+\infty} 2\pi \left\{ \sum_{k=-n}^{n} \widehat{f}(k)e^{-i\mu k^2 t}e^{-i\alpha t}\left(\frac{e^{-i\mu k^2 h}e^{-i\alpha h}-1}{h}\right)\widehat{\varphi}(-k) \right\} \\
&= 2\pi \sum_{k=-\infty}^{+\infty} \widehat{f}(k)e^{-i\mu k^2 t}e^{-i\alpha t}\left(\frac{e^{-i\mu k^2 h}e^{-i\alpha h}-1}{h}\right)\widehat{\varphi}(-k)\,. \qquad (3.5)
\end{aligned}$$

Let $h > 0$, we have

$$\begin{aligned}
e^{-i\mu k^2 h}e^{-i\alpha h}-1 &= \int_0^h [e^{-i\mu k^2 s}e^{-i\alpha s}]'\,ds \\
&= \int_0^h (-i\mu k^2 - i\alpha)e^{-i\mu k^2 s}e^{-i\alpha s}\,ds\,. \qquad (3.6)
\end{aligned}$$

Taking norm to equality (3.6) we obtain

$$\begin{aligned}
\left|e^{-i\mu k^2 h}e^{-i\alpha h}-1\right| &\leq \int_0^h \{\mu|k|^2+|\alpha|\}\underbrace{\left|e^{-i\mu k^2 s}\right|}_{=1}\underbrace{\left|e^{-i\alpha s}\right|}_{=1}\,ds \\
&= \{\mu|k|^2+|\alpha|\}\underbrace{\int_0^h ds}_{=h} \\
&= \{\mu|k|^2+|\alpha|\}h\,. \qquad (3.7)
\end{aligned}$$

That is, from (3.7) we get

$$\left|\frac{e^{-i\mu k^2 h}e^{-i\alpha h}-1}{h}\right| \leq \mu|k|^2+|\alpha|\,. \qquad (3.8)$$

Note that (3.8) is valid for $h \in \mathbb{R} - \{0\}$.

Using the inequality (3.8) and that $\hat{f} \in S'(\mathbb{Z})$ we obtain

$$\sum_{k=-\infty}^{+\infty} |\widehat{f}(k)| \underbrace{|e^{-i\mu k^2 t}|}_{=1} \underbrace{|e^{-i\alpha t}|}_{=1} |\widehat{\varphi}(-k)| \left| \frac{e^{-i\mu k^2 h} e^{-i\alpha h} - 1}{h} \right|$$
$$\leq \sum_{k=-\infty}^{+\infty} |\widehat{f}(k)||\widehat{\varphi}(-k)|\{\mu|k|^2 + |\alpha|\}$$
$$= \mu \sum_{k=-\infty}^{+\infty} |\widehat{f}(k)||\widehat{\varphi}(-k)||k|^2 + |\alpha| \sum_{k=-\infty}^{+\infty} |\widehat{f}(k)||\widehat{\varphi}(-k)|$$
$$\leq C \left\{ \mu \sum_{k=-\infty}^{+\infty} |k|^{N+2} |\widehat{\varphi}(\underbrace{-k}_{=J})| + |\alpha| \sum_{k=-\infty}^{+\infty} |k|^N |\widehat{\varphi}(\underbrace{-k}_{=J})| \right\}$$
$$= C \left\{ \mu \sum_{J=-\infty}^{+\infty} |J|^{N+2} |\widehat{\varphi}(J)| + |\alpha| \sum_{J=-\infty}^{+\infty} |J|^N |\widehat{\varphi}(J)| \right\} < \infty$$

since $\hat{\varphi} \in S(\mathbb{Z})$.

Using the Weierstrass M-Test, the series $I_{h,t}$ is absolute and uniformly convergent. Then we can take limit and get

$$\lim_{h\to 0} I_{h,t} = 2\pi \sum_{k=-\infty}^{+\infty} \widehat{f}(k) e^{-i\mu k^2 t} e^{-i\alpha t} \widehat{\varphi}(-k) \underbrace{\lim_{h\to 0} \left\{ \frac{e^{-i\mu k^2 h} e^{-i\alpha h} - 1}{h} \right\}}_{=-i\mu k^2 - i\alpha}$$
$$= (-i\mu) 2\pi \sum_{k=-\infty}^{+\infty} \widehat{f}(k) e^{-i\mu k^2 t} e^{-i\alpha t} \widehat{\varphi}(-k) k^2$$
$$-i\alpha 2\pi \sum_{k=-\infty}^{+\infty} \widehat{f}(k) e^{-i\mu k^2 t} e^{-i\alpha t} \widehat{\varphi}(-k)\,. \tag{3.9}$$

Using (3.9) and that $< T^{(2)}, \varphi >= (-1)^2 < T, \varphi^{(2)} >=< T, \varphi^{(2)} >$ for $\varphi \in P$, $T \in P'$, we have

$$\lim_{h\to 0} I_{h,t} = (-i\mu) 2\pi \sum_{k=-\infty}^{+\infty} \widehat{f}(k) e^{-i\mu k^2 t} e^{-i\alpha t} \underbrace{\widehat{\varphi}(-k)}_{=\frac{1}{2\pi} < \varphi, \phi_k >} k^2$$
$$-i\alpha 2\pi \sum_{k=-\infty}^{+\infty} \widehat{f}(k) e^{-i\mu k^2 t} e^{-i\alpha t} \underbrace{\widehat{\varphi}(-k)}_{=\frac{1}{2\pi} < \varphi, \phi_k >}$$
$$= i\mu \sum_{k=-\infty}^{+\infty} \widehat{f}(k) e^{-i\mu k^2 t} e^{-i\alpha t} < \varphi, \underbrace{-k^2 \phi_k}_{=(ik)^2 \phi_k} >$$
$$-i\alpha \sum_{k=-\infty}^{+\infty} \widehat{f}(k) e^{-i\mu k^2 t} e^{-i\alpha t} < \varphi, \phi_k >$$
$$= i\mu \sum_{k=-\infty}^{+\infty} \widehat{f}(k) e^{-i\mu k^2 t} e^{-i\alpha t} \underbrace{< \varphi, {\phi_k}^{(2)} >}_{=< \varphi^{(2)}, \phi_k >}$$
$$-i\alpha \sum_{k=-\infty}^{+\infty} \widehat{f}(k) e^{-i\mu k^2 t} e^{-i\alpha t} < \varphi, \phi_k >$$

$$
\begin{aligned}
&= i\mu \sum_{k=-\infty}^{+\infty} \widehat{f}(k) e^{-i\mu k^2 t} e^{-i\alpha t} < \phi_k, \varphi^{(2)} > \\
&\quad -i\alpha \sum_{k=-\infty}^{+\infty} \widehat{f}(k) e^{-i\mu k^2 t} e^{-i\alpha t} < \phi_k, \varphi > \\
&= i\mu \lim_{n\to+\infty} \sum_{k=-n}^{n} \widehat{f}(k) e^{-i\mu k^2 t} e^{-i\alpha t} < \phi_k, \varphi^{(2)} > \\
&\quad -i\alpha \lim_{n\to+\infty} \sum_{k=-n}^{n} \widehat{f}(k) e^{-i\mu k^2 t} e^{-i\alpha t} < \phi_k, \varphi > \\
&= i\mu \lim_{n\to+\infty} < \sum_{k=-n}^{n} \widehat{f}(k) e^{-i\mu k^2 t} e^{-i\alpha t} \phi_k, \varphi^{(2)} > \\
&\quad -i\alpha \lim_{n\to+\infty} < \sum_{k=-n}^{n} \widehat{f}(k) e^{-i\mu k^2 t} e^{-i\alpha t} \phi_k, \varphi > \\
&= i\mu < u(t), \varphi^{(2)} > -i\alpha < u(t), \varphi > \qquad (3.10) \\
&= i\mu < \partial_x^2 u(t), \varphi > \; -i\alpha < u(t), \varphi > \; .
\end{aligned}
$$

Therefore,

$$
< \partial_t u(t), \varphi > \;=\; i\mu < \partial_x^2 u(t), \varphi > -i\alpha < u(t), \varphi > , \quad \forall \varphi \in P\,, \quad \forall t \in I\!R\,.
$$

That is,

$$
\partial_t\, u(t) \;=\; i\mu\, \partial_x^2\, u(t) - i\alpha\; u(t) \qquad \text{in } P'\,, \quad \forall t \in I\!R\,.
$$

b) $u \in C(I\!R, P')$. That is, we will prove that

$$
u(t+h) \xrightarrow{P'} u(t) \;\; \text{when} \;\; h \to 0\,,\; \forall t \in I\!R\,.
$$

In effect, let $t \in I\!R$ and $\varphi \in P$, we will prove that

$$
H_{t,h} := < u(t+h) - u(t), \varphi > \longrightarrow 0\,, \qquad \text{when} \;\; h \to 0\,.
$$

We know that if $\varphi \in P$ then $\hat{\varphi} \in S(Z)$. Using (3.5) we have

$$
H_{t,h} = 2\pi \sum_{k=-\infty}^{+\infty} \widehat{f}(k) e^{-i\mu k^2 t} e^{-i\alpha t} \left(e^{-i\mu k^2 h} e^{-i\alpha h} - 1 \right) \widehat{\varphi}(-k)\,.
$$

Let $0 < |h| < 1$, from (3.8) we get

$$
\left| e^{-i\mu k^2 h} e^{-i\alpha h} - 1 \right| \le \mu |k|^2 |h| + |\alpha||h| < \mu |k|^2 + |\alpha|\,. \qquad (3.11)
$$

Using (3.11) and that $\hat{f} \in S'(Z)$ we obtain

$$
\begin{aligned}
&\sum_{k=-\infty}^{+\infty} |\widehat{f}(k)| \underbrace{|e^{-i\mu k^2 t}|}_{=1} \underbrace{|e^{-i\alpha t}|}_{=1} \left| e^{-i\mu k^2 h} e^{-i\alpha h} - 1 \right| |\widehat{\varphi}(-k)| \\
&\le C\mu \sum_{k=-\infty}^{+\infty} |k|^{N+2} |\widehat{\varphi}(\underbrace{-k}_{=J})| + C|\alpha| \sum_{k=-\infty}^{+\infty} |k|^{N} |\widehat{\varphi}(\underbrace{-k}_{=J})|
\end{aligned}
$$

$$= C\mu \sum_{J=-\infty}^{+\infty} |J|^{N+2}|\widehat{\varphi}(J)| + C|\alpha| \sum_{J=-\infty}^{+\infty} |J|^{N}|\widehat{\varphi}(J)| < \infty$$

since $\hat{\varphi} \in S(Z)$.

Using the Weierstrass M-Test we conclude that the series $H_{t,h}$ converges absolute and uniformly. Then it is possible to take limit and obtain

$$\lim_{h\to 0} H_{t,h} = 2\pi \sum_{k=-\infty}^{+\infty} \widehat{f}(k)e^{-i\mu k^2 t}e^{-i\alpha t}\widehat{\varphi}(-k) \underbrace{\lim_{h\to 0}\left\{e^{-i\mu k^2 h}e^{-i\alpha h} - 1\right\}}_{=0} = 0\,.$$

Since $t \in I\!R$ was taken arbitrarily, then we can conclude that

$$u \in C(I\!R, P')\,.$$

c) $\partial_t u \in C(I\!R, P')$. That is, we will prove that

$$\partial_t u(t+h) \xrightarrow{P'} \partial_t u(t) \ \text{ when } \ h \to 0\,,\ \forall t \in I\!R\,.$$

In effect, let $t \in I\!R$ and $\varphi \in P$, using item a) we have

$$\begin{aligned}
&< \partial_t u(t+h), \varphi > - < \partial_t u(t), \varphi > \\
&\quad = i\mu\{< \partial_x^2 u(t+h), \varphi > - < \partial_x^2 u(t), \varphi >\} \\
&\quad\quad -i\alpha\{< u(t+h), \varphi > - < u(t), \varphi >\} \\
&\quad = i\mu \underbrace{\{< u(t+h), \varphi^{(2)} > - < u(t), \varphi^{(2)} >\}}_{\longrightarrow 0} \\
&\quad\quad -i\alpha \underbrace{\{< u(t+h), \varphi > - < u(t), \varphi >\}}_{\longrightarrow 0} \longrightarrow 0 \qquad (3.12)
\end{aligned}$$

when $h \to 0$, since item b) is valid with $\varphi^{(r)} \in P$ for $r = 0, 2$.

From b) and c) we have that $u \in C^1(I\!R, P')$.

d) Now, if $f_n \xrightarrow{P'} f$ we will prove that:

$$u_n(t) \xrightarrow{P'} u(t)\,, \quad \forall t \in I\!R\,.$$

We know that if $f_n \xrightarrow{P'} f$ then $\widehat{f_n} \xrightarrow{S'(Z)} \widehat{f}$, that is

$$< \widehat{f_n} - \widehat{f}, \xi > \longrightarrow 0 \quad \text{when } n \to +\infty\,, \quad \forall \xi \in S(Z)\,. \qquad (3.13)$$

For $t \in I\!R$ fixed and arbitrary, we want to prove that

$$< u_n(t), \psi > \longrightarrow < u(t), \psi > \quad \text{when } n \to +\infty\,, \quad \forall \psi \in P\,.$$

Thus, let $t \in I\!R$ be fixed and $\psi \in P$, using the generalized Parseval identity, we obtain the following equalities:

$$< u_n(t), \psi > \ = \ 2\pi < \left(\widehat{f_n}(k)e^{-i\mu k^2 t}e^{-i\alpha t}\right)_{k\in Z}, \widetilde{\widehat{\psi}} > \qquad (3.14)$$

$$< u(t), \psi > \ = \ 2\pi < \left(\widehat{f}(k)e^{-i\mu k^2 t}e^{-i\alpha t}\right)_{k\in Z}, \widetilde{\widehat{\psi}} >\,. \qquad (3.15)$$

From (3.14) and (3.15) we obtain:

$$< u_n(t), \psi > - < u(t), \psi >$$
$$= 2\pi \sum_{k=-\infty}^{+\infty} \{\widehat{f_n}(k) - \widehat{f}(k)\} \underbrace{e^{-i\mu k^2 t} e^{-i\alpha t} \widetilde{\widehat{\psi}}(k)}_{\xi_k :=} \longrightarrow 0$$

when $n \to +\infty$, since $\xi := (\xi_k)_{k \in Z} \in S(Z)$ and (3.13) holds.

Corollary 3.1 *Let $\mu > 0$ and $\alpha > 0$, then the unique solution of (P_2) is*

$$u(t) = \sum_{k=-\infty}^{+\infty} \widehat{f}(k) e^{-i\mu k^2 t} e^{-i\alpha t} \phi_k = \left[\left(\widehat{f}(k) e^{-i\mu k^2 t} e^{-i\alpha t} \right)_{k \in Z} \right]^{\vee},$$

where $\phi_k(x) = e^{ikx}$, $x \in I\!R$.

3.2 Group of Operators in P'

In this subsection, we will introduce families of operators $\{T_{\mu,\alpha}(t)\}_{t \in I\!R}$ in P', with $\mu > 0$ and $\alpha > 0$; and we will prove that these operators are continuous in the weak sense and satisfy the group properties.

For simplicity, we will denote this family of operators by $\{T(t)\}_{t \in I\!R}$.

Theorem 3.2 *Let $t \in I\!R$, we define:*

$$\begin{aligned} T(t) :\ P' &\longrightarrow P' \\ f &\longrightarrow T(t)f := \left[\left(\widehat{f}(k) e^{-i\mu k^2 t} e^{-i\alpha t} \right)_{k \in Z} \right]^{\vee} \in P', \end{aligned}$$

then the following statements are satisfied:

1. *$T(0) = I$.*
2. *$T(t)$ is $\mathbb{C}$ - linear and continuous $\forall t \in I\!R$. That is, for every $t \in I\!R$, if $f_n \xrightarrow{P'} f$ then $T(t) f_n \xrightarrow{P'} T(t) f$.*
3. *$T(t + r) = T(t) \circ T(r)$, $\forall t, r \in I\!R$.*
4. *$T(t) f_n \xrightarrow{P'} f$ when $t \to 0$, $\forall f \in P'$.*

That is, for each $f \in P'$ fixed, the following is satisfied

$$< T(t)f, \psi > \longrightarrow < f, \psi >, \text{ when } t \to 0, \ \forall \psi \in P.$$

Proof.- Let $f \in P'$ then $\hat{f} \in S'(Z)$. Then, from (3.3) we have

$$\left(\widehat{f}(k) e^{-i\mu k^2 t} e^{-i\alpha t} \right)_{k \in Z} \in S'(Z);$$

taking the inverse Fourier transform, we obtain

$$\underbrace{\left[\left(\widehat{f}(k) e^{-i\mu k^2 t} e^{-i\alpha t} \right)_{k \in Z} \right]^{\vee}}_{= T(t) f} \in P', \quad \forall t \in I\!R.$$

That is, $T(t)$ is well defined for all $t \in I\!R$.

1. We easily obtain:

$$T(0)f = \left[\left(\widehat{f}(k)e^{-i\mu k^2 0}e^{-i\alpha 0}\right)_{k\in Z}\right]^{\vee} = \left[\left(\widehat{f}(k)\right)_{k\in Z}\right]^{\vee} = \left[\widehat{f}\right]^{\vee} = f\,, \quad \forall f \in P'\,.$$

2. Let $t \in \mathbb{R}$, we will prove that $T(t) : P' \to P'$ is $\mathbb{C}$-linear. In effect, let $a \in \mathbb{C}$ and $(\phi, \psi) \in P' \times P'$, we have

$$\begin{aligned} T(t)(a\phi + \psi) &= \left[\left(e^{-i\mu k^2 t}e^{-i\alpha t}[a\phi + \psi]^{\wedge}(k)\right)_{k\in Z}\right]^{\vee} \\ &= \left[\left(e^{-i\mu k^2 t}e^{-i\alpha t}[a\widehat{\phi}(k) + \widehat{\psi}(k)]\right)_{k\in Z}\right]^{\vee} \\ &= \left[a\left(e^{-i\mu k^2 t}e^{-i\alpha t}\widehat{\phi}(k)\right)_{k\in Z} + \left(e^{-i\mu k^2 t}e^{-i\alpha t}\widehat{\psi}(k)\right)_{k\in Z}\right]^{\vee} \\ &= a\left[\left(e^{-i\mu k^2 t}e^{-i\alpha t}\widehat{\phi}(k)\right)_{k\in Z}\right]^{\vee} + \left[\left(e^{-i\mu k^2 t}e^{-i\alpha t}\widehat{\psi}(k)\right)_{k\in Z}\right]^{\vee} \\ &= a\,T(t)\phi + T(t)\psi\,. \end{aligned}$$

Now, for $t \in \mathbb{R}$ we will prove that $T(t) : P' \to P'$ is continuous. That is, if $f_n \xrightarrow{P'} f$ then we will prove that $T(t)f_n \xrightarrow{P'} T(t)f$. Note that the case $t = 0$ is obvious. We know that if $f_n \xrightarrow{P'} f$ then $\hat{f}_n \xrightarrow{S'} \hat{f}$, that is,

$$< \widehat{f_n}, \xi > \to < \widehat{f}, \xi >, \quad \text{when } n \to +\infty\,, \quad \forall \xi \in S(Z)\,.$$

That is,

$$< \widehat{f_n} - \widehat{f}, \xi > \to 0\,, \quad \text{when } n \to +\infty\,, \quad \forall \xi \in S(Z)\,. \tag{3.16}$$

We want to prove that:

$$< T(t)f_n, \psi > \to < T(t)f, \psi > \ \text{when } n \to +\infty\,, \quad \forall \psi \in P\,.$$

Thus, let $t \in \mathbb{R}$ fixed and $\psi \in P$, using the generalized Parseval identity, we obtain the following equalities

$$\begin{aligned} < T(t)f_n, \psi > &= < \left[\left(\widehat{f_n}(k)e^{-i\mu k^2 t}e^{-i\alpha t}\right)_{k\in Z}\right]^{\vee}, \psi > \\ &= 2\pi < \left(\widehat{f_n}(k)e^{-i\mu k^2 t}e^{-i\alpha t}\right)_{k\in Z}, \widehat{\widetilde{\psi}} >, \end{aligned} \tag{3.17}$$

$$\begin{aligned} < T(t)f, \psi > &= < \left[\left(\widehat{f}(k)e^{-i\mu k^2 t}e^{-i\alpha t}\right)_{k\in Z}\right]^{\vee}, \psi > \\ &= 2\pi < \left(\widehat{f}(k)e^{-i\mu k^2 t}e^{-i\alpha t}\right)_{k\in Z}, \widehat{\widetilde{\psi}} > \,. \end{aligned} \tag{3.18}$$

From (3.17) and (3.18) we get

$$\begin{aligned} &< T(t)f_n, \psi > - < T(t)f, \psi > \\ &\quad = 2\pi\left\{< \left(\widehat{f_n}(k)e^{-i\mu k^2 t}e^{-i\alpha t}\right)_{k\in Z}, \widehat{\widetilde{\psi}} > - < \left(\widehat{f}(k)e^{-i\mu k^2 t}e^{-i\alpha t}\right)_{k\in Z}, \widehat{\widetilde{\psi}} >\right\} \\ &\quad = 2\pi\left\{\sum_{k=-\infty}^{+\infty} \widehat{f_n}(k)e^{-i\mu k^2 t}e^{-i\alpha t}\widehat{\widetilde{\psi}}(k) - \sum_{k=-\infty}^{+\infty} \widehat{f}(k)e^{-i\mu k^2 t}e^{-i\alpha t}\widehat{\widetilde{\psi}}(k)\right\} \end{aligned}$$

$$= 2\pi \sum_{k=-\infty}^{+\infty} \{\widehat{f_n}(k) - \widehat{f}(k)\} \underbrace{e^{-i\mu k^2 t} e^{-i\alpha t} \widetilde{\widetilde{\psi}}(k)}_{\xi_k :=} \longrightarrow 0$$

when $n \to +\infty$, since $\xi := (\xi_k)_{k\in Z} \in S(Z)$ and (3.16) holds, that is $< \hat{f}_n - \hat{f}, \xi > \to 0$ when $n \to +\infty$.

3. Let $t, r \in I\!R - \{0\}$, we will prove that $T(t) \circ T(r) = T(t+r)$. In effect, let $\phi \in P'$,

$$\begin{aligned} T(t+r)\phi &= \left[\left(\widehat{\phi}(k)e^{-i\mu k^2(t+r)}e^{-i\alpha(t+r)}\right)_{k\in Z}\right]^{\vee} \\ &= \left[\left(\underbrace{\widehat{\phi}(k)e^{-i\mu k^2 r}e^{-i\alpha r}} \cdot e^{-i\mu k^2 t}e^{-i\alpha t}\right)_{k\in Z}\right]^{\vee} . \end{aligned} \tag{3.19}$$

Since $\phi \in P'$, using (3.3) we hav`e that

$$\left(\widehat{\phi}(k)e^{-i\mu k^2 r}e^{-i\alpha r}\right)_{k\in Z} \in S'(Z) , \quad \forall r \in I\!R . \tag{3.20}$$

Then, taking the inverse Fourier transform, we get:

$$\left[\left(\widehat{\phi}(k)e^{-i\mu k^2 r}e^{-i\alpha r}\right)_{k\in Z}\right]^{\vee} \in P' , \quad \forall r \in I\!R .$$

Thus, we define:

$$g_r := \left[\left(\widehat{\phi}(k)e^{-i\mu k^2 r}e^{-i\alpha r}\right)_{k\in Z}\right]^{\vee} \in P' .$$

That is,

$$g_r := T(r)\phi . \tag{3.21}$$

Taking the Fourier transform to g_r we get:

$$\widehat{g_r} = \left(\widehat{\phi}(k)e^{-i\mu k^2 r}e^{-i\alpha r}\right)_{k\in Z} ,$$

that is,

$$\widehat{g_r}(k) = \widehat{\phi}(k)e^{-i\mu k^2 r}e^{-i\alpha r} , \quad \forall k \in Z . \tag{3.22}$$

Using (3.22) in (3.19) and from (3.21) we have:

$$\begin{aligned} T(t+r)\phi &= \left[\left(\widehat{g_r}(k)e^{-i\mu k^2 t}e^{-i\alpha t}\right)_{k\in Z}\right]^{\vee} \in P' \\ &= T(t)g_r \\ &= T(t)(T(r)\phi) \\ &= [T(t)\circ T(r)](\phi) , \quad \forall t, r \in I\!R - \{0\} . \end{aligned}$$

So we have proven,

$$T(t+r) = T(t)\circ T(r)\,,\ \forall t,r\in I\!R-\{0\}\,. \tag{3.23}$$

If $t = 0$ or $r = 0$ then equality (3.23) is also true, with this we conclude the proof of

$$T(t+r) = T(t)\circ T(r)\,, \quad \forall t,r\in I\!R\,. \tag{3.24}$$

4. Let $f \in P'$, we will prove that:

$$T(t)f \xrightarrow{P'} f \ \text{ when } t\to 0\,.$$

That is, we will prove that

$$< T(t)f,\varphi > \longrightarrow < f,\varphi > \ \text{ when } t\to 0\,, \quad \forall\varphi\in P\,.$$

In effect, for $t \in I\!R - \{0\}$ and $\varphi \in P$, we have

$$\begin{aligned}
\mathcal{H}_t &:= < T(t)f,\varphi > - < f,\varphi > \\
&= \lim_{n\to+\infty}\left\{ < \sum_{k=-n}^{n}\widehat{f}(k)e^{-i\mu k^2 t}e^{-i\alpha t}\phi_k,\varphi > - < \sum_{k=-n}^{n}\widehat{f}(k)\phi_k,\varphi > \right\} \\
&= \lim_{n\to+\infty} < \sum_{k=-n}^{n}\widehat{f}(k)\left(e^{-i\mu k^2 t}e^{-i\alpha t}-1\right)\phi_k,\varphi > \\
&= \lim_{n\to+\infty} \sum_{k=-n}^{n}\widehat{f}(k)\left(e^{-i\mu k^2 t}e^{-i\alpha t}-1\right) < \phi_k,\varphi > \\
&= \lim_{n\to+\infty} 2\pi\sum_{k=-n}^{n}\widehat{f}(k)\left(e^{-i\mu k^2 t}e^{-i\alpha t}-1\right)\widehat{\varphi}(-k) \\
&= 2\pi\sum_{k=-\infty}^{+\infty}\widehat{f}(k)\left(e^{-i\mu k^2 t}e^{-i\alpha t}-1\right)\widehat{\varphi}(-k)\,. \qquad (3.25)
\end{aligned}$$

Since $t \in I\!R - \{0\}$, from (3.8) we get

$$\left|\frac{e^{-i\mu k^2 t}e^{-i\alpha t}-1}{t}\right| \le \mu|k|^2+|\alpha|\,. \tag{3.26}$$

From (3.26) we obtain

$$\left|e^{-i\mu k^2 t}e^{-i\alpha t}-1\right| \le \{\mu|k|^2+|\alpha|\}|t|\,, \quad \forall t\in I\!R\,. \tag{3.27}$$

From (3.27) with $0 < |t| < 1$, we have

$$\left|e^{-i\mu k^2 t}e^{-i\alpha t}-1\right| \le \mu|k|^2+|\alpha|\,. \tag{3.28}$$

Then using (3.28) and that $f \in P'$, we obtain

$$\sum_{k=-\infty}^{+\infty}|\widehat{f}(k)|\left|e^{-i\mu k^2 t}e^{-i\alpha t}-1\right||\widehat{\varphi}(-k)|$$

$$\leq C\left\{\mu\sum_{k=-\infty}^{+\infty}|k|^{N+2}|\widehat{\varphi}(\underbrace{-k}_{=J})|+|\alpha|\sum_{k=-\infty}^{+\infty}|k|^{N}|\widehat{\varphi}(\underbrace{-k}_{=J})|\right\}$$
$$=C\left\{\mu\sum_{J=-\infty}^{+\infty}|J|^{N+2}|\widehat{\varphi}(J)|+|\alpha|\sum_{J=-\infty}^{+\infty}|J|^{N}|\widehat{\varphi}(J)|\right\}\ <\ \infty$$

since $\hat{\varphi} \in S(\mathbb{Z})$.

Using the Weierstrass M-Test we conclude that the H_t series converges absolute and uniformly. So,

$$\begin{aligned}\lim_{t\to 0}\mathcal{H}_t &= 2\pi\sum_{k=-\infty}^{+\infty}\widehat{f}(k)\widehat{\varphi}(-k)\underbrace{\lim_{t\to 0}\{e^{-i\mu k^2 t}e^{-i\alpha t}-1\}}_{=0}\\ &= 0\,.\end{aligned}$$

Thus, we have proved

$$\lim_{t\to 0} < T(t)f,\varphi > = < f,\varphi > \ .$$

Theorem 3.3 *For each $f \in P'$ fixed and the family of operators $\{T(t)\}_{t\in\mathbb{R}}$ from Theorem 3.2, then the application*

$$\begin{aligned}\mathcal{M}\,:\,\mathbb{R} &\longrightarrow P'\\ t &\longrightarrow T(t)f\end{aligned}$$

is continuous in $\mathbb{R}$. That is,

$$T(t+h)f \xrightarrow{P'} T(t)f \ \ when\ h\to 0\,,\ \forall t\in\mathbb{R}\,. \tag{3.29}$$

(is the continuity at t).

That is, (3.29) tell us that for each $t \in \mathbb{R}$ fixed, the following is satisfied

$$< T(t+h)f,\psi > \longrightarrow < T(t)f,\psi >,\quad when\ \ h\to 0\,,\ \forall\,\psi\in P\,.$$

And if $t = 0$, we have the continuity of M at 0, which is item 4) of Theorem 3.2.

Proof.- Let $t \in \mathbb{R} - \{0\}$, arbitrary fixed and $f \in P'$ then $g := T(t)f \in P'$, using item 4) of Theorem 3.2, we have that $T(h)g \xrightarrow{P'} g$ when $h \to 0$. That is,

$$\underbrace{T(h)(T(t)f)}_{=\underbrace{[T(h)\circ T(t)]f}_{=T(h+t)f}} \xrightarrow{P'} T(t)f \text{ when } h\to 0\,,$$

where we use item 3) of Theorem 3.2.

Remark 3.1 *The results obtain in Theorems 3.2 and 3.3 are also valid for the family of operators $\{S(t)\}_{t\in\mathbb{R}}$, defined as*

$$S(t) : P' \ \longrightarrow\ P'$$

$$f \quad \to \quad S(t)f := \left[\left(e^{i\mu k^2 t}e^{-i\alpha t}\widehat{f}(k)\right)_{k\in Z}\right]^{\vee},$$

for $t \in I\!R$. Its proof is similar.

3.3 Version of Theorem 3.1 using the family $\{T(t)\}_{t\in I\!R}$

We improve the statement of theorem 3.1, using a family of weakly continuous Operators $\{T(t)\}_{t\in I\!R}$.

Theorem 3.4 *Let $f \in P'$ and the family of operators $\{T(t)\}_{t\in I\!R}$ from Theorem 3.2, defining $u(t) := T(t)f \in P'$, $\forall t \in I\!R$, then $u \in C(I\!R, P')$ is the unique solution of (P_2). Furthermore, u continuously depends on f. That is, given f_n, $f \in P'$ with $f_n \xrightarrow{P'} f$ implies $u_n(t) \xrightarrow{P'} u(t)$, $\forall t \in I\!R$, where $u_n(t) := T(t)f_n$, $\forall t \in I\!R$ (that is, u_n is a solution of (P_2) with initial data f_n).*

Proof.- It is analogous to the proof of Theorem 3.1.

Corollary 3.2 *Let $f \in P'$ be fixed and the family of operators $\{T(t)\}_{t\in I\!R}$ from Theorem 3.4, then $\exists\ \partial_t T(t)f$, $\forall t \in I\!R$ and the mapping*

$$\begin{aligned} \tilde{\eta}:\ I\!R &\ \to\ P' \\ t &\ \to\ \partial_t T(t)f = i\mu\partial_x^2 T(t)f - i\alpha T(t)f \end{aligned}$$

is continuous at $I\!R$. That is,

$$\partial_t T(t+h)f \xrightarrow{P'} \partial_t T(t)f \quad when\ h \to 0\,, \quad \forall t \in I\!R\,. \tag{3.30}$$

(3.30) tells us that for each $t \in I\!R$ fixed, it holds:

$$< \partial_t T(t+h)f, \varphi > \longrightarrow < \partial_t T(t)f, \varphi > \quad when\ h \to 0\,, \quad \forall \varphi \in P\,.$$

Proof.- Indeed,

$$\begin{aligned} &< \partial_t T(t+h)f, \varphi > \ - \ < \partial_t T(t)f, \varphi > \\ &\quad = i\mu\{< \partial_x^2 T(t+h)f, \varphi > - < \partial_x^2 T(t)f, \varphi >\} \\ &\qquad -i\alpha\{< T(t+h)f, \varphi > - < T(t)f, \varphi >\} \\ &\quad = i\mu\{\underbrace{< T(t+h)f, \varphi^{(2)} > - < T(t)f, \varphi^{(2)} >}_{\longrightarrow 0}\} \\ &\qquad -i\alpha\{\underbrace{< T(t+h)f, \varphi > - < T(t)f, \varphi >}_{\longrightarrow 0}\} \ \longrightarrow \ 0 \end{aligned}$$

when $h \to 0$, due to Theorem 3.3 with $\psi := \varphi^{(J)} \in P$, for $J = 0, 2$.

Corollary 3.3 *Let $f \in P'$ be fixed and the family of operators $\{T(t)\}_{t\in I\!R}$ from Theorem 3.4, then the solution of (P_2): $u(t) := T(t)f$, $\forall t \in I\!R$, satisfies $u \in C^1(I\!R, P')$.*

Proof.- It comes out as a consequence of Corollary 3.2.

Remark 3.2 *If the order of the equation is even and not multiple of four, we can obtain similar results.*

4 I CONCLUSIONS

In our study of the Schrödinger type homogeneous model in the periodic distributional space P', we have obtained the following results:

1. We prove the existence, uniqueness of the solution of the problem (P_2). Thus we also prove the continuous dependence of the solution respect to the initial data.

2. We introduce families of operators in P': $\{T(t)\}_{t\in\mathbb{R}}$ and we prove that they are linear and weakly continuous in P'. Furthermore, we proved that they form a group of weakly continuous operators in P'.

3. With the family of operators $\{T(t)\}_{t\in\mathbb{R}}$ we improve Theorem 3.1.

4. Finally, we must indicate that this study can be applied to other evolution equations in P'.

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