SEMIGROUP OF WEAKLY CONTINUOUS OPERATORS ASSOCIATED TO A SCHRÖDINGER EQUATION

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ABSTRACT: In this article, we prove the existence and uniqueness of the solution of the Schrödinger equation in the periodic distributional space *P'*. Furthermore, we prove that the solution depends continuously respect to the initial data in *P'*. Introducing a family of weakly continuous operators, we prove that this family is a semigroup in *P'*. Then, with this family of operators, we get a fine version of the existence and dependency continuous theorem obtained. Finally, we give some remarks derived from this study.
KEYWORDS: Semigroups theory, weakly continuous operators, existence of solution, Schrödinger equation, distributional problem, periodic distributional space.

1 I INTRODUCTION

We know from [5], with $m = 2$, that the Schrödinger equation

$$u_t - i\mu\partial_x^2 u = 0 \ \in \ P' \qquad (1.1)$$

with initial data in the periodic distributional sapce: *P'*, has a solution in *P'*. So we set up the model:

$$u_t - i\mu\partial_x^2 u = \beta\partial_x^2 u \ \in \ P' \qquad (1.2)$$

with initial data in *P'*, which we will solve following the ideas of [4] and [5].

That is, we will prove that (1.2) has a solution and that it is unique. Furthermore, we will demonstrate that the solution depends continuously with respect to the initial data in *P'*, considering the weak convergence in *P'*. And we will prove that the introduced family of operators forms a semigroup of weakly continuous operators. Thus, with this family we will rewrite our result in an elegant version.

Our article is organized as follows. In section 2, we indicate the methodology used and cite the references used. In section 3, we put the results obtained from our study. This section is divided into three subsections. Thus, in 3.1 we prove that the problem (P_2) has a unique solution and also demonstrate that the solution depends continuously with respect to the initial data. In subsection 3.2, we introduce families of weakly continuous linear operators in P' that manage to form a semigroup. In subsection 3.3 we improve Theorem 3.1.

Finally, in section 4 we give the conclusions of this study.

2 I METHODOLOGY

As theoretical framework in this article we use the references [1], [2], [3], [4] and [5] for Fourier Theory in periodic distributional space, periodic Sobolev spaces, topological vector spaces, weakly continuous operators and existence of solution of a distributional differential equation.

3 I MAIN RESULTS

The presentation of the results obtained has been organized in subsections and is as follows.

3.1 Solution of the Schrödinger Equation (P_2)

In this subsection we will study the existence of a solution to the problem (P_2) and the continuous dependence of the solution with respect to the initial data in P'.

Theorem 3.1 *Let $\mu > 0$, $\beta > 0$ and the distributional problem*

$$(P_2) \quad \left| \begin{array}{l} u \in C([0,+\infty), P') \\ \partial_t u - i\mu \partial_x^2 u = \beta \partial_x^2 u \;\; \in P' \\ u(0) = f \in P' . \end{array} \right.$$

then (P_2) has a unique solution $u \in C^1((0, +\infty), P')$. Furthermore, the solution depends continuously on the initial data. That is, given f_n, $f \in P'$ such that $f_n \xrightarrow{P'} f$ P' implies $u_n(t) \xrightarrow{P'} u(t)$, $\forall t \in [0, +\infty)$, where u_n is solution of (P_2) with initial data f_n and u is solution of (P_2) with initial data f.

Proof.- We have organized the proof as follows.

1. Suppose there exists $u \in C([0, +\infty), P')$ satisfying (P_2), then taking the Fourier transform to the equation

$$\partial_t u - i\mu \partial_x^2 u \quad = \quad \beta \partial_x^2 u \, ,$$

we get

$$-\beta k^2 \hat{u} = \beta (ik)^2 \hat{u} \quad = \quad \partial_t \hat{u} - i\mu (ik)^2 \hat{u} \; = \; \partial_t \hat{u} + i\mu k^2 \hat{u},$$

which for each $k \in Z$ is an ODE with initial data $\hat{u}(k, 0) = \hat{f}\,(k)$.

Thus, we propose an uncoupled system of homogeneous first-order ordinary differential equations

$$(\Omega_k)\left|\begin{array}{l}\hat{u}\in C((0,+\infty),S'(Z))\\ \partial_t\hat{u}(k,t)+i\mu k^2\hat{u}(k,t)=-\beta k^2\hat{u}(k,t)\\ \hat{u}(k,0)=\hat{f}(k)\text{ with }\hat{f}\in S'(Z)\,,\end{array}\right.$$

$\forall k \in Z$ and we get

$$\hat{u}(k,t)=e^{-i\mu k^2 t}e^{-\beta k^2 t}\hat{f}(k)\,,$$

from where we obtain the explicit expression of u, candidate for solution:

$$u(t) = \sum_{k=-\infty}^{+\infty}\hat{u}(k,t)\phi_k = \sum_{k=-\infty}^{+\infty} e^{-i\mu k^2 t}e^{-\beta k^2 t}\hat{f}(k)\phi_k\,, \tag{3.1}$$

$$= \left[\left(\widehat{f}(k)e^{-i\mu k^2 t}e^{-\beta k^2 t}\right)_{k\in Z}\right]^{\vee}. \tag{3.2}$$

Since $f \in P'$ then $\hat{f} \in S'(Z)$. Thus, we affirm that

$$\left(\widehat{f}(k)e^{-i\mu k^2 t}e^{-\beta k^2 t}\right)_{k\in Z}\in S'(Z)\,,\quad \forall t\geq 0\,. \tag{3.3}$$

Indeed, let $t \geq 0$, since $\hat{f} \in S'(Z)$ then satisfies: $\exists C > 0$, $\exists N \in IN$ such that $|\hat{f}(k)| \leq C|k|^N$, $\forall k \in Z - \{0\}$, using this we get

$$|\widehat{f}(k)e^{-i\mu k^2 t}e^{-\beta k^2 t}| = |\widehat{f}(k)e^{-i\mu k^2 t}|\underbrace{e^{-\beta k^2 t}}_{\leq 1}\leq|\widehat{f}(k)|\underbrace{|e^{-i\mu k^2 t}|}_{=1}=|\widehat{f}(k)|\leq C|k|^N\,.$$

Then,

$$\left(\widehat{f}(k)e^{-i\mu k^2 t}e^{-\beta k^2 t}\right)_{k\in Z}\in S'(Z)\,.$$

If we define

$$u(t) := \left[\left(\widehat{f}(k)e^{-i\mu k^2 t}e^{-\beta k^2 t}\right)_{k\in Z}\right]^{\vee}\,,\qquad \text{for all } t\geq 0\,, \tag{3.4}$$

we have that $u(t) \in P'$, $\forall t \geq 0$, since we apply the inverse Fourier transform to $(\hat{f}(k)e^{-i\mu k^2 t}e^{-\beta k^2 t})_{k\in Z} \in S'(Z)$.

2. We will prove that u defined in (3.4) is solution of (P_2).

Evaluating (3.2) at $t = 0$, we obtain

$$u(0) = \left[\left(\widehat{f}(k)\right)_{k\in Z}\right]^{\vee} = \left[\widehat{f}\right]^{\vee} = f\,.$$

In addition, the following statements are verified.

a) $\partial_t u(t) = i\mu\partial_x^2 u(t) + \beta\,\partial_x^2 u(t)$ in P', $\forall t \geq 0$. That is, we will prove that it is satisfied:

$$\underbrace{\lim_{h\to 0} < \frac{u(t+h)-u(t)}{h},\varphi >}_{<\partial_t u(t),\varphi>:=} = i\mu < \partial_x^2 u(t),\varphi > +\beta < \partial_x^2 u(t),\varphi >\,,\ \forall\varphi\in P$$

and for all $t \geq 0$.

Indeed, let $t > 0$, $\varphi \in P$ and $0 < |h| < t$, we denote

$$I_{h,t} :=< \frac{u(t+h) - u(t)}{h}, \varphi > \ .$$

Thus, we get

$$\begin{aligned}
I_{h,t} &= \frac{1}{h}\{< u(t+h), \varphi > - < u(t), \varphi >\} \\
&= \frac{1}{h}\left\{ \lim_{n\to+\infty} < \sum_{k=-n}^{n} \widehat{f}(k) e^{-i\mu k^2(t+h)} e^{-\beta k^2(t+h)} \phi_k, \varphi > \right. \\
&\qquad \left. - \lim_{n\to+\infty} < \sum_{k=-n}^{n} \widehat{f}(k) e^{-i\mu k^2 t} e^{-\beta k^2 t} \phi_k, \varphi > \right\} \\
&= \frac{1}{h}\left\{ \lim_{n\to+\infty} < \sum_{k=-n}^{n} \widehat{f}(k) e^{-i\mu k^2 t} e^{-\beta k^2 t} \left(e^{-i\mu k^2 h} e^{-\beta k^2 h} - 1 \right) \phi_k, \varphi > \right\} \\
&= \lim_{n\to+\infty} < \sum_{k=-n}^{n} \widehat{f}(k) e^{-i\mu k^2 t} e^{-\beta k^2 t} \left(\frac{e^{-i\mu k^2 h} e^{-\beta k^2 h} - 1}{h} \right) \phi_k, \varphi > \\
&= \lim_{n\to+\infty} \left\{ \sum_{k=-n}^{n} \widehat{f}(k) e^{-i\mu k^2 t} e^{-\beta k^2 t} \left(\frac{e^{-i\mu k^2 h} e^{-\beta k^2 h} - 1}{h} \right) \underbrace{< \phi_k, \varphi >}_{=2\pi\widehat{\varphi}(-k)} \right\} \\
&= \lim_{n\to+\infty} 2\pi \left\{ \sum_{k=-n}^{n} \widehat{f}(k) e^{-i\mu k^2 t} e^{-\beta k^2 t} \left(\frac{e^{-i\mu k^2 h} e^{-\beta k^2 h} - 1}{h} \right) \widehat{\varphi}(-k) \right\} \\
&= 2\pi \sum_{k=-\infty}^{+\infty} \widehat{f}(k) e^{-i\mu k^2 t} e^{-\beta k^2 t} \left(\frac{e^{-i\mu k^2 h} e^{-\beta k^2 h} - 1}{h} \right) \widehat{\varphi}(-k) \, . \qquad (3.5)
\end{aligned}$$

Let $h > 0$, we have

$$\begin{aligned}
e^{-i\mu k^2 h} e^{-\beta k^2 h} - 1 &= \int_0^h [e^{-i\mu k^2 s} e^{-\beta k^2 s}]' \, ds \\
&= \int_0^h (-i\mu k^2 - \beta k^2) e^{-i\mu k^2 s} e^{-\beta k^2 s} \, ds \, . \qquad (3.6)
\end{aligned}$$

Taking norm to equality (3.6) we obtain

$$\begin{aligned}
\left| e^{-i\mu k^2 h} e^{-\beta k^2 h} - 1 \right| &\leq \int_0^h \{\mu|k|^2 + \beta|k|^2\} \underbrace{|e^{-i\mu k^2 s}|}_{=1} \underbrace{e^{-\beta k^2 s}}_{\leq 1} \, ds \\
&= \{\mu|k|^2 + \beta|k|^2\} \underbrace{\int_0^h ds}_{=h} \\
&= \{\mu|k|^2 + \beta|k|^2\} h \, . \qquad (3.7)
\end{aligned}$$

That is, from (3.7) we get

$$\left| \frac{e^{-i\mu k^2 h} e^{-\beta k^2 h} - 1}{h} \right| \leq \mu |k|^2 + \beta |k|^2 \, . \tag{3.8}$$

Using the inequality (3.8) and that $\hat{f} \in S'(Z)$ we obtain

$$\sum_{k=-\infty}^{+\infty} |\widehat{f}(k)| \underbrace{|e^{-i\mu k^2 t}|}_{=1} \underbrace{e^{-\beta k^2 t}}_{\leq 1} |\widehat{\varphi}(-k)| \left| \frac{e^{-i\mu k^2 h} e^{-\beta k^2 h} - 1}{h} \right|$$

$$\leq \sum_{k=-\infty}^{+\infty} |\widehat{f}(k)| |\widehat{\varphi}(-k)| \{\mu |k|^2 + \beta |k|^2\}$$

$$= \mu \sum_{k=-\infty}^{+\infty} |\widehat{f}(k)| |\widehat{\varphi}(-k)| |k|^2 + \beta \sum_{k=-\infty}^{+\infty} |\widehat{f}(k)| |\widehat{\varphi}(-k)| |k|^2$$

$$\leq C \left\{ \mu \sum_{k=-\infty}^{+\infty} |k|^{N+2} |\widehat{\varphi}(\underbrace{-k}_{=J})| + \beta \sum_{k=-\infty}^{+\infty} |k|^{N+2} |\widehat{\varphi}(\underbrace{-k}_{=J})| \right\}$$

$$= C \left\{ \mu \sum_{J=-\infty}^{+\infty} |J|^{N+2} |\widehat{\varphi}(J)| + \beta \sum_{J=-\infty}^{+\infty} |J|^{N+2} |\widehat{\varphi}(J)| \right\} < \infty$$

since $\hat{\varphi} \in S(Z)$.

Using the Weierstrass M-Test, the series $I_{h,t}$ is absolutely and uniformly convergent. Then we can take limit and get

$$\begin{aligned}
\lim_{h \to 0} I_{h,t} &= 2\pi \sum_{k=-\infty}^{+\infty} \widehat{f}(k) e^{-i\mu k^2 t} e^{-\beta k^2 t} \widehat{\varphi}(-k) \underbrace{\lim_{h \to 0} \left\{ \frac{e^{-i\mu k^2 h} e^{-\beta k^2 h} - 1}{h} \right\}}_{=-i\mu k^2 - \beta k^2} \\
&= (-i\mu) 2\pi \sum_{k=-\infty}^{+\infty} \widehat{f}(k) e^{-i\mu k^2 t} e^{-\beta k^2 t} \widehat{\varphi}(-k) k^2 \\
&\quad -\beta 2\pi \sum_{k=-\infty}^{+\infty} \widehat{f}(k) e^{-i\mu k^2 t} e^{-\beta k^2 t} \widehat{\varphi}(-k) k^2 \, .
\end{aligned} \tag{3.9}$$

Using (3.9) and that $< T^{(2)}, \varphi > = (-1)^2 < T, \varphi^{(2)} > = < T, \varphi^{(2)} >$ for $\varphi \in P$, $T \in P'$, we have

$$\begin{aligned}
\lim_{h \to 0} I_{h,t} &= (-i\mu) 2\pi \sum_{k=-\infty}^{+\infty} \widehat{f}(k) e^{-i\mu k^2 t} e^{-\beta k^2 t} \underbrace{\widehat{\varphi}(-k)}_{=\frac{1}{2\pi} < \varphi, \phi_k >} k^2 \\
&\quad -\beta 2\pi \sum_{k=-\infty}^{+\infty} \widehat{f}(k) e^{-i\mu k^2 t} e^{-\beta k^2 t} \underbrace{\widehat{\varphi}(-k)}_{=\frac{1}{2\pi} < \varphi, \phi_k >} k^2 \\
&= i\mu \sum_{k=-\infty}^{+\infty} \widehat{f}(k) e^{-i\mu k^2 t} e^{-\beta k^2 t} < \varphi, \underbrace{-k^2 \phi_k}_{=(ik)^2 \phi_k} >
\end{aligned}$$

$$
\begin{aligned}
& +\beta \sum_{k=-\infty}^{+\infty} \widehat{f}(k) e^{-i\mu k^2 t} e^{-\beta k^2 t} < \varphi, \underbrace{-k^2\phi_k}_{=(ik)^2\phi_k} > \\
= \;& i\mu \sum_{k=-\infty}^{+\infty} \widehat{f}(k) e^{-i\mu k^2 t} e^{-\beta k^2 t} \underbrace{< \varphi, {\phi_k}^{(2)} >}_{=<\varphi^{(2)},\phi_k>} \\
& +\beta \sum_{k=-\infty}^{+\infty} \widehat{f}(k) e^{-i\mu k^2 t} e^{-\beta k^2 t} \underbrace{< \varphi, {\phi_k}^{(2)} >}_{=<\varphi^{(2)},\phi_k>} \\
= \;& i\mu \sum_{k=-\infty}^{+\infty} \widehat{f}(k) e^{-i\mu k^2 t} e^{-\beta k^2 t} < \phi_k, \varphi^{(2)} > \\
& +\beta \sum_{k=-\infty}^{+\infty} \widehat{f}(k) e^{-i\mu k^2 t} e^{-\beta k^2 t} < \phi_k, \varphi^{(2)} > \\
= \;& i\mu \lim_{n\to+\infty} \sum_{k=-n}^{n} \widehat{f}(k) e^{-i\mu k^2 t} e^{-\beta k^2 t} < \phi_k, \varphi^{(2)} > \\
& +\beta \lim_{n\to+\infty} \sum_{k=-n}^{n} \widehat{f}(k) e^{-i\mu k^2 t} e^{-\beta k^2 t} < \phi_k, \varphi^{(2)} > \\
= \;& i\mu \lim_{n\to+\infty} < \sum_{k=-n}^{n} \widehat{f}(k) e^{-i\mu k^2 t} e^{-\beta k^2 t} \phi_k, \varphi^{(2)} > \\
& +\beta \lim_{n\to+\infty} < \sum_{k=-n}^{n} \widehat{f}(k) e^{-i\mu k^2 t} e^{-\beta k^2 t} \phi_k, \varphi^{(2)} > \\
= \;& i\mu < u(t), \varphi^{(2)} > +\beta < u(t), \varphi^{(2)} > \qquad (3.10) \\
= \;& i\mu < \partial_x^2 u(t), \varphi > +\beta < \partial_x^2 u(t), \varphi > \;.
\end{aligned}
$$

Therefore,

$$
< \partial_t u(t), \varphi > \;=\; i\mu < \partial_x^2 u(t), \varphi > +\beta < \partial_x^2 u(t), \varphi >, \quad \forall \varphi \in P, \quad \forall t \geq 0 .
$$

That is,

$$
\partial_t\, u(t) \,=\, i\mu\, \partial_x^2\, u(t) + \beta\, \partial_x^2\, u(t) \qquad \text{in } P', \quad \forall t \geq 0 .
$$

b) $u \in C([0, +\infty), P')$. That is, we will prove that

$$
u(t+h) \xrightarrow{P'} u(t) \;\text{ when }\; h \to 0, \; \forall t \geq 0 .
$$

In effect, let $t > 0$ and $\varphi \in P$, we will prove that

$$
H_{t,h} := < u(t+h) - u(t), \varphi > \longrightarrow 0 , \qquad \text{when }\; h \to 0 .
$$

We know that if $\varphi \in P$ then $\hat{\varphi} \in S(Z)$. Using (3.5) we have

$$H_{t,h} = 2\pi \sum_{k=-\infty}^{+\infty} \widehat{f}(k) e^{-i\mu k^2 t} e^{-\beta k^2 t} \left(e^{-i\mu k^2 h} e^{-\beta k^2 h} - 1 \right) \widehat{\varphi}(-k) \,.$$

Let $0 < h < 1$, form (3.8) we get

$$\left| e^{-i\mu k^2 h} e^{-\beta k^2 h} - 1 \right| \leq \mu |k|^2 |h| + \beta k^2 |h| < \mu |k|^2 + \beta |k|^2 \,. \tag{3.11}$$

Using (3.11) and that $\hat{f} \in S'(Z)$ we obtain

$$\begin{aligned} &\sum_{k=-\infty}^{+\infty} |\widehat{f}(k)| \underbrace{|e^{-i\mu k^2 t}|}_{=1} \underbrace{e^{-\beta k^2 t}}_{\leq 1} \left| e^{-i\mu k^2 h} e^{-\beta k^2 h} - 1 \right| |\widehat{\varphi}(-k)| \\ &\leq C\mu \sum_{k=-\infty}^{+\infty} |k|^{N+2} |\widehat{\varphi}(\underbrace{-k}_{=J})| + C\beta \sum_{k=-\infty}^{+\infty} |k|^{N+2} |\widehat{\varphi}(\underbrace{-k}_{=J})| \\ &= C\mu \sum_{J=-\infty}^{+\infty} |J|^{N+2} |\widehat{\varphi}(J)| + C\beta \sum_{J=-\infty}^{+\infty} |J|^{N+2} |\widehat{\varphi}(J)| \; < \; \infty \end{aligned}$$

since $\hat{\varphi} \in S(Z)$.

Using the Weierstrass M-Test we conclude that the series $H_{t,h}$ converges absolutely and uniformly. Then it is possible to take limit and obtain

$$\lim_{h\to 0} H_{t,h} = 2\pi \sum_{k=-\infty}^{+\infty} \widehat{f}(k) e^{-i\mu k^2 t} e^{-\beta k^2 t} \widehat{\varphi}(-k) \underbrace{\lim_{h\to 0} \left\{ e^{-i\mu k^2 h} e^{-\beta k^2 h} - 1 \right\}}_{=0} \; = \; 0 \,.$$

Since $t \in I\!R^+$ was taken arbitrarily, then we can conclude that

$$u \in C([0, \infty), P') \,.$$

c) $\partial_t u \in C(I\!R^+, P')$. That is, we will prove that

$$\partial_t u(t+h) \overset{P'}{\longrightarrow} \partial_t u(t) \;\; \text{when} \;\; h \to 0 \,, \;\; \forall t \in I\!R^+ \,.$$

In effect, let $t \in I\!R^+$ and $\varphi \in P$, using item a) we have

$$\begin{aligned} &< \partial_t u(t+h), \varphi > - < \partial_t u(t), \varphi > \\ &\quad = i\mu \{ < \partial_x^2 u(t+h), \varphi > - < \partial_x^2 u(t), \varphi > \} \\ &\qquad + \beta \{ < \partial_x^2 u(t+h), \varphi > - < \partial_x^2 u(t), \varphi > \} \\ &\quad = i\mu \underbrace{\{ < u(t+h), \varphi^{(2)} > - < u(t), \varphi^{(2)} > \}}_{\longrightarrow 0} \\ &\qquad + \beta \underbrace{\{ < u(t+h), \varphi^{(2)} > - < u(t), \varphi^{(2)} > \}}_{\longrightarrow 0} \; \longrightarrow \; 0 \end{aligned} \tag{3.12}$$

when $h \to 0$, since item b) is valid with $\varphi^{(2)} \in P$.

From b) and c) we have that $u \in C^1(I\!R^+, P')$.

d) Now, if $f_n \overset{P'}{\longrightarrow} f$ we will prove that:

$$u_n(t) \xrightarrow{P'} u(t)\,, \quad \forall t \in I\!R^+\,.$$

We know that if $f_n \xrightarrow{P'} f$ then $\hat{f}_n \xrightarrow{S'(Z)} \hat{f}_n$, that is

$$< \widehat{f_n} - \widehat{f}, \xi > \longrightarrow 0 \quad \text{when } n \to +\infty\,, \quad \forall \xi \in S(Z)\,. \tag{3.13}$$

For $t \in I\!R^+$ fixed and arbitrary, we want to prove that

$$< u_n(t), \psi > \longrightarrow < u(t), \psi > \quad \text{when } n \to +\infty\,, \quad \forall \psi \in P\,.$$

Thus, let $t \in I\!R^+$ be fixed and $\psi \in P$, using the generalized Parseval identity, we obtain the following equalities:

$$< u_n(t), \psi > \;=\; 2\pi < \left(\widehat{f_n}(k) e^{-i\mu k^2 t} e^{-\beta k^2 t}\right)_{k \in Z}, \widetilde{\widehat{\psi}} > \tag{3.14}$$

$$< u(t), \psi > \;=\; 2\pi < \left(\widehat{f}(k) e^{-i\mu k^2 t} e^{-\beta k^2 t}\right)_{k \in Z}, \widetilde{\widehat{\psi}} >\,. \tag{3.15}$$

From (3.14) and (3.15) we obtain:

$$< u_n(t), \psi > - < u(t), \psi > \\ = 2\pi \sum_{k=-\infty}^{+\infty} \{\widehat{f_n}(k) - \widehat{f}(k)\} \underbrace{e^{-i\mu k^2 t} e^{-\beta k^2 t} \widetilde{\widehat{\psi}}(k)}_{\xi_k :=} \longrightarrow 0$$

when $n \to +\infty$, since $\xi := (\xi_k)_{k \in Z} \in S(Z)$ and (3.13) holds.

Corollary 3.1 *Let $\mu > 0$ and $\beta > 0$, then the unique solution of (P_2) is*

$$u(t) = \sum_{k=-\infty}^{+\infty} \widehat{f}(k) e^{-i\mu k^2 t} e^{-\beta k^2 t} \phi_k = \left[\left(\widehat{f}(k) e^{-i\mu k^2 t} e^{-\beta k^2 t}\right)_{k \in Z}\right]^{\vee}\,,$$

where $\phi_k(x) = e^{ikx}$, $x \in I\!R$.

3.2 Semigroup of Operators in P'

Let's remember that P' is the topological dual of P, where P is a complete metric space.

In this subsection, we will introduce families of operators $\{T_{\mu,\beta}(t)\}_{t \geq 0}$ in P', with $\mu > 0$ and $\beta > 0$; and we will prove that these operators are continuous in the weak sense and satisfy the semigroup properties.

For simplicity, we will denote this family of operators by $\{T(t)\}_{t \geq 0}$.

Theorem 3.2 *Let $t \geq 0$, $\mu > 0$ and $\beta > 0$, we define:*

$$\begin{aligned} T(t) : P' &\longrightarrow P' \\ f &\longrightarrow T(t)f := \left[\left(\widehat{f}(k) e^{-i\mu k^2 t} e^{-\beta k^2 t}\right)_{k \in Z}\right]^{\vee} \in P'\,, \end{aligned}$$

then the following statements are satisfied:

1. *$T(0) = I$.*
2. *$T(t)$ is $\mathbb{C}$ - linear and continuous $\forall t \geq 0$. That is, for every $t \geq 0$, if $f_n \xrightarrow{P'} f$ then $T(t)$*

$f_n \xrightarrow{P'} f$ $T(t)f$.

3. $T(t+r) = T(t) \circ T(r)$, $\forall t, r \geq 0$.

4. $T(t)f \xrightarrow{P'} f$ when $t \to 0^+$, $\forall f \in P'$.

That is, for each $f \in P'$ fixed, the following is satisfied

$$< T(t)f, \psi > \longrightarrow < f, \psi >, \quad when \ t \to 0^+ , \ \forall \psi \in P .$$

Proof.- Let $f \in P'$ then $\hat{f} \in S'(Z)$. Then, from (3.3) we have

$$\left(\widehat{f}(k) e^{-i\mu k^2 t} e^{-\beta k^2 t}\right)_{k\in Z} \in S'(Z) ;$$

taking the inverse Fourier transform, we obtain

$$\underbrace{\left[\left(\widehat{f}(k) e^{-i\mu k^2 t} e^{-\beta k^2 t}\right)_{k\in Z}\right]^{\vee}}_{=T(t)f} \in P' , \quad \forall t \geq 0 .$$

That is, $T(t)$ is well defined for all $t \geq 0$.

1. We easily obtain:

$$T(0)f = \left[\left(\widehat{f}(k) e^{-i\mu k^2 0} e^{-\beta k^2 0}\right)_{k\in Z}\right]^{\vee} = \left[\left(\widehat{f}(k)\right)_{k\in Z}\right]^{\vee} = \left[\widehat{f}\right]^{\vee} = f , \quad \forall f \in P' .$$

2. Let $t \in \mathbb{R}^+$, we will prove that $T(t)$: $P' \to P'$ is $\mathbb{C}$-linear. In effect, let $\alpha \in \mathbb{C}$ and $(\phi, \psi) \in P' \times P'$, we have

$$\begin{aligned}
T(t)(a\phi + \psi) &= \left[\left(e^{-i\mu k^2 t} e^{-\beta k^2 t} [a\phi + \psi]^{\wedge}(k)\right)_{k\in Z}\right]^{\vee} \\
&= \left[\left(e^{-i\mu k^2 t} e^{-\beta k^2 t} [a\widehat{\phi}(k) + \widehat{\psi}(k)]\right)_{k\in Z}\right]^{\vee} \\
&= \left[a\left(e^{-i\mu k^2 t} e^{-\beta k^2 t} \widehat{\phi}(k)\right)_{k\in Z} + \left(e^{-i\mu k^2 t} e^{-\beta k^2 t} \widehat{\psi}(k)\right)_{k\in Z}\right]^{\vee} \\
&= a\left[\left(e^{-i\mu k^2 t} e^{-\beta k^2 t} \widehat{\phi}(k)\right)_{k\in Z}\right]^{\vee} + \left[\left(e^{-i\mu k^2 t} e^{-\beta k^2 t} \widehat{\psi}(k)\right)_{k\in Z}\right]^{\vee} \\
&= a\, T(t)\phi + T(t)\psi .
\end{aligned}$$

Now, for $t \in \mathbb{R}^+$ we will prove that $T(t) : P' \to P'$ is continuous. That is, if $f_n \xrightarrow{P'} f$ we will prove that $T(t)f_n \xrightarrow{P'} T(t)f$.

We know that if $f_n \xrightarrow{P'} f$ then $\hat{f}_n \xrightarrow{S'} \hat{f}$, that is,

$$< \widehat{f_n}, \xi > \to < \widehat{f}, \xi > , \quad \text{when } n \to +\infty , \quad \forall \xi \in S(Z) .$$

That is,

$$< \widehat{f_n} - \widehat{f}, \xi > \to 0 , \quad \text{when } n \to +\infty , \quad \forall \xi \in S(Z) . \tag{3.16}$$

We want to prove that:

$$< T(t)f_n, \psi > \to < T(t)f, \psi > \ \text{when } n \to +\infty , \quad \forall \psi \in P .$$

Thus, let $t \in \mathbb{R}^+$ fixed and $\psi \in P$, using the generalized Parseval identity, we obtain the following equalities

$$\begin{aligned} < T(t)f_n, \psi > &= < \left[\left(\widehat{f_n}(k)e^{-i\mu k^2 t}e^{-\beta k^2 t}\right)_{k\in Z}\right]^{\vee}, \psi > \\ &= 2\pi < \left(\widehat{f_n}(k)e^{-i\mu k^2 t}e^{-\beta k^2 t}\right)_{k\in Z}, \widetilde{\widehat{\psi}} >, \end{aligned} \tag{3.17}$$

$$\begin{aligned} < T(t)f, \psi > &= < \left[\left(\widehat{f}(k)e^{-i\mu k^2 t}e^{-\beta k^2 t}\right)_{k\in Z}\right]^{\vee}, \psi > \\ &= 2\pi < \left(\widehat{f}(k)e^{-i\mu k^2 t}e^{-\beta k^2 t}\right)_{k\in Z}, \widetilde{\widehat{\psi}} > . \end{aligned} \tag{3.18}$$

From (3.17) and (3.18) we get

$$\begin{aligned} &< T(t)f_n, \psi > - < T(t)f, \psi > \\ &= 2\pi \left\{ < \left(\widehat{f_n}(k)e^{-i\mu k^2 t}e^{-\beta k^2 t}\right)_{k\in Z}, \widetilde{\widehat{\psi}} > - < \left(\widehat{f}(k)e^{-i\mu k^2 t}e^{-\beta k^2 t}\right)_{k\in Z}, \widetilde{\widehat{\psi}} > \right\} \\ &= 2\pi \left\{ \sum_{k=-\infty}^{+\infty} \widehat{f_n}(k)e^{-i\mu k^2 t}e^{-\beta k^2 t}\widetilde{\widehat{\psi}}(k) - \sum_{k=-\infty}^{+\infty} \widehat{f}(k)e^{-i\mu k^2 t}e^{-\beta k^2 t}\widetilde{\widehat{\psi}}(k) \right\} \\ &= 2\pi \sum_{k=-\infty}^{+\infty} \{\widehat{f_n}(k) - \widehat{f}(k)\} \underbrace{e^{-i\mu k^2 t}e^{-\beta k^2 t}\widetilde{\widehat{\psi}}(k)}_{\xi_k :=} \longrightarrow 0 \end{aligned}$$

when $n \to +\infty$, since $\xi := (\xi_k)_{k\in Z} \in S(Z)$ and (3.16) holds, that is $< \hat{f}_n - \hat{f}, \xi > \to 0$ when $n \to +\infty$.

3. Let $t, r \in \mathbb{R}^+$, we will prove that $T(t) \circ T(r) = T(t + r)$. In effect, let $\phi \in P'$,

$$\begin{aligned} T(t+r)\phi &= \left[\left(\widehat{\phi}(k)e^{-i\mu k^2(t+r)}e^{-\beta k^2(t+r)}\right)_{k\in Z}\right]^{\vee} \\ &= \left[\left(\underbrace{\widehat{\phi}(k)e^{-i\mu k^2 r}e^{-\beta k^2 r}}\cdot e^{-i\mu k^2 t}e^{-\beta k^2 t}\right)_{k\in Z}\right]^{\vee} . \end{aligned} \tag{3.19}$$

Since $\phi \in P'$, using (3.3) we hav`e that

$$\left(\widehat{\phi}(k)e^{-i\mu k^2 r}e^{-\beta k^2 r}\right)_{k\in Z} \in S'(Z), \quad \forall r \in [0, +\infty) . \tag{3.20}$$

Then, taking the inverse Fourier transform, we get:

$$\left[\left(\widehat{\phi}(k)e^{-i\mu k^2 r}e^{-\beta k^2 r}\right)_{k\in Z}\right]^{\vee} \in P', \quad \forall r \in [0, +\infty) .$$

Thus, we define:

$$g_r := \left[\left(\widehat{\phi}(k)e^{-i\mu k^2 r}e^{-\beta k^2 r}\right)_{k\in Z}\right]^{\vee} \in P' .$$

That is,

$$g_r := T(r)\phi . \tag{3.21}$$

Taking the Fourier transform to g_r we get:

$$\widehat{g_r} = \left(\widehat{\phi}(k)e^{-i\mu k^2 r}e^{-\beta k^2 r}\right)_{k\in Z},$$

that is,

$$\widehat{g_r}(k) = \widehat{\phi}(k)e^{-i\mu k^2 r}e^{-\beta k^2 r}, \quad \forall k \in Z. \tag{3.22}$$

Using (3.22) in (3.19) and from (3.21) we have:

$$\begin{aligned} T(t+r)\phi &= \left[\left(\widehat{g_r}(k)e^{-i\mu k^2 t}e^{-\beta k^2 t}\right)_{k\in Z}\right]^{\vee} \in P' \\ &= T(t)g_r \\ &= T(t)(T(r)\phi) \\ &= [T(t)\circ T(r)](\phi), \quad \forall t, r \in I\!R^+. \end{aligned}$$

So we have proven,

$$T(t+r) = T(t)\circ T(r), \ \forall t, r \in I\!R^+. \tag{3.23}$$

If $t = 0$ or $r = 0$ then equality (3.23) is also true, with this we conclude the proof of

$$T(t+r) = T(t)\circ T(r), \quad \forall t, r \in [0, +\infty). \tag{3.24}$$

4. Let $f \in P'$, we will prove that:

$$T(t)f \xrightarrow{P'} f \ \text{ when } t \to 0^+.$$

That is, we will prove that

$$< T(t)f, \varphi > \longrightarrow < f, \varphi > \ \text{ when } t \to 0^+, \quad \forall \varphi \in P.$$

In effect, for $t > 0$ and $\varphi \in P$, we have

$$\begin{aligned} \mathcal{H}_t &:= < T(t)f, \varphi > - < f, \varphi > \\ &= \lim_{n\to+\infty}\left\{< \sum_{k=-n}^{n} \widehat{f}(k)e^{-i\mu k^2 t}e^{-\beta k^2 t}\phi_k, \varphi > - < \sum_{k=-n}^{n} \widehat{f}(k)\phi_k, \varphi >\right\} \\ &= \lim_{n\to+\infty} < \sum_{k=-n}^{n} \widehat{f}(k)\left(e^{-i\mu k^2 t}e^{-\beta k^2 t} - 1\right)\phi_k, \varphi > \\ &= \lim_{n\to+\infty} \sum_{k=-n}^{n} \widehat{f}(k)\left(e^{-i\mu k^2 t}e^{-\beta k^2 t} - 1\right) < \phi_k, \varphi > \\ &= \lim_{n\to+\infty} 2\pi \sum_{k=-n}^{n} \widehat{f}(k)\left(e^{-i\mu k^2 t}e^{-\beta k^2 t} - 1\right)\widehat{\varphi}(-k) \\ &= 2\pi \sum_{k=-\infty}^{+\infty} \widehat{f}(k)\left(e^{-i\mu k^2 t}e^{-\beta k^2 t} - 1\right)\widehat{\varphi}(-k). \end{aligned} \tag{3.25}$$

Since $t > 0$, from (3.7) we get

$$\left|e^{-i\mu k^2 t}e^{-\beta k^2 t} - 1\right| \leq \{\mu|k|^2 + \beta|k|^2\}t\,. \tag{3.26}$$

From (3.26) we obtain

$$\left|e^{-i\mu k^2 t}e^{-\beta k^2 t} - 1\right| \leq \{\mu|k|^2 + \beta|k|^2\}t\,, \quad \forall t \in [0, +\infty)\,. \tag{3.27}$$

From (3.27) with $0 < t < 1$, we have

$$\left|e^{-i\mu k^2 t}e^{-\beta k^2 t} - 1\right| \leq \mu|k|^2 + \beta|k|^2\,. \tag{3.28}$$

Then using (3.28) and that $f \in P'$, we obtain

$$\begin{aligned}
&\sum_{k=-\infty}^{+\infty} |\widehat{f}(k)| \left|e^{-i\mu k^2 t}e^{-\beta k^2 t} - 1\right| |\widehat{\varphi}(-k)| \\
&\quad \leq C\left\{\mu \sum_{k=-\infty}^{+\infty} |k|^{N+2}|\widehat{\varphi}(\underbrace{-k}_{=J})| + \beta \sum_{k=-\infty}^{+\infty} |k|^{N+2}|\widehat{\varphi}(\underbrace{-k}_{=J})|\right\} \\
&\quad = C\left\{\mu \sum_{J=-\infty}^{+\infty} |J|^{N+2}|\widehat{\varphi}(J)| + \beta \sum_{J=-\infty}^{+\infty} |J|^{N+2}|\widehat{\varphi}(J)|\right\} \; < \; \infty
\end{aligned}$$

since $\hat{\varphi} \in S(Z)$.

Using the Weierstrass M-Test we conclude that the H_t series converges absolutely and uniformly. So,

$$\begin{aligned}
\lim_{t \to 0^+} \mathcal{H}_t &= 2\pi \sum_{k=-\infty}^{+\infty} \widehat{f}(k)\widehat{\varphi}(-k) \underbrace{\lim_{t \to 0^+} \{e^{-i\mu k^2 t}e^{-\beta k^2 t} - 1\}}_{=0} \\
&= 0\,.
\end{aligned}$$

Thus, we have proved

$$\lim_{t \to 0^+} < T(t)f, \varphi > = < f, \varphi > \,.$$

Theorem 3.3 *For each $f \in P'$ fixed and the family of operators $\{T(t)\}_{t \geq 0}$ from Theorem 3.2, then the application*

$$\begin{aligned}
\zeta \,:\, [0, +\infty) &\longrightarrow P' \\
t &\longrightarrow T(t)f
\end{aligned}$$

is continuous in $[0, +\infty)$. That is,

$$T(t+h)f \xrightarrow{P'} T(t)f \;\; when \; h \to 0\,, \; \forall t \in [0, +\infty)\,. \tag{3.29}$$

(is the continuity at t).

That is, (3.29) tell us that for each $t \in (0, +\infty)$ fixed, the following is satisfied

$$< T(t+h)f, \psi > \longrightarrow < T(t)f, \psi >\,, \quad when \;\; h \to 0\,, \; \forall\, \psi \in P\,.$$

If $t = 0$, we have the continuity of ζ at 0 on the right, which is item 4) of Theorem 3.2.

Proof.- Let $t > 0$, arbitrary fixed, then $g := T(t)f \in P'$, using item 4) of Theorem 3.2, we have that $T(h)g \xrightarrow{P'} g$ when $h \to 0^+$. That is,

$$\underbrace{\underbrace{T(h)(T(t)f)}_{=[T(h)\circ T(t)]f}}_{=T(h+t)f} \xrightarrow{P'} T(t)f \text{ when } h \to 0^+,$$

where we use item 3) of Theorem 3.2. With this we have proved that

$$T(t+h)f \xrightarrow{P'} T(t)f \text{ when } h \to 0^+, \ \forall t \in (0, +\infty). \tag{3.30}$$

Now, we will prove that $T(t+v)f \xrightarrow{P'} T(t)f$ when $v \to 0^-$. That is, we will demonstrate

$$< T(t-h)f, \varphi > \longrightarrow < T(t)f, \varphi > \quad \text{when } h \to 0^+, \quad \forall \varphi \in P. \tag{3.31}$$

In effect, for $t > h > 0$ and $\varphi \in P$, we have

$$\begin{aligned}
\mathcal{L}_{t,h} &:= < T(t-h)f, \varphi > - < T(t)f, \varphi > \\
&= \lim_{n\to+\infty} \Bigg\{ < \sum_{k=-n}^{n} \widehat{f}(k) e^{-i\mu k^2(t-h)} e^{-\beta k^2(t-h)} \phi_k, \varphi > \\
&\qquad - < \sum_{k=-n}^{n} \widehat{f}(k) e^{-i\mu k^2 t} e^{-\beta k^2 t} \phi_k, \varphi > \Bigg\} \\
&= \lim_{n\to+\infty} < \sum_{k=-n}^{n} \widehat{f}(k) e^{-i\mu k^2 t} e^{-\beta k^2 t} \left(e^{i\mu k^2 h} e^{\beta k^2 h} - 1 \right) \phi_k, \varphi > \\
&= \lim_{n\to+\infty} \sum_{k=-n}^{n} \widehat{f}(k) e^{-i\mu k^2 t} e^{-\beta k^2 t} \left(e^{i\mu k^2 h} e^{\beta k^2 h} - 1 \right) < \phi_k, \varphi > \\
&= \lim_{n\to+\infty} 2\pi \sum_{k=-n}^{n} \widehat{f}(k) e^{-i\mu k^2 t} e^{-\beta k^2 t} \left(e^{i\mu k^2 h} e^{\beta k^2 h} - 1 \right) \widehat{\varphi}(-k) \\
&= 2\pi \sum_{k=-\infty}^{+\infty} \widehat{f}(k) e^{-i\mu k^2 t} e^{-\beta k^2 t} \left(e^{i\mu k^2 h} e^{\beta k^2 h} - 1 \right) \widehat{\varphi}(-k). \qquad (3.32)
\end{aligned}$$

In the series (3.32), we need to delimit the expression $e^{-i\mu k}2he^{-\beta k}2h - 1$. So, we have

$$\begin{aligned}
e^{i\mu k^2 h} e^{\beta k^2 h} - 1 &= \int_0^h [e^{(i\mu k^2 + \beta k^2)s}]' ds \\
&= (i\mu k^2 + \beta k^2) \int_0^h e^{(i\mu k^2 + \beta k^2)s} ds. \qquad (3.33)
\end{aligned}$$

Taking norm to equality (3.33) and using: $\int_0^h e^{(\beta k2)s}\, ds \leq e^{(\beta k2)h} h$ for $h > 0$, we obtain

$$\begin{aligned}
|e^{i\mu k^2 h} e^{\beta k^2 h} - 1| &\leq |i\mu k^2 + \beta k^2| \int_0^h e^{(\beta k^2)s}\, ds \\
&\leq \{\mu|k|^2 + \beta|k|^2\} e^{(\beta k^2)h} \cdot h \\
&\leq \{\mu|k|^2 + \beta|k|^2\} e^{(\beta k^2)h} \qquad (3.34)
\end{aligned}$$

whenever $0 < h < 1$.

Using inequality (3.34) and $e^{-\beta k2(t-h)} \leq 1$ for $0 < h < t$ with $h << 1$, we have

$$\begin{aligned}
&\sum_{k=-\infty}^{+\infty} |\widehat{f}(k)||e^{-i\mu k^2 t}|e^{-\beta k^2 t}\left|e^{i\mu k^2 h}e^{\beta k^2 h}-1\right||\widehat{\varphi}(-k)| \\
&\leq \sum_{k=-\infty}^{+\infty} |\widehat{f}(k)|e^{-\beta k^2(t-h)}\left\{\mu|k|^2+\beta|k|^2\right\}|\widehat{\varphi}(-k)| \\
&\leq \sum_{k=-\infty}^{+\infty} |\widehat{f}(k)||\widehat{\varphi}(-k)|\left\{\mu|k|^2+\beta|k|^2\right\} \\
&\leq \mu \sum_{k=-\infty}^{+\infty} |\widehat{f}(k)||\widehat{\varphi}(-k)||k|^2 + \beta \sum_{k=-\infty}^{+\infty} |\widehat{f}(k)||\widehat{\varphi}(-k)||k|^2 \\
&\leq C\left\{\mu \sum_{k=-\infty}^{+\infty} |k|^{N+2}|\widehat{\varphi}(-k)| + \beta \sum_{k=-\infty}^{+\infty} |k|^{N+2}|\widehat{\varphi}(-k)|\right\} \\
&\leq C\left\{\mu \sum_{J=-\infty}^{+\infty} |J|^{N+2}|\widehat{\varphi}(J)| + \beta \sum_{J=-\infty}^{+\infty} |J|^{N+2}|\widehat{\varphi}(J)|\right\} < \infty \qquad (3.35)
\end{aligned}$$

since $\widehat{\varphi} \in S(Z)$.

Using the Weierstrass M-Test e obtain that the series $\mathcal{L}_{h,t}$ is absolutely and uniformly convergent.

Then, we can take limit and get:

$$\lim_{h\to 0^+} \mathcal{L}_{h,t} = 2\pi \sum_{k=-\infty}^{+\infty} \widehat{f}(k)e^{-i\mu k^2 t}e^{-\beta k^2 t} \underbrace{\lim_{h\to 0^+}\left\{e^{i\mu k^2 h}e^{\beta k^2 h}-1\right\}}_{=0} \widehat{\varphi}(-k) = 0\,,$$

with this (3.31) is proved.

From (3.30) and (3.31) we conclude that

$$T(t+h)f \xrightarrow{P'} T(t)f \text{ when } h \to 0\,,\ \forall t \in (0,+\infty)\,. \qquad (3.36)$$

Remark 3.1 *The results obtain in Theorems 3.2 and 3.3 are also valid for the family of operators* $\{S(t)\}_{t\geq 0}$, *defined as*

$$\begin{aligned}
S(t) : P' &\longrightarrow P' \\
f &\rightarrow S(t)f := \left[\left(e^{i\mu k^2 t}e^{-\beta k^2 t}\widehat{f}(k)\right)_{k\in Z}\right]^{\vee},
\end{aligned}$$

for $t \in [0, +\infty)$. *Its proof is similar.*

3.3 Version of Theorem 3.1 using the family $\{T(t)\}_{t\geq 0}$

We improve the statement of theorem 3.1, using a family of weakly continuous Operators $\{T(t)\}_{t\geq 0}$.

Theorem 3.4 *Let* $f \in P'$ *and the family of operators* $\{T(t)\}_{t\geq 0}$ *from Theorem 3.2, defining* $u(t) := T(t)f \in P'$, $\forall t \in [0, +\infty)$, *then* $u \in C([0, +\infty), P')$ *is the unique solution of* (P_2). *Furthermore, u continuously depends on f. That is, given* f_n, $f \in P'$ *with* $f_n \xrightarrow{P'} f$ *implies* $u_n(t)$

$\xrightarrow{P'} u(t)$, $\forall t \in [0, +\infty)$, *where* $u_n(t) := T(t)f_n$, $\forall t \in [0, +\infty)$ *(that is,* u_n *is a solution of* (P_2) *with initial data* f_n*).*

Proof.- It is analogous to the proof of Theorem 3.1.

Corollary 3.2 *Let* $f \in P'$ *be fixed and the family of operators* $\{T(t)\}_{t\geq 0}$ *from Theorem 3.4, then* $\exists\, \partial_t T(t)f$, $\forall t \in (0, +\infty)$ *and the mapping*

$$\begin{array}{rcl} \eta: \ (0,+\infty) & \to & P' \\ t & \to & \partial_t T(t)f = i\mu\partial_x^2 T(t)f + \beta\partial_x^2 T(t)f \end{array}$$

is continuous at $(0, +\infty)$*. That is,*

$$\partial_t T(t+h)f \xrightarrow{P'} \partial_t T(t)f \quad when\ h \to 0\,, \quad \forall t \in (0,+\infty)\,. \tag{3.37}$$

(3.37) tells us that for each $t \in (0, +\infty)$ *fixed, it holds:*

$$< \partial_t T(t+h)f, \varphi > \longrightarrow < \partial_t T(t)f, \varphi > \quad when\ h \to 0\,, \quad \forall \varphi \in P\,.$$

Proof.- Indeed,

$$\begin{aligned} &< \partial_t T(t+h)f, \varphi > - < \partial_t T(t)f, \varphi > \\ &\quad = i\mu\{< \partial_x^2 T(t+h)f, \varphi > - < \partial_x^2 T(t)f, \varphi >\} \\ &\quad\quad +\beta\{< \partial_x^2 T(t+h)f, \varphi > - < \partial_x^2 T(t)f, \varphi >\} \\ &\quad = i\mu\{\underbrace{< T(t+h)f, \varphi^{(2)} > - < T(t)f, \varphi^{(2)} >}_{\longrightarrow 0}\} \\ &\quad\quad +\beta\{\underbrace{< T(t+h)f, \varphi^{(2)} > - < T(t)f, \varphi^{(2)} >}_{\longrightarrow 0}\} \longrightarrow 0 \end{aligned}$$

when $h \to 0$, due to Theorem 3.3 with $\psi := \varphi^{(2)} \in P$.

Corollary 3.3 *Let* $f \in P'$ *be fixed and the family of operators* $\{T(t)\}_{t\geq 0}$ *from Theorem 3.4, then the solution of* (P_2)*:* $u(t) := T(t)f$, $\forall t \in [0, +\infty)$, *satisfies* $u \in C^1((0, +\infty), P')$.

Proof.- It comes out as a consequence of Corollary 3.2.

Remark 3.2 *We can generalize this study to the even case not multiple of four, obtaining analogous results.*

4 I CONCLUSIONS

In our study of the Schrödinger equation in the periodic distributional space P', for the case (P_2) we have obtained the following results:

1. We prove the existence, uniqueness of the solution of the problem (P_2). Thus we also prove the continuous dependence of the solution respect to the initial data.

2. We introduce families of operators in P': $\{T(t)\}_{t\geq 0}$ and we prove that they are linear and weakly continuous in P'. Furthermore, we proved that they form a semigroup of weakly continuous operators in P'.

3. With the family of operators $\{T(t)\}_{t\geq 0}$ we improve Theorem 3.1.

4. Finally, we must indicate that this study (or technique) can be applied to other evolution equations in P'.

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