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# USING GEOGEBRA 3D IN THE COMPOSITION/ DECOMPOSITION OF CONVEX POLYHEDRA FOR VOLUME CALCULATION

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**Abstract:** In this work, we present the use of GeoGebra 3D in the composition/decomposition of convex polyhedra to establish strategies for calculating the volume of these solids. We basically employ two operations on polyhedra for compositions/decompositions: accumulation and truncation, and we use accumulation to calculate the volume of the tetrakis hexahedron and the triakis octahedron, two Catalan polyhedra. We conclude that GeoGebra 3D is an excellent tool to establish strategies for calculating the volume of convex polyhedra and that it can be explored in this sense both in High School and in University, particularly in the Mathematics-Teaching Degree Course and in the Professional Master's in Mathematics in National Network – PROFMAT.
**Keywords:** Tetrakis hexahedron. Triakis octahedron. Classes of polyhedra. Operations on polyhedra. Nested radicals.

## INTRODUCTION

The calculation of the volume of a convex polyhedron is a theme present in the high school mathematics curriculum (LIMA et al., 2006) and also in university, such as in the Mathematics-Teaching Degree (DOLCE; POMPEO, 2013) and in PROFMAT (NETO, 2013). This calculation is a complex task when it depends on measures of the dual polyhedron (COXETER, 1973; SILVA; NÓS, 2018) and the transformation of nested radicals into simple radicals (NÓS; SILVA, 2019; NÓS; SAITO; SANTOS, 2017; SAITO; NÓS; SANTOS, 2017). In addition, "[...] there are crystals and living organisms with polyhedral shapes in nature. The volume of a gemstone and a viral mass, for example, are measures that must, under certain circumstances, be calculated" (SILVA; NÓS, 2018, p. 17).

Thus, given a polyhedron of Plato, Archimedean, Catalan or Johnson, the first task in calculating the volume is to establish a strategy to visualize the parts of the polyhedron, that is, if we can separate it into parts of which we know how to calculate the volume, such as prisms and pyramids, or if we can obtain it from another known solid, such as the cube for example, eliminating parts of which we also know how to calculate the volume. In this task, we can use dynamic geometry applications, such as GeoGebra 3D (GEOGEBRA, 2022), which allow the composition/decomposition of the polyhedron and, consequently, the determination of an effective strategy for calculating the volume.

In this work, we basically approach two strategies: in decomposition, the operation on polyhedra called truncation and, in composition, the operation called accumulation.

Truncation consists of cutting the vertices or edges of the polyhedron, as illustrated in Figure 01(a), where we truncate triangular pyramids from the vertices of a cube, a Plato polyhedron, to obtain the truncated cube, an Archimedean polyhedron. Accumulation is the dual operation of truncation that couples pyramids to the faces of the polyhedron, as illustrated in Figure 01(b), where we accumulate the regular tetrahedron, a Plato polyhedron, to obtain the triakis tetrahedron, a Catalan polyhedron.

## METHODOLOGY

We used GeoGebra 3D to build all the figures in this work. Once the compositions/decompositions were performed with the dynamic geometry application, the results obtained for the volumes of the polyhedra were checked in Rechneronline (2022), WolframAlpha (2022) and WolframMathWorld (2022).

## DEVELOPMENT

In this section, we present some compositions – Figures 02 to 05 - and

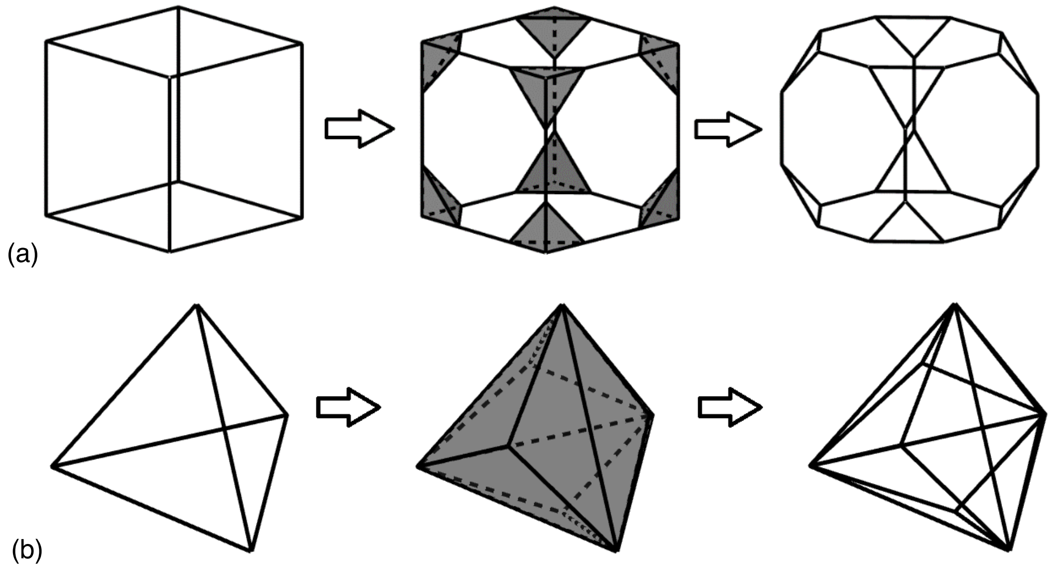


Figure 01 – Operations on polyhedra: (a) cube truncation; (b) accumulation of the regular tetrahedron

Source: Silva and Nós (2018).

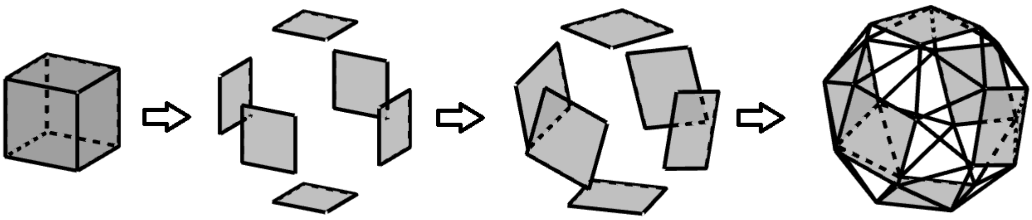

Figure 02 – The snub cube obtained by snubifying the cube

Source: Silva and Nós (2018).

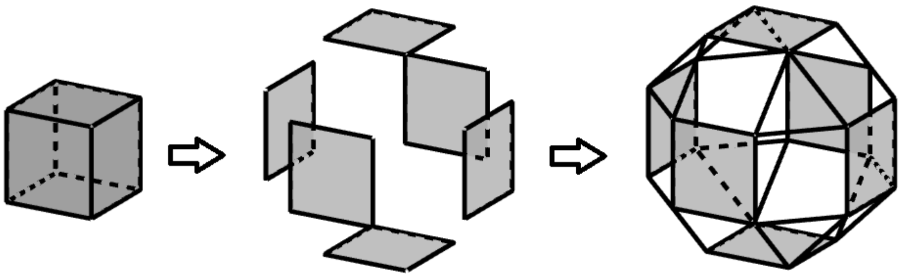

Figure 03 – The rhombicuboctahedron obtained by expanding the cube

Source: Silva and Nós (2018).

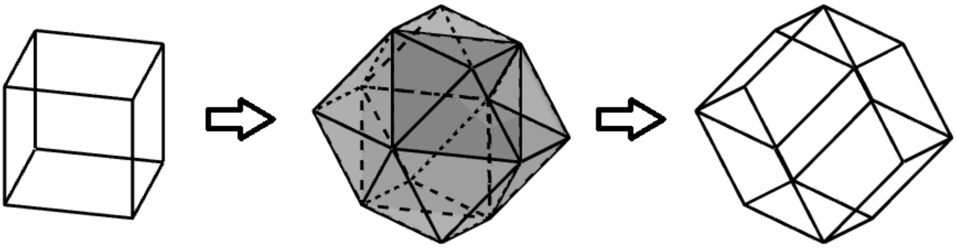

Figure 04 - The rhombic dodecahedron obtained by accumulating the cube

Source: Siva and Nós (2018).


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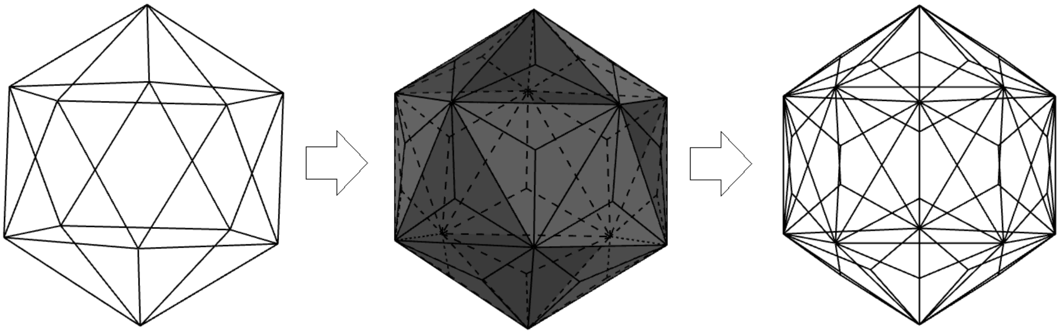


Figure 05 – The triakis icosahedron obtained by accumulating the regular icosahedron

Source: Nós and Silva (2019).

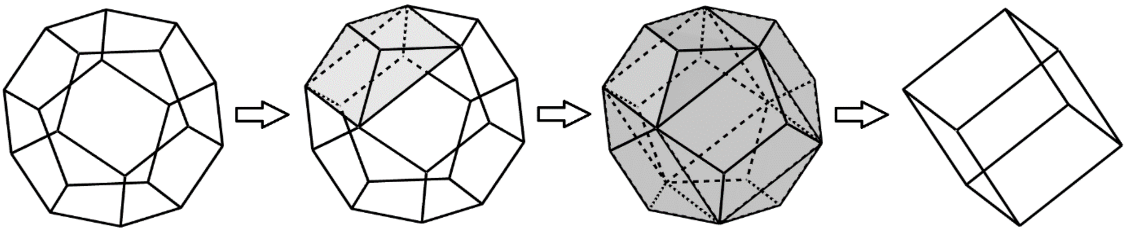


Figure 06 - The decomposition of the regular dodecahedron

Source: Silva and Nós (2018).

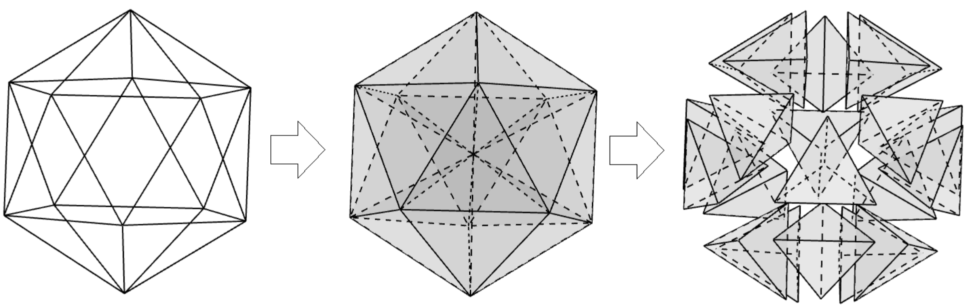


Figure 07 - The decomposition of the regular icosahedron

Source: Nós and Silva (2019).

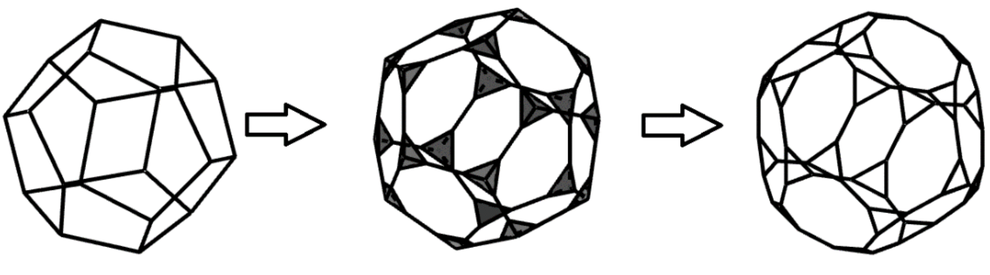


Figure 08 - The truncated dodecahedron obtained from truncating the regular dodecahedron

Source: Silva and Nós (2018).

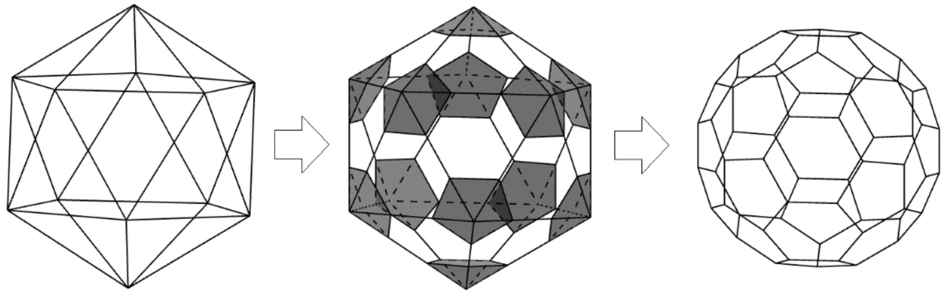


Figure 09 – The truncated icosahedron obtained from the truncation of the regular icosahedron

Source: Nós and Silva (2019).

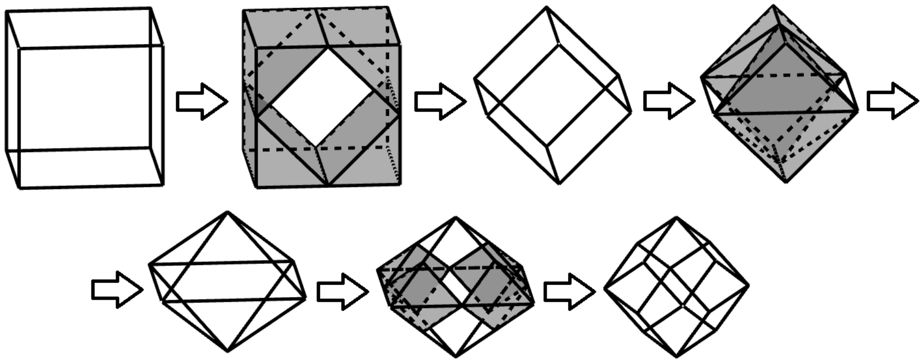


Figure 10 – The rhombic dodecahedron obtained from successive truncations of the cube

Source: Silva and Nós (2018).

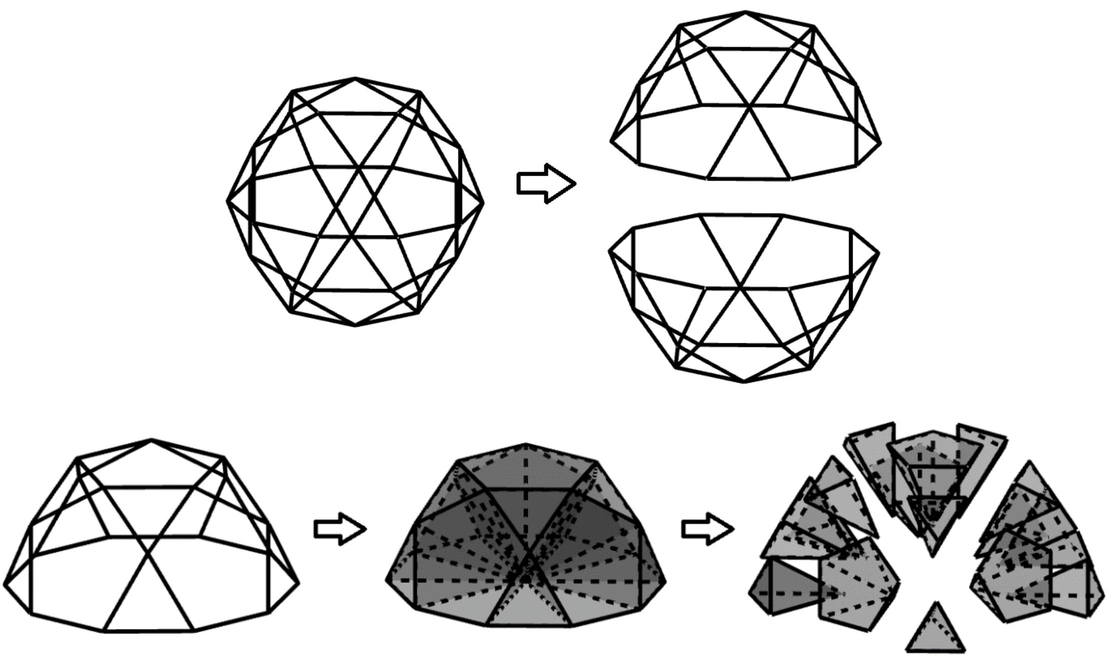


Figure 11 – The decomposition of the pentagonal rotunda (J6)

Source: Silva and Nós (2018).

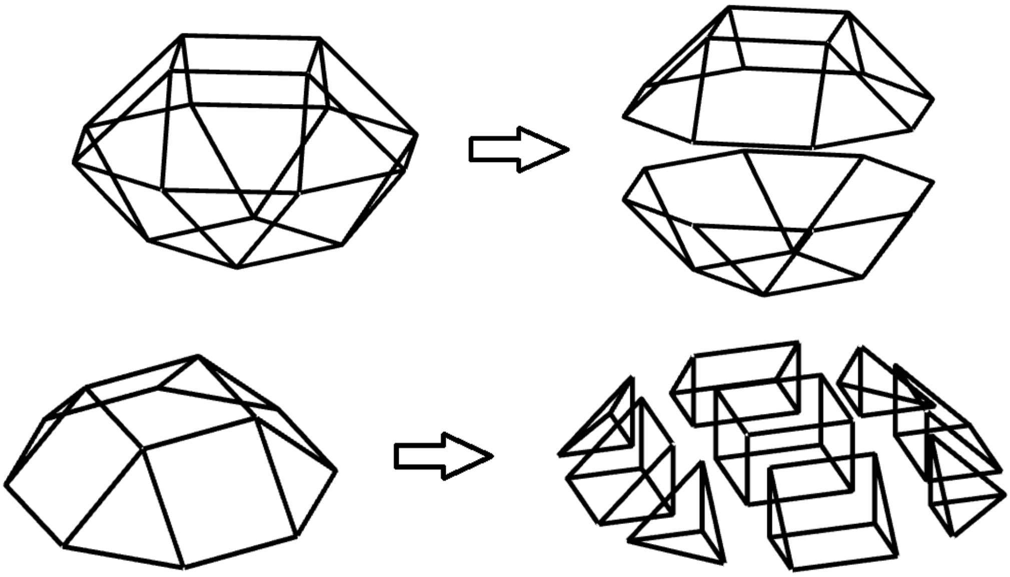


Figure 12 - The decomposition of the square gyrobicupola (J29)

Source: Silva and Nós (2018).

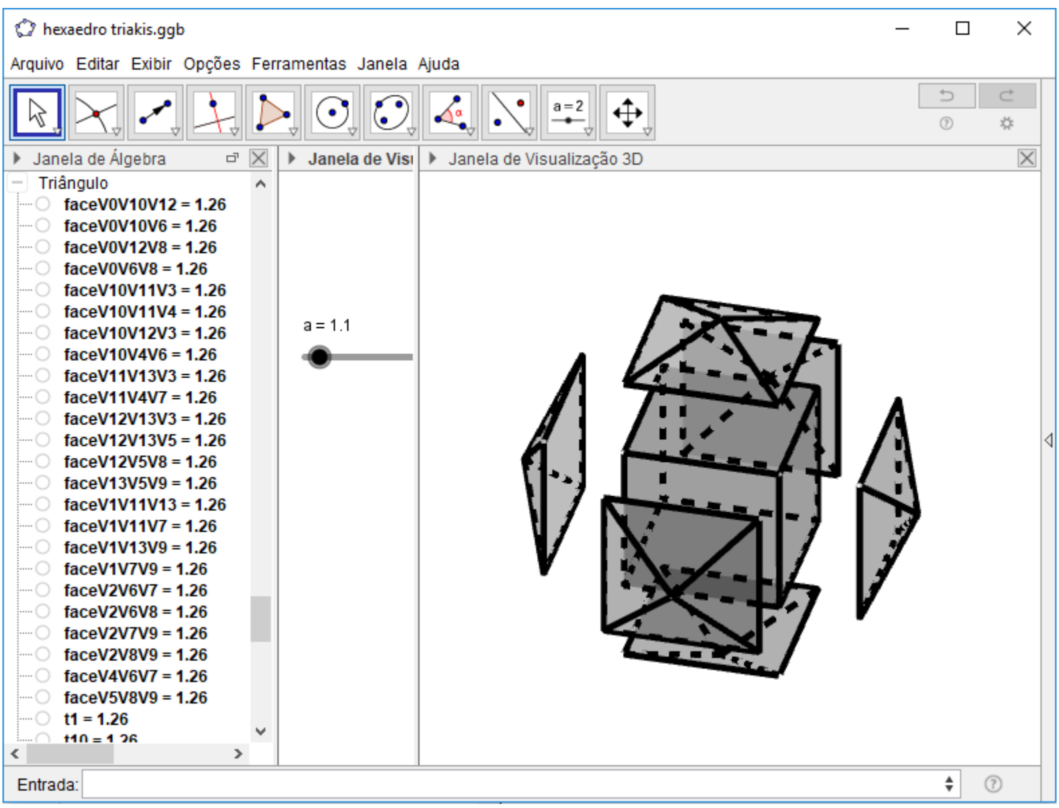



Figure 13 – The decomposition of the tetrakis hexahedron

Source: Silva and Nós (2018).

decompositions – Figures 06 to 12 – carried out with GeoGebra 3D. To illustrate the calculation of volume, we selected the tetrakis hexahedron, a Catalan polyhedron obtained by accumulating the regular hexahedron (or cube), and the triakis octahedron, a Catalan polyhedron obtained by accumulating the regular octahedron.

## VOLUME OF TETRAKIS HEXAHEDRON

The tetrakis hexahedron, a Catalan polyhedron (SILVA; NÓS, 2018), can be obtained by accumulating the regular hexahedron, a Plato polyhedron (COXETER, 1973), by attaching a square pyramid to each face, as illustrated in Figure 13.

Thus, the volume of the tetrakis hexahedron is given by

$$V(tetrakis\ hexahedron) = V(regular\ hexahedron) + 6V(square\ pyramid). \quad (1)$$

The tetrakis hexahedron is a convex polyhedron composed of 24 faces, 36 edges and 14 vertices. Its faces are formed by isosceles triangles and its edges have two lengths:

1. the twelve largest are the edges of the regular hexahedron on whose faces the accumulation takes place;
2. the twenty-four smaller ones are the lateral edges of the quadrangular pyramids coupled to the faces of the regular hexahedron in the accumulation.

According to WolframMathWorld (2022), when we form the triakis hexahedron from its dual (SILVA; NÓS, 2018), the truncated octahedron, with unitary edge, we obtain a tetrakis hexahedron with the largest edge measuring $\frac{3}{2}\sqrt{2}$ and the smallest edge measuring $\frac{9}{8}\sqrt{2}$. Considering the largest and smallest edges of the tetrakis hexahedron with arbitrary measures $a$ and $x$, respectively, we have, by similarity of triangles, that:

$$\frac{a}{\frac{3}{2}\sqrt{2}} = \frac{x}{\frac{9}{8}\sqrt{2}};$$

$$x = \frac{3}{4}a; \quad (2)$$

$$a = \frac{4}{3}x. \quad (3)$$

The quadrangular pyramid of accumulation has the shape and dimensions of the pyramid illustrated in Figure 14, where $a$ is the measure of the edge of the regular hexahedron and $x$ is the measure of the lateral edge of the coupled pyramid.

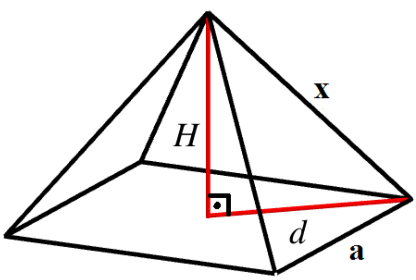



Figure 14 – The quadrangular pyramid in the accumulation of the regular hexahedron

Source: Silva and Nós (2018).

We know that the base of the coupled quadrangular pyramid is a square. Applying the Pythagorean theorem to the right triangle of Figure 15, with hypotenuse $x$, and legs $d = \frac{a\sqrt{2}}{2}$, and $H$, being $H$ the height of the pyramid, we obtain, using (2):

$$x^2 = H^2 + d^2;$$

$$\left(\frac{3}{4}a\right)^2 = H^2 + \left(\frac{a\sqrt{2}}{2}\right)^2;$$

$$H^2 = \frac{9}{16}a^2 - \frac{2}{4}a^2;$$

$$H^2 = \frac{1}{16}a^2;$$

$$H = \frac{a}{4}. \quad (4)$$

Thus, using (4) in (1) and the relation for calculating the volume of the pyramid

(NETO, 2013), we conclude that the volume of the tetrakis hexahedron is equal to:

$$\mathcal{V}(tetrakis\ hexahedron) = a^3 + 6\frac{1}{3}a^2\frac{a}{4};$$

$$\mathcal{V}(tetrakis\ hexahedron) = a^3 + \frac{a^3}{2};$$

$$\mathcal{V}(tetrakis\ hexahedron) = \frac{3}{2}a^3, \qquad (5)$$

where $a$ is the length of the longest edge of the polyhedron (or the length of the edge of the regular hexahedron).

We can also determine the volume of the tetrakis hexahedron from the measure of its smallest edge $x$. Substituting (3) into (5), we get:

$$\mathcal{V}(tetrakis\ hexahedron) = \frac{3}{2}\left(\frac{4}{3}x\right)^3;$$

$$\mathcal{V}(tetrakis\ hexahedron) = \frac{3}{2}\frac{64}{27}x^3;$$

$$\mathcal{V}(tetrakis\ hexahedron) = \frac{32}{9}x^3, \qquad (6)$$

where $x$ is the measure of the smallest edge of the polyhedron (or the lateral edge of the quadrangular pyramid coupled in the accumulation of the regular hexahedron).

## VOLUME OF THE TRIAKIS OCTAHEDRON

The triakis octahedron, a Catalan polyhedron (SILVA; NÓS, 2018), can be obtained by accumulating the regular octahedron, a Plato polyhedron (COXETER, 1973), by attaching a triangular pyramid to each face, as illustrated in Figure 15.

This way, the volume of the triakis octahedron is given by

$$\mathcal{V}(octaedro\ triakis) = \mathcal{V}(regular\ octahedron) + 8\mathcal{V}(triangular\ pyramid). \qquad (7)$$

The triakis octahedron – Figure 16(a) - is a convex polyhedron composed of 24 faces, 36 edges and 14 vertices – Figure 16(b). Its faces are formed by isosceles triangles and its edges have two lengths:

3. the twelve largest are the edges of the regular octahedron on whose faces the accumulation takes place;
4. the twenty-four smaller ones are the lateral edges of the triangular pyramids attached to the faces of the regular octahedron in the accumulation.

According to WolframMathWorld (2022), when we form the triakis octahedron from its dual (SILVA; NÓS, 2018), the truncated cube, with unitary edge, we obtain a triakis octahedron with the largest edge measuring $2+\sqrt{2}$ and the smallest edge measuring 2. Considering the largest and smallest edges of the triakis octahedron with arbitrary measures $a$ and $x$, respectively, we have, by similarity of triangles, that:

$$x = (2-\sqrt{2})a; \qquad (8)$$

$$a = \frac{x}{2-\sqrt{2}}. \qquad (9)$$

The triangular pyramid in the accumulation has the shape and dimensions of the pyramid illustrated in Figure 17, where is the measure of the edge of the regular octahedron and $x$ is the measure of the lateral edge of the coupled pyramid.

We know that the base of the coupled pyramid is an equilateral triangle. Applying the Pythagorean theorem in the right triangle of Figure 17, with hypotenuse $x$, and legs $b = \frac{\sqrt{3}}{3}a$, and $H$, where $H$ the height of the pyramid, we obtain with the use of (8):

$$x^2 = H^2 + b^2;$$

$$\left((2-\sqrt{2})a\right)^2 = H^2 + \left(\frac{a\sqrt{3}}{3}\right)^2;$$

$$H^2 = (6-4\sqrt{2})a^2 - \frac{1}{3}a^2;$$

$$H^2 = \frac{17-12\sqrt{2}}{3}a^2;$$

$$H = \frac{\sqrt{17-12\sqrt{2}}}{\sqrt{3}}a;$$

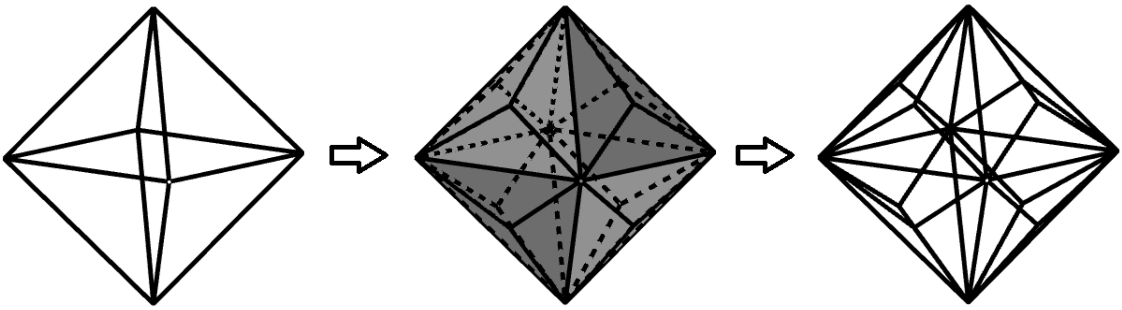
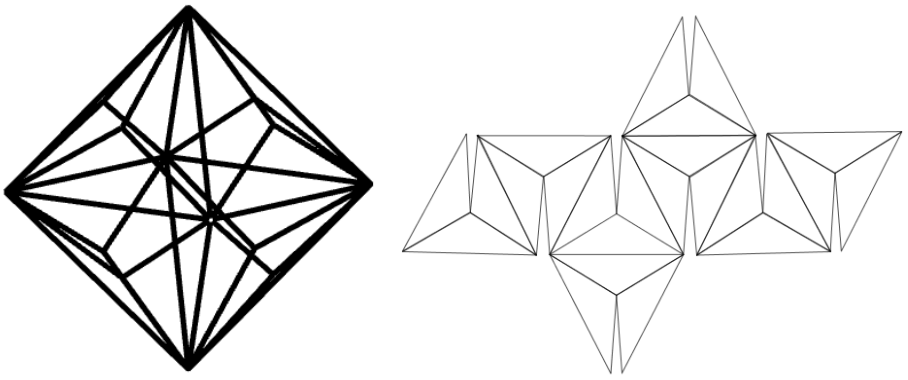


Figure 15 – The triakis octahedron obtained by accumulating the regular octahedron

Source: Silva and Nós (2018), Nós and Silva (2020).



(a) (b)

Figure 16 – Triakis octahedron: (a) solid; (b) planning

Source: Silva and Nós (2018), Nós and Silva (2020).

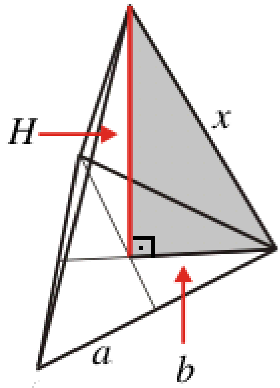



Figure 17 – The triangular pyramid in the accumulation of the regular octahedron

Source: Silva and Nós (2018), Nós and Silva (2020).

$$H = \frac{\sqrt{51 - 36\sqrt{2}}}{3}a. \quad (10)$$

The nested radical in (10) can be written as:

$$\sqrt{51 - 36\sqrt{2}} = \sqrt{51 - \sqrt{2592}}. \quad (11)$$

Employing the transformation relation described in Nós, Saito and Santos (2017) and Saito, Nós and Santos (2017) to transform the nested radical (11), with A=51 and B=2592, into a difference of simple radicals, we obtained:

$$\sqrt{51 - \sqrt{2592}} = \sqrt{\frac{51 + \sqrt{51^2 - 2592}}{2}} - \sqrt{\frac{51 - \sqrt{51^2 - 2592}}{2}};$$

$$\sqrt{51 - \sqrt{2592}} = \sqrt{\frac{51 + \sqrt{9}}{2}} - \sqrt{\frac{51 - \sqrt{9}}{2}};$$

$$\sqrt{51 - \sqrt{2592}} = \sqrt{27} - \sqrt{24};$$

$$\sqrt{51 - \sqrt{2592}} = 3\sqrt{3} - 2\sqrt{6}. \quad (12)$$

Replacing (12) in (10), we have:

$$H = \frac{3\sqrt{3} - 2\sqrt{6}}{3}a. \quad (13)$$

This way, using (13), we have that the volume of the triangular pyramid, whose base area is represented by $A_b$, is given by:

$$\mathcal{V}(triangular\ pyramid) = \frac{1}{3}\mathcal{A}_b H;$$

$$\mathcal{V}(triangular\ pyramid) = \frac{1}{3}\frac{\sqrt{3}}{4}a^2\frac{3\sqrt{3} - 2\sqrt{6}}{3}a;$$

$$\mathcal{V}(triangular\ pyramid) = \frac{9 - 6\sqrt{2}}{36}a^3;$$

$$\mathcal{V}(triangular\ pyramid) = \frac{3 - 2\sqrt{2}}{12}a^3. \quad (14)$$

The volume of the regular octahedron (SILVA; NÓS, 2018) is given by

$$\mathcal{V}(regular\ octahedron) = \frac{\sqrt{2}}{3}a^3. \quad (15)$$

Replacing (14) and (15) in (7), we conclude that the volume of the triakis octahedron is equal to:

$$\mathcal{V}(triakis\ octahedron) = \frac{\sqrt{2}}{3}a^3 + 8\frac{3 - 2\sqrt{2}}{12}a^3;$$

$$\mathcal{V}(triakis\ octahedron) = (2 - \sqrt{2})a^3, \quad (16)$$

where $a$ is the measure of the longest edge of the polyhedron (or the measure of the edge of the regular octahedron).

We can also determine the volume of the triakis octahedron from the measure of the shortest edge $x$. Replacing (9) in (16), we conclude that:

$$\mathcal{V}(triakis\ octahedron) = (2 - \sqrt{2})\left(\frac{2 + \sqrt{2}}{2}x\right)^3;$$

$$\mathcal{V}(triakis\ octahedron) = (2 - \sqrt{2})\left(\frac{x}{2 - \sqrt{2}}\right)^3;$$

$$\mathcal{V}(triakis\ octahedron) = \frac{x^3}{(2 - \sqrt{2})^2};$$

$$\mathcal{V}(triakis\ octahedron) = \frac{x^3}{2(3 - 2\sqrt{2})}\frac{3 + 2\sqrt{2}}{3 + 2\sqrt{2}};$$

$$\mathcal{V}(triakis\ octahedron) = \frac{3 + 2\sqrt{2}}{2}x^3, \quad (17)$$

where $x$ is the measure of the smallest edge of the polyhedron (or the lateral edge of the triangular pyramid coupled in the accumulation of the regular octahedron).

## RESULTS AND DISCUSSION

Results (5), (6), (16) and (17) agree with Rechneronline (2022), WolframAlpha (2022) and WolframMathWorld (2022).

## FINAL CONSIDERATIONS

In this work, we present strategies for calculating the volume of convex polyhedra using the dynamic geometry application GeoGebra 3D. With this application, we carry out compositions and decompositions of polyhedra, which are important for calculating the volume of these solids. The work contributes to the scarce references on the subject and we hope that it will encourage mathematics teachers in Basic Education and University to use and explore dynamic geometry applications in the classroom, particularly in calculating the volume of convex polyhedra.

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