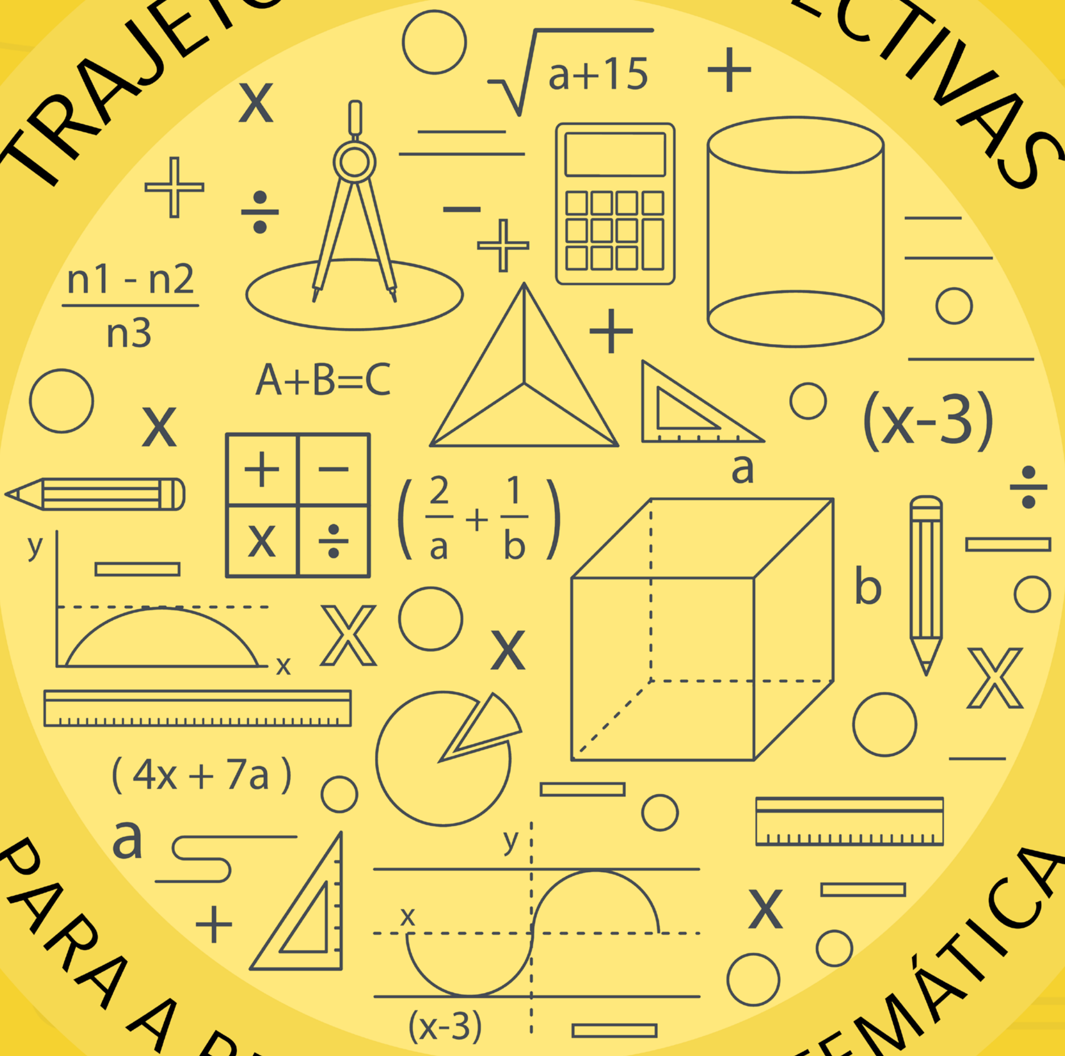
AMÉRICO JUNIOR NUNES DA SILVA
(Organizador)

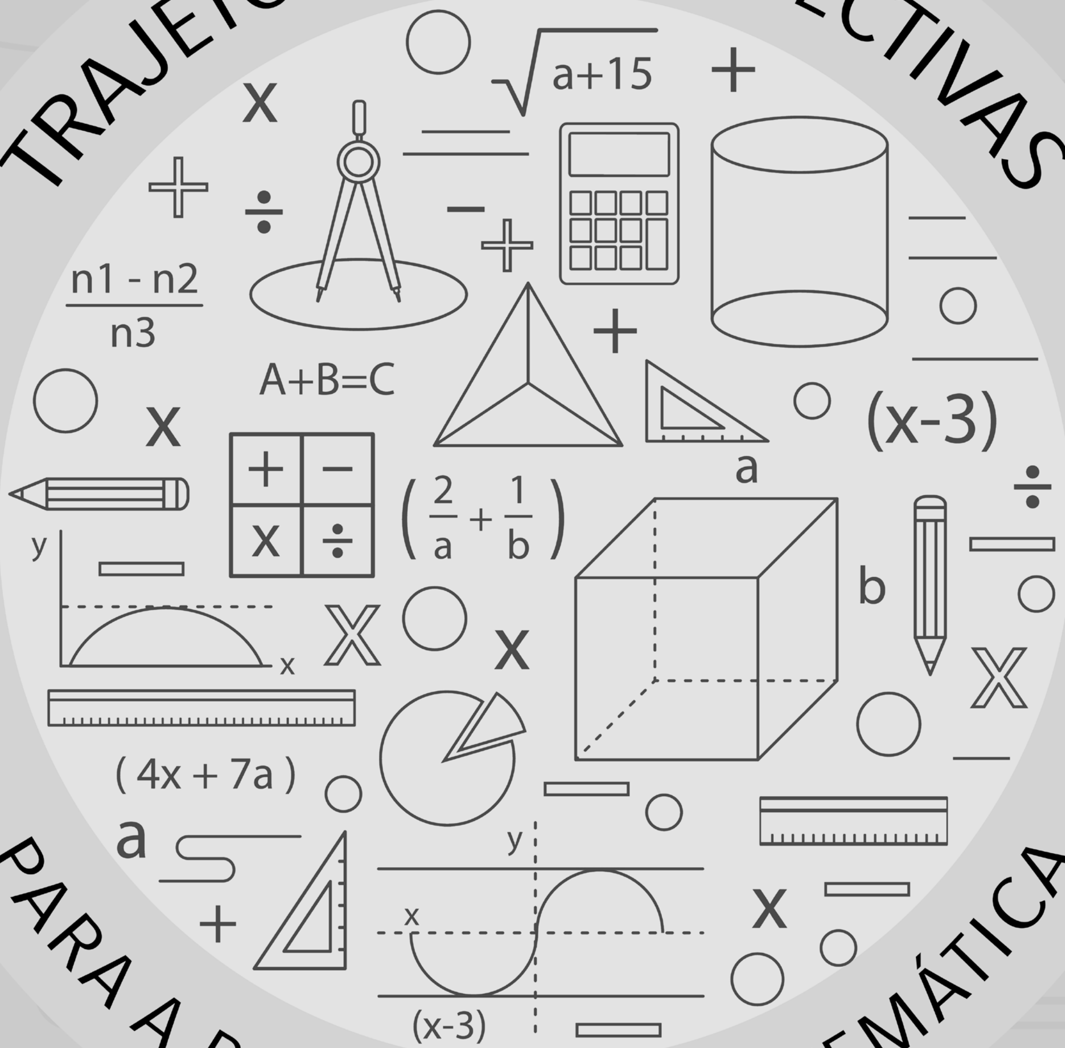
TRAJETÓRIAS E PERSPECTIVAS PARA A PESQUISA EM MATEMÁTICA



Atena
Editora
Ano 2022

AMÉRICO JUNIOR NUNES DA SILVA
(Organizador)

TRAJETÓRIAS E PERSPECTIVAS PARA A PESQUISA EM MATEMÁTICA



Atena
Editora
Ano 2022

Editora chefe
Profª Drª Antonella Carvalho de Oliveira
Editora executiva
Natalia Oliveira
Assistente editorial
Flávia Roberta Barão
Bibliotecária
Janaina Ramos
Projeto gráfico
Bruno Oliveira
Camila Alves de Cremo
Luiza Alves Batista
Imagens da capa
iStock
Edição de arte
Luiza Alves Batista

CIÊNCIAS EXATAS E DA TERRA E ENGENHARIAS

Trajetórias e perspectivas para a pesquisa em matemática

Diagramação: Camila Alves de Cremo
Correção: Maiara Ferreira
Indexação: Amanda Kelly da Costa Veiga
Revisão: Os autores
Organizador: Américo Junior Nunes da Silva

Dados Internacionais de Catalogação na Publicação (CIP)

T768 Trajetórias e perspectivas para a pesquisa em matemática / Organizador Américo Junior Nunes da Silva. – Ponta Grossa - PR: Atena, 2022.

Formato: PDF
Requisitos de sistema: Adobe Acrobat Reader
Modo de acesso: World Wide Web
Inclui bibliografia
ISBN 978-65-258-0854-3
DOI: https://doi.org/10.22533/at.ed.543220612

1. Matemática – Pesquisa. I. Silva, Américo Junior Nunes da (Organizador). II. Título.

CDD 510.07

Elaborado por Bibliotecária Janaina Ramos – CRB-8/9166

Atena Editora
Ponta Grossa – Paraná – Brasil
Telefone: +55 (42) 3323-5493
www.atenaeditora.com.br
contato@atenaeditora.com.br

DECLARAÇÃO DOS AUTORES

Os autores desta obra: 1. Atestam não possuir qualquer interesse comercial que constitua um conflito de interesses em relação ao artigo científico publicado; 2. Declaram que participaram ativamente da construção dos respectivos manuscritos, preferencialmente na: a) Concepção do estudo, e/ou aquisição de dados, e/ou análise e interpretação de dados; b) Elaboração do artigo ou revisão com vistas a tornar o material intelectualmente relevante; c) Aprovação final do manuscrito para submissão.; 3. Certificam que os artigos científicos publicados estão completamente isentos de dados e/ou resultados fraudulentos; 4. Confirmam a citação e a referência correta de todos os dados e de interpretações de dados de outras pesquisas; 5. Reconhecem terem informado todas as fontes de financiamento recebidas para a consecução da pesquisa; 6. Autorizam a edição da obra, que incluem os registros de ficha catalográfica, ISBN, DOI e demais indexadores, projeto visual e criação de capa, diagramação de miolo, assim como lançamento e divulgação da mesma conforme critérios da Atena Editora.

DECLARAÇÃO DA EDITORA

APRESENTAÇÃO

O contexto social, político e cultural tem demandado questões muito particulares para a escola e, sobretudo, para a formação, desenvolvimento e prática docente. Isso, de certa forma, tem levado os gestores a se aterem aos cursos de licenciatura e Educação Básica com atenção. Importante olhar mais atentamente para os espaços formativos, em um movimento dialógico e pendular de (re)pensar as diversas formas de se fazer ciências no país, sobretudo considerando as problemáticas evidenciadas em um mundo de pós-pandemia. A pesquisa, nesse interim, tem se constituído como um importante lugar de ampliar o olhar acerca das problemáticas reveladas, sobretudo no que tange ao conhecimento matemático.

O fazer Matemática vai muito além de aplicar fórmulas e regras. Existe uma dinâmica em sua construção que precisa ser percebida. Importante, nos processos de ensino e aprendizagem dessa ciência, priorizar e não perder de vista o prazer da descoberta, algo peculiar e importante no processo de matematizar. Isso, a que nos referimos anteriormente, configura-se como um dos principais desafios do educador matemático; e sobre isso, de uma forma muito particular, os autores e autoras abordaram nesta obra.

É neste sentido, que o livro "***Trajetórias e perspectivas para a pesquisa em matemática***" nasceu, como forma de permitir que as diferentes experiências do professor e professora pesquisadora que ensina Matemática sejam apresentadas e constituam-se enquanto canal de formação para educadores/as da Educação Básica e outros sujeitos. Reunimos aqui trabalhos de pesquisa e relatos de experiências de diferentes práticas que surgiram no interior da universidade e escola, por estudantes e professores/as pesquisadores/as de diferentes instituições do país.

Esperamos que esta obra, da forma como a organizamos, desperte nos leitores provocações, inquietações, reflexões e o (re)pensar da própria prática docente, para quem já é docente, e das trajetórias de suas formações iniciais para quem encontra-se matriculado em algum curso de licenciatura. Que, após esta leitura, possamos olhar para a sala de aula e para o ensino de Matemática com outros olhos, contribuindo de forma mais significativa com todo o processo educativo. Desejamos, portanto, uma ótima leitura.

Américo Junior Nunes da Silva

SUMÁRIO

ON THE WELLPOSEDNESS OF THE KDV-K-S EQUATION IN PERIODIC SOBOLEV SPACES

Data de aceite: 01/12/2022

Yolanda Silvia Santiago Ayala
Universidad Nacional Mayor de San Marcos
Lima, Peru
https://orcid.org/0000-0003-2516-0871

2020 Mathematics Subject Classification: 35G10, 35Q53, 35B40, 47D06.

ABSTRACT: In this work we prove that the Cauchy problem associated to the KdV-Kuramoto-Sivashinky (KdV-K-S) equation is globally well posed. We do this in an intuitive way using Fourier theory and in a fine version using Semigroups theory. Also, we study the corresponding nonhomogeneous problem and prove it is locally well posed and even more we obtain the continuous dependence of the solution with respect to the initial data and the non homogeneity. Finally, we prove the uniqueness solution of the homogeneous KdV- K-S equation using its dissipative property.

KEYWORDS: Semigroups theory, existence of solution, KdV - Ku- ramoto - Sivashinski equation, nonhomogeneous equation, periodic Sobolev spaces, Fourier theory.

1 I INTRODUCTION

We will study the KdV-Kuramoto-Sivashinsky equation:

$$(P_1): \ u_t + u_{xxx} + \beta(u_{xxxx} + u_{xx}) = 0 \text{ in } H^{s-4}_{per}, \text{ with } u(0) = \phi \in H^s_{per},$$

considering β a positive constant, *s* a real number and denoting by H^s_{per} to the periodic Sobolev space. For physical support to the model, we cite [15]. In this work we will make a complete study of the existence, uniqueness and continuous dependence of the solution of the KdV-K-S equation and its corresponding non-homogeneous problem, giving more properties, improvement of results and additional proofs. Thus, this is a unified study with improvements of [10] and [13].

We can cite [2] and [1] for this class of equations. We also cite some works about existence by semigroups [3], [4], [6], [7], [8] and take support in some results of [9].

Our article is organized as follows. In section 2 we state the preliminary results. In section 3, we prove that problem (P_1) is well posed and has regularity H^∞. Moreover, we introduce a family of operators, known as semigroups of contraction of class C_0 to state the result and prove it in a fine version. In section 4, we prove that the non homogeneous problem has a unique local solution and it continuously depends respect to the initial data and respect to the non homogeneity. In section 5, we study the uniqueness of the for homogeneous case using another technique that involves the dissipative property of the problem.

Finally, in section 6, we give the conclusions of our study.

2 I METHODOLOGY

We use the references [2], [10], [11], [12], [14] and [5] for the Fourier theory in periodic Sobolev spaces, and differential and integral calculus in Banach spaces.

3 I PROBLEM (P_1) IS WELL POSED

Theorem 3.1. *Let s a fixed real number,* $\beta > 0$ *and the problem*

$$(P_1)\quad \left|\begin{array}{l} u \in C([0,+\infty), H^s_{per}) \\ \partial_t u + \partial_x^3 u + \beta(\partial_x^4 u + \partial_x^2 u) = 0 \in H^{s-4}_{per} \\ u(0) = \phi \in H^s_{per} \end{array}\right.$$

then (P_1) *is globally well posed, that is*

$$\exists!\, u \in C([\,0,\infty),\ H^s_{per}) \cap C^1([0,\infty), H^{s-4}_{per})$$

satisfying equation (P_1*) so that the application:* $\phi \to u$ *, which to every initial data* ϕ *assigns the solution u of the IVP (*P_1*), is continuous. That is, for* ϕ *and* $\tilde{\phi}$ *initial data close in* H^s_{per} *, their corresponding solutions u and* $\tilde{u}$ *respectively, are also close in the solution space. Also,*

$$\|u(t) - \widetilde{u}(t)\|_s \le \|\phi - \widetilde{\phi}\|_s\,,\ \forall t \ge 0\,,$$

and

$$\sup_{t>0} \|u(t) - \widetilde{u}(t)\|_s \le \|\phi - \widetilde{\phi}\|_s\,.$$

Moreover, solution u satisfies the regularity:

$$u(t) \in H^\infty\,,\ \forall t > 0$$

with $\| u(t) \|_r \le C \| \phi \|_s$ *,* $\forall r \in \mathbb{R}$ *and* $t > 0$*, where*

$$H^\infty := \bigcap_{r\in\mathbb{R}} H^r_{per}$$

The application: $\phi \to \partial_t u$, which for every initial data ϕ assigns the derivative of solution u of the IVP (P_1) is continuous. That is, for ϕ and $\tilde{\phi}$ initial data close in H^s_{per} their corresponding $\partial_t u$ and $\partial_t \tilde{u}$, respectively, are also close in the solution space.

Also, the following inequalities are verified

$$\|\partial_t u(t) - \partial_t \widetilde{u}(t)\|_{s-4} \leq (1+2\beta)\|\phi - \widetilde{\phi}\|_s\,,\ \forall t \geq 0\,,\ ,$$

$$\sup_{t\geq 0} \|\partial_t u(t) - \partial_t \widetilde{u}(t)\|_{s-4} \leq (1+2\beta)\|\phi - \widetilde{\phi}\|_s\,.$$

Proof. We prove it in the following way.

1. First, we obtain the candidate to the solution. In order to get it we apply the Fourier transform to the equation

$$\partial_t u \quad = \quad -\partial_x^3 u - \beta(\partial_x^4 u + \partial_x^2 u)$$

and have

$$\begin{aligned}
\partial_t \hat{u} &= -(ik)^3\hat{u} - \beta((ik)^4\hat{u} + (ik)^2\hat{u}) \\
&= ik^3\hat{u} - \beta(k^4\hat{u} - k^2\hat{u}) \\
&= (ik^3 - \beta(k^4 - k^2))\hat{u} \\
&= (ik - \beta(k^2-1))k^2\hat{u}\,,
\end{aligned}$$

which for every k is an ODE with initial data $\hat{u}(k, 0) = \hat{\phi}(k)$. Thus, solving the IVP's

$$(\Omega_k) \left| \begin{array}{l} \hat{u} \in C([0,+\infty), l_s^2(Z)) \\ \partial_t \hat{u}(k,t) = (ik - \beta(k^2-1))k^2\hat{u}(k,t) \\ \hat{u}(k,0) = \hat{\phi}(k) \end{array} \right.$$

we obtain

$$\hat{u}(k,t) = e^{(ik-\beta(k^2-1))k^2 t}\hat{\phi}(k)\,,$$

from which we get our candidate to the solution:

$$\begin{aligned}
u(t) &= \sum_{k=-\infty}^{+\infty} \hat{u}(k,t)\phi_k \\
&= \sum_{k=-\infty}^{+\infty} e^{ik^3 t} F_k \hat{\phi}(k)\phi_k \qquad (3.1)
\end{aligned}$$

here we are denoting $\phi_k(x) = e^{ikx}$ and $F_k := e^{-\beta(k^2-1)k^2 t}$. We remark that when $k \in Z$ and $|k| = 1$ or $k = 0$, F_k is 1. When $k \in Z$ and $0 \neq |k| \neq 1$, we have that $(k^2 - 1)k^2 > 0$, and since β

> 0 have $F_k \to 0$ when $t \to +\infty$. Also $|e^{ik^3t}| = 1$.

2. Second, we prove:

$$u(t) \in H^s_{per} \quad \text{and} \quad \|u(t)\|_s \le \|\phi\|_s\,, \ \forall t \in [0,\infty)\,. \tag{3.2}$$

In effect, let $t > 0$, $\phi \in H^s_{per}$ and remarking that $e^{-2\beta(k^2-1)k^2t} < 1$, we have

$$\begin{aligned}
\|u(t)\|^2_{H^s_{per}} &= 2\pi \sum_{k=-\infty}^{+\infty} (1+k^2)^s |e^{ik^3t} e^{-\beta(k^2-1)k^2t} \hat{\phi}(k)|^2 \\
&= 2\pi \sum_{k=-\infty}^{+\infty} (1+k^2)^s |e^{-\beta(k^2-1)k^2t} \hat{\phi}(k)|^2 \\
&= 2\pi \sum_{k=-\infty}^{+\infty} (1+k^2)^s |\hat{\phi}(k)|^2 |e^{-\beta(k^2-1)k^2t}|^2 \\
&= 2\pi \sum_{k=-\infty}^{+\infty} (1+k^2)^s |\hat{\phi}(k)|^2 e^{-2\beta(k^2-1)k^2t} \qquad (3.3) \\
&\le 2\pi \sum_{k=-\infty}^{+\infty} (1+k^2)^s |\hat{\phi}(k)|^2 \quad < \infty \\
&= \|\phi\|^2_{H^s_{per}}\,.
\end{aligned}$$

Obviously it holds (3.2) for $t = 0$.

3. We will prove that $u(\cdot)$ is continuous in $[\,0, +\infty)$.

Let $t' \in [\,0, \infty)$,

$$\begin{aligned}
&\|u(t) - u(t')\|^2_{H^s_{per}} \\
&\quad = 2\pi \sum_{k=-\infty}^{+\infty} (1+k^2)^s |(e^{ik^3t} e^{-\beta(k^2-1)k^2t} \\
&\qquad\qquad -e^{ik^3t'} e^{-\beta(k^2-1)k^2t'}) \hat{\phi}(k)|^2 \\
&\quad = 2\pi \sum_{k=-\infty}^{+\infty} (1+k^2)^s |\hat{\phi}(k)|^2 |H(t)|^2 \qquad (3.4)
\end{aligned}$$

where $H(t) := e^{(ik^3-\beta(k^2-1)k^2)t} - e^{(ik^3-\beta(k^2-1)k^2)t'}$.

We see that $\lim_{t\to t'} H(t) = 0$. In order to interchange limits, we need the uniform convergence of the series. For this, we take the k-th term of the series and bound it by a convergent series, i.e.

$$
\begin{aligned}
I_{k,t} : &= 2\pi(1+k^2)^s|\hat{\phi}(k)|^2 \left| e^{(ik^3-\beta(k^2-1)k^2)t} - e^{(ik^3-\beta(k^2-1)k^2)t'} \right|^2 \\
&\leq 8\pi(1+k^2)^s|\hat{\phi}(k)|^2 ,
\end{aligned}
$$

there we have used the triangular inequality (property of the norm) and the inequality $e^{-\theta} \leq 1$ for $\theta \geq 0$. Thus,

$$
\sum_{k=-\infty}^{+\infty} I_{k,t} \leq 4\|\phi\|^2_{H^s_{per}} < \infty ,
$$

and using the M-Test of Weierstrass Theorem, we have the series converges uniformly. Now, we can interchange limits, that is

$$
\lim_{t \to t'} \|u(t) - u(t')\|^2_{H^s_{per}} = \sum_{k=-\infty}^{+\infty} \lim_{t \to t'} I_{k,t} = 0
$$

and then we conclude

$$
\lim_{t \to t'} \|u(t) - u(t')\|_{H^s_{per}} = 0 .
$$

4. We will prove

$$
\left\| \frac{u(t+h) - u(t)}{h} + \left(\partial_x^3 + \beta(\partial_x^4 + \partial_x^2)\right) u(t) \right\|_{H^{s-4}_{per}} \longrightarrow 0 \text{ when } h \to 0 .
$$

In effect,

$$
\begin{aligned}
&\left\| \frac{u(t+h) - u(t)}{h} + \partial_x^3 u(t) + \beta(\partial_x^4 u(t) + \partial_x^2 u(t)) \right\|^2_{H^{s-4}_{per}} \\
&= 2\pi \sum_{k=-\infty}^{+\infty} (1+k^2)^{s-4} \left| \hat{\phi}(k) \right|^2 \cdot \\
&\quad \left| \frac{e^{(ik^3-\beta(k^2-1)k^2)(t+h)} - e^{(ik^3-\beta(k^2-1)k^2)t}}{h} \right. \\
&\quad \left. + (ik)^3 e^{(ik^3-\beta(k^2-1)k^2)t} + \beta((ik)^4 + (ik)^2) e^{(ik^3-\beta(k^2-1)k^2)t} \right|^2 \\
&= 2\pi \sum_{k=-\infty}^{+\infty} (1+k^2)^{s-4} \left| \hat{\phi}(k) \right|^2 \left| e^{(ik^3-\beta(k^2-1)k^2)t} \cdot M(h) \right|^2 \qquad (3.5)
\end{aligned}
$$

where $M(h) := \left\{ \frac{e^{(ik^3-\beta(k^2-1)k^2)h} - 1}{h} + (ik)^3 + \beta((ik)^4 + (ik)^2) \right\}$.

Using L'Hospital we have $M(h) \to 0$ when $h \to 0$.

Again, to interchange limits, we need the uniform convergence of the series. For this we will bound the k-th term of the series. Previously, we observe for $h > 0$:

$$\begin{aligned}&\frac{e^{(ik^3-\beta(k^2-1)k^2)h}-1}{h}\\&=\int_0^h \frac{1}{h}\frac{\partial}{\partial s}\left\{e^{(ik^3-\beta(k^2-1)k^2)s}\right\}ds\\&=\int_0^h \frac{1}{h}\left[ik^3-\beta(k^2-1)k^2\right]e^{(ik^3-\beta(k^2-1)k^2)s}ds\end{aligned}$$

and taking norm we have

$$\begin{aligned}&\left|\frac{e^{(ik^3-\beta(k^2-1)k^2)h}-1}{h}\right|\\&\leq \frac{1}{h}|ik^3-\beta(k^2-1)k^2|\int_0^h\left|e^{(ik^3-\beta(k^2-1)k^2)s}\right|ds\\&\leq \frac{1}{h}\left\{|k|^3+\beta|k|^4+\beta|k|^2\right\}\cdot h\\&=\left\{|k|^3+\beta|k|^4+\beta|k|^2\right\}. \qquad (3.6)\end{aligned}$$

Using the inequalities (3.6), we are going to bound $|M(h)|^2$ as follows:

$$\begin{aligned}|M(h)|^2 &\leq \left\{2\left[|k|^3+\beta|k|^4+\beta|k|^2\right]\right\}^2\\&\leq \left[C_4|k|^4\right]^2\\&\leq C_5\left[|k|^2\right]^4\\&\leq C_5\left[1+|k|^2\right]^4. \qquad (3.7)\end{aligned}$$

Let us bound the k-th term of the series. Here we will use the estimation (3.7)

$$\begin{aligned}&(1+k^2)^{s-4}\left|\hat{\phi}(k)\right|^2 e^{-2\beta(k^2-1)k^2t}[M(h)]^2\\&\leq (1+k^2)^{s-4}\left|\hat{\phi}(k)\right|^2[M(h)]^2\\&\leq (1+k^2)^{s-4}\left|\hat{\phi}(k)\right|^2 C_5(1+|k|^2)^4\\&= C_5(1+k^2)^s\left|\hat{\phi}(k)\right|^2\end{aligned}$$

and, since $2\pi\sum_{k=-\infty}^{+\infty}(1+k^2)^s\left|\hat{\phi}(k)\right|^2=\|\phi\|^2_{H^s_{per}}<\infty$ for $\phi\in H^s_{per}$, using the M-Test of Weierstrass we get the series (3.5) converges uniformly and then it is possible to interchange limits and obtain

$$\left\| \frac{u(t+h)-u(t)}{h} + \left(\partial_x^3 + \beta(\partial_x^4 + \partial_x^2)\right) u(t) \right\|_{H^{s-4}_{per}}^2 \longrightarrow 0 \qquad (3.8)$$

when $h \to 0^+$.

Considering $h<0$, for the case $t>0$ we get

$$\begin{aligned} &\frac{e^{(ik^3-\beta(k^2-1)k^2)h}-1}{h} \\ &\quad = -\int_h^0 \frac{1}{h}\frac{\partial}{\partial s}\left\{e^{(ik^3-\beta(k^2-1)k^2)s}\right\} ds \\ &\quad = -\int_h^0 \frac{1}{h}\left[ik^3-\beta(k^2-1)k^2\right] e^{(ik^3-\beta(k^2-1)k^2)s} ds\,, \end{aligned}$$

taking norm, we have

$$\begin{aligned} &\left|\frac{e^{(ik^3-\beta(k^2-1)k^2)h}-1}{h}\right| \\ &\quad \le \frac{1}{|h|}|ik^3-\beta(k^2-1)k^2| \int_h^0 \left|e^{(ik^3-\beta(k^2-1)k^2)s}\right| ds \\ &\quad \le \frac{1}{|h|}|ik^3-\beta(k^2-1)k^2| \int_h^0 e^{(-\beta(k^2-1)k^2)s}\, ds\,. \qquad (3.9) \end{aligned}$$

We know that the area under the graph of the function $G(t) := e^{-\beta(k^2-1)k^2 t}$ from h to 0 is less than or equal to the area of the rectangle with base $|h|$ and height $e^{-\beta(k^2-1)k^2 h}$, that is

$$\int_h^0 e^{-\beta(k^2-1)k^2 s}\, ds \le |h| e^{-\beta(k^2-1)k^2 h}\,. \qquad (3.10)$$

Using (3.10) in (3.9) we get

$$\left|\frac{e^{[ik^3-\beta(k^2-1)k^2]h}-1}{h}\right| \le \{|k|^3+\beta|k|^4+\beta|k|^2\} e^{-\beta(k^2-1)k^2 h}\,. \quad (3.11)$$

Using (3.11) we have

$$|M(h)| \quad \le \quad \{|k|^3+\beta|k|^4+\beta|k|^2\}\{e^{-\beta(k^2-1)k^2 h}+1\}\,. \quad (3.12)$$

Using (3.12) we get

$$
\begin{aligned}
H(t,h) &= 2\pi \sum_{k=-\infty}^{+\infty} (1+k^2)^{s-4}|\widehat{\phi}(k)|^2|e^{[ik^3-\beta(k^2-1)k^2]t}M(h)|^2 \\
&= 2\pi \sum_{k=-\infty}^{+\infty} (1+k^2)^{s-4}|\widehat{\phi}(k)|^2|e^{i2k^3t}||e^{[-\beta(k^2-1)k^2]t}M(h)|^2 \\
&= 2\pi \sum_{k=-\infty}^{+\infty} (1+k^2)^{s-4}|\widehat{\phi}(k)|^2 \underbrace{\{|k|^3+\beta|k|^4+\beta|k|^2\}^2}_{\leq \max\{1,\beta\}^2\{|k|^2\}^4} \cdot \\
&\qquad \left\{e^{-\beta(k^2-1)k^2(t+h)} + e^{-\beta(k^2-1)k^2t}\right\}^2 . \qquad (3.13)
\end{aligned}
$$

Taking $h < 0$ with $|h|$ small enough such that $0 < t + h < t$, we have

$$
e^{-\beta(k^2-1)k^2(t+h)} \leq 1 . \qquad (3.14)
$$

Thus using (3.14) in (3.13) and that $\phi \in H^s_{per}$ we obtain

$$
\begin{aligned}
H(t,h) &\leq 2\pi \sum_{k=-\infty}^{+\infty} (1+k^2)^{s-4}|\widehat{\phi}(k)|^2 \max\{1,\beta\}^2\{1+|k|^2\}^4 \cdot 4 \\
&= (4\max\{1,\beta\}^2) \cdot 2\pi \sum_{k=-\infty}^{+\infty} (1+k^2)^{s-4}|\widehat{\phi}(k)|^2 \\
&= (4\max\{1,\beta\}^2)\|\phi\|_s^2 < \infty .
\end{aligned}
$$

Using the M-Test of Weierstrass, the series (3.5) is convergent absolutely and uniformly. Then it is possible to interchange limits and obtain

$$
\left\| \frac{u(t+h)-u(t)}{h} + \left(\partial_x^3 + \beta(\partial_x^4 + \partial_x^2)\right) u(t) \right\|^2_{H^{s-4}_{per}} \longrightarrow 0 \qquad (3.15)
$$

when $h \to 0^-$.

From (3.8) and (3.15) we have

$$
\left\| \frac{u(t+h)-u(t)}{h} + \left(\partial_x^3 + \beta(\partial_x^4 + \partial_x^2)\right) u(t) \right\|^2_{H^{s-4}_{per}} \longrightarrow 0
$$

when $h \to 0$, $\forall t > 0$.

This is also true for the case $t = 0$, there we use (3.6) only.

5. We will prove the continuous dependency of the solution with respect to the initial data, that is, let ϕ and $\tilde{\phi}$ be close data in H^s_{per}, then their corresponding solutions u and $\tilde{u}$, respectively, are also close in the solution space. Let $t \geq 0$,

$$
\begin{aligned}
&\|u(t) - \widetilde{u}(t)\|^2_{H^s_{per}} \\
&= 2\pi \sum_{k=-\infty}^{+\infty} \left| e^{ik^3 t} e^{-\beta(k^2-1)k^2 t} (\widehat{\phi}(k) - \widehat{\widetilde{\phi}}(k)) \right|^2 (1+k^2)^s \\
&= 2\pi \sum_{k=-\infty}^{+\infty} e^{-2\beta(k^2-1)k^2 t} \left| \widehat{\phi}(k) - \widehat{\widetilde{\phi}}(k) \right|^2 (1+k^2)^s \\
&\leq 2\pi \sum_{k=-\infty}^{+\infty} \left| \widehat{\phi}(k) - \widehat{\widetilde{\phi}}(k) \right|^2 (1+k^2)^s \\
&= 2\pi \sum_{k=-\infty}^{+\infty} (1+k^2)^s \left| \widehat{\phi}(k) - \widehat{\widetilde{\phi}}(k) \right|^2 \\
&= \|\phi - \widetilde{\phi}\|^2_{H^s_{per}} .
\end{aligned}
$$

Taking supremum over $(0, +\infty)$ we have

$$
\sup_{t \in (0,+\infty)} \|u(t) - \widetilde{u}(t)\|_{H^s_{per}} \leq \|\phi - \widetilde{\phi}\|_{H^s_{per}} . \tag{3.16}
$$

Hence, we have: if $\phi \to \tilde{\phi}$ then $u \to \tilde{u}$.

6. Uniqueness of Solution. Inequality (3.16) will allow us to prove the solution is unique. In effect, let $\phi \in H^s_{per}$ and suppose there are u and $\tilde{u}$ two solutions, then using (3.16) we have,

$$
\|u(r) - \widetilde{u}(r)\|_{H^s_{per}} \leq \sup_{t \in [0,\infty)} \|u(t) - \widetilde{u}(t)\|_{H^s_{per}} \leq \|\phi - \phi\|_{H^s_{per}} = 0 ,
$$

$\forall r \in [0, \infty)$, from where we conclude that $u = \tilde{u}$.

Thus, problem (P_1) is well posed and its unique solution, which depends continuously on the initial data, is

$$
u(t) = \sum_{k=-\infty}^{+\infty} e^{ik^3 t - \beta(k^2-1)k^2 t} \hat{\phi}(k) \phi_k .
$$

7. Let $t > 0$. From (3.3) we have for $r > s$:

$$
\|u(t)\|^2_r = 2\pi \sum_{k=-\infty}^{+\infty} (1+k^2)^r |\widehat{\phi}(k)|^2 |e^{-\beta(k^2-1)k^2 t}|^2
$$

$$
\begin{aligned}
&= \quad 2\pi \sum_{k=-\infty}^{+\infty} (1+k^2)^s |\widehat{\phi}(k)|^2 F(k,t) \\
&\leq \quad C^* 2\pi \sum_{k=-\infty}^{+\infty} (1+k^2)^s |\widehat{\phi}(k)|^2 \ < \ \infty \\
&= \quad C^* \|\phi\|_s^2 \,,
\end{aligned}
$$

where $F(k, t) := e^{-2\beta(k^2-1)k^2 t} \cdot (1+k^2)^{r-s}$ and satisfies $|F(k, t)| \leq C^*$, $\forall k \in Z$, $t > 0$. Thus,

$$
u(t) \in H^r_{per} \,, \ \ \forall r \in (s, +\infty) \,. \tag{3.17}
$$

The case $r = s$ was already proven on the item 2.

8. Now, we consider the case $r < s$. Here we have $H^s_{per} \subset H^r_{per}$ and since the initial data $\phi \in H^s_{per}$, then $\phi \in H^r_{per}$ and satisfies

$$
\|\phi\|_r \leq \|\phi\|_s \,. \tag{3.18}
$$

From (3.3) for r and using (3.18) we get

$$
\|u(t)\|_r^2 \leq \|\phi\|_r^2 \leq \|\phi\|_s^2 \ < \ \infty \,.
$$

That is,

$$
u(t) \in H^r_{per}, \forall r \in (-\infty, s) \,. \tag{3.19}
$$

Therefore, from (3.17), (3.2) and (3.19), we conclude for $t > 0$

$$
u(t) \in H^r_{per} \,, \ \ \forall r \in \mathbb{R} \,,
$$

and there exists $C := \max\{1, \sqrt{C^*}\}$ such that $\| u(t) \|_r \leq C \| \phi \|_s$, $\forall r \in$ R and $\forall t > 0$.

9. We will prove that $\partial_t u(\cdot)$ is continuous in $[0, \infty)$. Let $t, t' \in [0, \infty)$, using the inequality $\| \partial^m_x u(t) \|_{s-m} \leq \| u(t) \|_s$ and continuity of $u(\cdot)$, we obtain

$$
\begin{aligned}
&\|\partial_t u(t) - \partial_t u(t')\|_{H^{s-4}_{per}} \\
&= \| - \partial_x^3 u(t) - \beta(\partial_x^4 u(t) + \partial_x^2 u(t)) + \partial_x^3 u(t') \\
&\qquad + \beta(\partial_x^4 u(t') + \partial_x^2 u(t'))\|_{s-4} \\
&= \| - (\partial_x^3 u(t) - \partial_x^3 u(t')) - \beta(\partial_x^4 u(t) - \partial_x^4 u(t')) \\
&\qquad - \beta(\partial_x^2 u(t) - \partial_x^2 u(t'))\|_{s-4} \\
&= \|\partial_x^3 (u(t) - u(t'))\|_{s-4} + \beta\|\partial_x^4 (u(t) - u(t'))\|_{s-4} \\
&\qquad + \beta\|\partial_x^2 (u(t) - u(t'))\|_{s-4}
\end{aligned}
$$

$$\leq \|u(t)-u(t')\|_{s-1}+\beta\|u(t)-u(t')\|_{s}+\beta\|u(t)-u(t')\|_{s-2}$$
$$= (1+2\beta)\|u(t)-u(t')\|_{s} \to 0 \qquad (3.20)$$

when $t \to t'$. That is $\partial_t u \in C([0,\infty), H_{per}^{s-4})$.

10.Let $\phi \in H^s_{per}$, if we define

$$W(t)\phi := \sum_{k=-\infty}^{+\infty} (k^3 i - \beta(k^2-1)k^2) e^{[k^3 i - \beta(k^2-1)k^2]t} \widehat{\phi}(k)\phi_k$$

then $W(t)\phi \in H_{per}^{s-4}$ and $\| W(t)\phi \|_{s-4} \leq (1+2\beta) \| \phi \|_s$, $\forall t \geq 0$. That is, $W(t) \in L(H^s_{per}, H_{per}^{s-4})$ with $\| W(t) \| \leq (1+2\beta)$.

In effect, using $|k^3 i - \beta(k^2-1)k^2|^2 \leq (1+2\beta)^2(|k|^4)^2 = (1+2\beta)^2(|k|^2)^4 \leq (1+2\beta)^2(1+|k|^2)^4$, $\forall k \in Z$ and $e^{-\theta} \leq 1$, $\forall \theta \geq 0$, we have

$$\begin{aligned}
&\|W(t)\phi\|_{s-4}^2 \\
&= 2\pi \sum_{k=-\infty}^{+\infty} (1+k^2)^{s-4} |(k^3 i - \beta(k^2-1)k^2) e^{[k^3 i - \beta(k^2-1)k^2]t} \widehat{\phi}(k)|^2 \\
&= 2\pi \sum_{k=-\infty}^{+\infty} (1+k^2)^{s-4} |k^3 i - \beta(k^2-1)k^2|^2 e^{[-2\beta(k^2-1)k^2]t} |\widehat{\phi}(k)|^2 \\
&\leq (1+2\beta)^2 2\pi \sum_{k=-\infty}^{+\infty} (1+k^2)^{s} e^{[-2\beta(k^2-1)k^2]t} |\widehat{\phi}(k)|^2 \\
&\leq (1+2\beta)^2 2\pi \sum_{k=-\infty}^{+\infty} (1+k^2)^{s} |\widehat{\phi}(k)|^2 < \infty \\
&= (1+2\beta)^2 \|\phi\|_s^2 .
\end{aligned}$$

1. From item 4 and 10, we have $\partial_t u(t) = W(t)\phi$.

Next, we have the following result

Corollary 3.2. *The unique solution of* (P_1) *is*

$$u(t) = \sum_{k=-\infty}^{+\infty} e^{ik^3 t - \beta(k^2-1)k^2 t} \hat{\phi}(k)\phi_k ,$$

where $\phi_k(x) := e^{ikx}$ *for* $x \in \mathbb{R}$.

Corollary 3.3. *With the hypothesis of preceding Theorem, we obtain*

1. $u \in C([0,\infty), H^r_{per}) \cap C^1([0,\infty), H^{r-4}_{per})$, $\forall r \leq s$.

2. Also,

$$\begin{aligned} \|u(t)-\widetilde{u}(t)\|_r &\le \|\phi-\widetilde{\phi}\|_s\,,\ \forall t\ge 0\,,\ \forall r\le s\,,\\ \sup_{t>0}\|u(t)-\widetilde{u}(t)\|_r &\le \|\phi-\widetilde{\phi}\|_s\,,\ \forall r\le s\,. \end{aligned}$$

3. Moreover,

$$\begin{aligned} \|\partial_t u(t)-\partial_t\widetilde{u}(t)\|_{r-4} &\le (1+2\beta)\|\phi-\widetilde{\phi}\|_s\,,\ \forall t\ge 0\,,\ \forall r\le s\,,\\ \sup_{t\ge 0}\|\partial_t u(t)-\partial_t\widetilde{u}(t)\|_{r-4} &\le (1+2\beta)\|\phi-\widetilde{\phi}\|_s\,,\ \forall r\le s\,. \end{aligned}$$

4. If $r > s$ then $\| u(t) \|_r \le \sqrt{C^*} \| \phi \|_s$, $\forall t > 0$, $\| \partial_t u(t) \|_{r-4} \le (1 + 2\beta)\sqrt{C^*} \| \phi \|_s$, $\forall t > 0$, where C^* is such that $|G(k, t)| \le C^*$, $\forall k \in Z$, $\forall t > 0$, with $G(k, t) := e^{-2\beta(k^2-1)k^2 t} \cdot (1 + k^2)^{r-s}$.

5. Finally, $\forall r \in \mathbb{R}$

$$\|u(t)\|_r \le \min\{1,\sqrt{C^*}\}\|\phi\|_s\,,\ \forall t>0\,, \tag{3.21}$$

$$\|\partial_t u(t)\|_{r-4} \le (1+2\beta)\min\{1,\sqrt{C^*}\}\|\phi\|_s\,,\ \forall t>0\,, \tag{3.22}$$

Proof. The inequality (3.21) follows of the Sobolev continuous imbedding. We will use the Sobolev continuous Imbedding and item 10 for prove that if $\phi \in H^s_{per}$ then $W(t)\phi \in H_{per}^{r-4}$ and $\| W(t)\phi \|_{r-4} \le (1 + 2\beta) \| \phi \|_s$, $\forall t \ge 0$, $\forall r \le s$. That is, $W(t) \in L(H^s_{per}, H_{per}^{r-4})$ with $\| W(t) \| \le (1 + 2\beta)$, $\forall r \le s$.

In effect, using $|k^3 i - \beta(k^2 - 1)k^2|^2 \le (1 + 2\beta)^2(1 + |k|^2)^4$, $\forall k \in Z$ and $e^{-\theta} \le 1$, $\forall \theta \ge 0$, we have

$$\begin{aligned} &\|W(t)\phi\|^2_{r-4}\\ &= 2\pi\sum_{k=-\infty}^{+\infty}(1+k^2)^{r-4}|k^3 i-\beta(k^2-1)k^2|^2 e^{[-2\beta(k^2-1)k^2]t}|\widehat{\phi}(k)|^2\\ &\le (1+2\beta)^2 2\pi\sum_{k=-\infty}^{+\infty}(1+k^2)^{r}e^{[-2\beta(k^2-1)k^2]t}|\widehat{\phi}(k)|^2\\ &\le (1+2\beta)^2 2\pi\sum_{k=-\infty}^{+\infty}(1+k^2)^{r}|\widehat{\phi}(k)|^2\\ &\le (1+2\beta)^2 2\pi\sum_{k=-\infty}^{+\infty}(1+k^2)^{s}|\widehat{\phi}(k)|^2<\infty\\ &= (1+2\beta)^2\|\phi\|^2_s\,. \end{aligned} \tag{3.23}$$

Also, we will prove that if $\phi \in H_{per}^{s}$ then $W(t)\phi \in H_{per}^{r-4}$ and $\| W(t)\phi \|_{r-4} \le (1 + 2\beta)\sqrt{C^*} \| \phi \|_s$, $\forall t > 0$, $\forall r > s$. That is, $W(t) \in L(H_{per}^{s}, H_{per}^{r-4})$ with $\| W(t) \| \le (1 + 2\beta)\sqrt{C^*}$, $\forall r > s$.

In effect, using $|k^3 i - \beta(k^2 - 1)k^2|^2 \le (1 + 2\beta)^2(1 + |k|^2)^4$, $\forall k \in Z$ and that $|G(k, t)| \le C^*$, $\forall k \in Z$, $\forall t > 0$ with $r > s$, we have

$$
\begin{aligned}
&\|W(t)\phi\|_{r-4}^2 \\
&= 2\pi \sum_{k=-\infty}^{+\infty} (1+k^2)^{r-4} |k^3 i - \beta(k^2-1)k^2|^2 e^{[-2\beta(k^2-1)k^2]t} |\widehat{\phi}(k)|^2 \\
&\leq (1+2\beta)^2 2\pi \sum_{k=-\infty}^{+\infty} (1+k^2)^r e^{[-2\beta(k^2-1)k^2]t} |\widehat{\phi}(k)|^2 \\
&= (1+2\beta)^2 2\pi \sum_{k=-\infty}^{+\infty} (1+k^2)^s \underbrace{(1+k^2)^{r-s} e^{[-2\beta(k^2-1)k^2]t}}_{G(k,t):=} |\widehat{\phi}(k)|^2 \\
&\leq (1+2\beta)^2 C^* 2\pi \sum_{k=-\infty}^{+\infty} (1+k^2)^s |\widehat{\phi}(k)|^2 < \infty \\
&= (1+2\beta)^2 C^* \|\phi\|_s^2 \,. \qquad (3.24)
\end{aligned}
$$

From (3.23) and (3.24), we have (3.22).

Now, we will introduce a family of operators which verify the condition of being a semigroup of contraction of class C_0.

Theorem 3.4. *Let* $\beta > 0$ *and* $s \in \mathbb{R}$. *The application*

$$
\begin{aligned}
S : [0, +\infty) &\rightarrow L(H^s_{per}) \\
t &\rightarrow S(t)
\end{aligned}
$$

such that $S(t) = e^{-(\partial_x^3 + \beta(\partial_x^4 + \partial_x^2))t}$, *that is, applies*

$$
S(t)\phi = \left[\left(e^{(ik^3 - \beta(k^4 - k^2))t} \hat{\phi}(k) \right)_{k \in Z} \right]^{\vee}, \ \forall \phi \in H^s_{per},
$$

then $\{S(t)\}_{t\geq 0}$ *is a semigroup of class* C_0 *of contraction on* H_{per}^s.

Moreover, the following assertions hold:

1. *If* $\phi \in H_{per}^s$ *then* $S(\cdot)\phi \in C([0, \infty), H_{per}^s)$.

2. *The application* $\phi \rightarrow S(\cdot)\phi$ *is continuous and verifies:*

$$
\|S(t)\psi_1 - S(t)\psi_2\|_{H^s_{per}} \leq \|\psi_1 - \psi_2\|_{H^s_{per}}, \ \forall t \geq 0
$$

and

$$
\sup_{t>0} \|S(t)\psi_1 - S(t)\psi_2\|_{H^s_{per}} \leq \|\psi_1 - \psi_2\|_{H^s_{per}}
$$

with $\psi_i \in H_{per}^s$ *for* $i = 1, 2$.

3. *If* $\phi \in H_{per}^s$ *then* $\partial_t S(t)\phi \in H_{per}^{s-4}$ *and* $\|\partial_t S(t)\phi\|_{s-4} \leq (1 + 2\beta)\|\phi\|_s$, *that is,* $\partial_t S(t) \in L(H_{per}^s, H_{per}^{s-4})$, $\forall t \geq 0$, *where*

$$\partial_t S(t)\phi = \left[\left((ik^3 - \beta(k^2-1)k^2)e^{[ik^3-\beta(k^2-1)k^2]t}\widehat{\phi}(k)\right)_{k\in Z}\right]^{\vee} \in H^{s-4}_{per},$$
$\forall \phi \in H^s_{per}$.

4. If $\phi \in H_{per}^{\ s}$ then $\partial_t S(\cdot)\phi \in C([0, \infty), H_{per}^{\ s-4})$.

5. The application: $\psi \to \partial_t S(\cdot)\psi$ is continuous and verifies:

$$\begin{aligned} \|\partial_t S(t)\psi_1 - \partial_t S(t)\psi_2\|_{s-4} &\leq (1+2\beta)\|\psi_1 - \psi_2\|_s \,, \ \forall t \geq 0\,, \\ \sup_{t\geq 0} \|\partial_t S(t)\psi_1 - \partial_t S(t)\psi_2\|_{s-4} &\leq (1+2\beta)\|\psi_1 - \psi_2\|_s \end{aligned}$$

with $\psi_i \in H_{per}^{\ s}$ for $i = 1, 2$.

Proof. We first observe that $S(0)\phi = \left[\left(\widehat{\phi}(k)\right)_{k\in Z}\right]^{\vee} = \left[\widehat{\phi}\right]^{\vee} = \phi,\ \forall \phi \in H_{per}^{\ s}$, thus $S(0) = I$. From linearity of Fourier transformation and its inverse, we have that $S(t)$ is linear. In effect, let be $a \in \mathbb{C}$, ϕ, $\psi \in H_{per}^{\ s}$, we have

$$\begin{aligned} & S(t)(a\phi + \psi) \\ &= \left[\left(e^{[ik^3-\beta(k^2-1)k^2]t}[a\phi+\psi]^{\wedge}(k)\right)_{k\in Z}\right]^{\vee} \\ &= \left[\left(e^{[ik^3-\beta(k^2-1)k^2]t}[a\widehat{\phi}(k)+\widehat{\psi}(k)]\right)_{k\in Z}\right]^{\vee} \\ &= \left[a\left(e^{[ik^3-\beta(k^2-1)k^2]t}\widehat{\phi}(k)\right)_{k\in Z} + \left(e^{[ik^3-\beta(k^2-1)k^2]t}\widehat{\psi}(k)\right)_{k\in Z}\right]^{\vee} \\ &= a\left[\left(e^{[ik^3-\beta(k^2-1)k^2]t}\widehat{\phi}(k)\right)_{k\in Z}\right]^{\vee} + \left[\left(e^{[ik^3-\beta(k^2-1)k^2]t}\widehat{\psi}(k)\right)_{k\in Z}\right]^{\vee} \\ &= a\,S(t)\phi + S(t)\psi\,, \end{aligned}$$

for $t \geq 0$.

If $\phi \in H_{per}^{\ s}$ and $t > 0$, we will prove that $S(t)\phi \in H_{per}^{\ s}$ and $\|S(t)\phi\|_s \leq \|\phi\|_s$, that is $\|S(t)\| \leq 1$. In effect, similar to (3.3) we have

$$\begin{aligned} \|S(t)\phi\|^2_{H^s_{per}} &= 2\pi \sum_{k=-\infty}^{+\infty} (1+k^2)^s |e^{ik^3 t} e^{-\beta(k^2-1)k^2 t}\hat{\phi}(k)|^2 \\ &= 2\pi \sum_{k=-\infty}^{+\infty} (1+k^2)^s |\hat{\phi}(k)|^2 e^{-2\beta(k^2-1)k^2 t} \qquad (3.25) \\ &\leq 2\pi \sum_{k=-\infty}^{+\infty} (1+k^2)^s |\hat{\phi}(k)|^2 = \|\phi\|^2_{H^s_{per}} \quad < \infty\ . \end{aligned}$$

Then, $S(t)\phi \in H_{per}^{\ s}$ and $\|S(t)\phi\|_s \leq \|\phi\|_s$, $\forall t > 0$, that is $S(t) \in L(H_{per}^{\ s})$ with $\|S(t)\| \leq 1$, $\forall t > 0$.

Therefore,

$$\|S(t)\phi\|_s \leq \|\phi\|_s\,,\ \forall t \geq 0\,,\ \forall \phi \in H^s_{per}\,. \tag{3.26}$$

That is,

$$S(t) \in L(H^s_{per})\,, \quad \text{with } \|S(t)\| \leq 1\,,\ \forall t \geq 0\,. \tag{3.27}$$

Now we will prove that $S(t + r) = S(t) \circ S(r)$, $\forall t,\ r \geq 0$. In effect, let be $f \in H_{per}{}^s$ and t, $r \in (0, \infty)$,

$$\begin{aligned} S(t+r)f &= \left[\left(e^{[ik^3-\beta(k^2-1)k^2](t+r)}\hat{f}(k)\right)_{k\in Z}\right]^\vee \\ &= \left[\left(e^{[ik^3-\beta(k^2-1)k^2]t}e^{[ik^3-\beta(k^2-1)k^2]r}\hat{f}(k)\right)_{k\in Z}\right]^\vee . \end{aligned} \tag{3.28}$$

We know that if $f \in H_{per}{}^s$ then $\hat{f} \in l_s^2$, that is

$$\sum_{k=-\infty}^{+\infty} (1+k^2)^s |\widehat{f}(k)|^2 < \infty\,. \tag{3.29}$$

We affirm that

$$\left(e^{[ik^3-\beta(k^2-1)k^2]r}\hat{f}(k)\right)_{k\in Z} \in l^2_s\,,\ \forall r \geq 0\,. \tag{3.30}$$

Indeed, when $r = 0$ it is evident that the statement is true. Thus, we will prove the case $r > 0$. For this, it is enough to observe that

$$\begin{aligned} &\sum_{k=-\infty}^{+\infty} (1+k^2)^s |e^{ik^3r}e^{-\beta(k^2-1)k^2r}\widehat{f}(k)|^2 \\ &\qquad = \sum_{k=-\infty}^{+\infty} (1+k^2)^s \underbrace{|e^{i2k^3r}|}_{=1}\underbrace{e^{-2\beta(k^2-1)k^2r}}_{\leq 1}|\widehat{f}(k)|^2 \\ &\qquad = \sum_{k=-\infty}^{+\infty} (1+k^2)^s |\widehat{f}(k)|^2 < \infty\,, \end{aligned}$$

since it worth (3.29).

Then, from (3.30) and taking the inverse Fourier transform, we have

$$\left[\left(e^{[ik^3-\beta(k^2-1)k^2]r}\hat{f}(k)\right)_{k\in Z}\right]^\vee \in H^s_{per}\,,\ \forall r \geq 0\,.$$

This motivates us to define

$$g_r := \left[\left(e^{[ik^3-\beta(k^2-1)k^2]r}\hat{f}(k)\right)_{k\in Z}\right]^\vee \in H^s_{per}\,.$$

That is,

$$g_r = S(r)f\,. \tag{3.31}$$

Also, taking the Fourier transform to g_r we obtain

$$\widehat{g_r} = \left(e^{[ik^3-\beta(k^2-1)k^2]r}\hat{f}(k)\right)_{k\in Z},$$

that is

$$\widehat{g_r}(k) = e^{[ik^3-\beta(k^2-1)k^2]r}\hat{f}(k)\,,\ \forall k \in Z\,. \tag{3.32}$$

Using (3.32) in (3.28) and from (3.31) we have

$$\begin{aligned} S(t+r)f &= \left[\left(e^{[ik^3-\beta(k^2-1)k^2]t}\hat{g}_r(k)\right)_{k\in Z}\right]^{\vee} \\ &= S(t)g_r \\ &= S(t)[S(r)f] \\ &= [S(t)\circ S(r)]f\,,\ \forall t,r \in (0,\infty)\,. \end{aligned}$$

Thus,

$$S(t+r) = S(t)\circ S(r)\,,\ \forall t,r \in (0,\infty)\,. \tag{3.33}$$

If $t = 0$ or $r = 0$, then the equality of (3.33) is also true, with this we conclude the proof of

$$S(t+r) = S(t)\circ S(r)\,,\ \forall t,r \in [0,\infty)\,. \tag{3.34}$$

Now, we will prove the continuity of $t \to S(t)\phi$

$$\|S(t+h)\phi - S(t)\phi\|_{H^s_{per}} \to 0 \ \text{ when } h \to 0\,. \tag{3.35}$$

In effect, using item 3 of the proof of the preceding theorem, we have

$$\begin{aligned} &\|S(t+h)\phi - S(t)\phi\|^2_{H^s_{per}} \\ &= 2\pi \sum_{k=-\infty}^{+\infty}(1+k^2)^s|(e^{ik^3(t+h)}e^{-\beta(k^2-1)k^2(t+h)} \\ &\qquad -e^{ik^3t}e^{-\beta(k^2-1)k^2t})\hat{\phi}(k)|^2 \\ &= 2\pi \sum_{k=-\infty}^{+\infty}(1+k^2)^s|\hat{\phi}(k)|^2|H(t+h)|^2 \end{aligned} \tag{3.36}$$

where $H(t+h) := e^{(ik^3-\beta(k^2-1)k^2)(t+h)} - e^{(ik^3-\beta(k^2-1)k^2)t}$.

We observe that $\lim_{h\to 0} H(t+h) = 0$.

Now, we again need the uniform convergence of the series in order to interchange the limits. For this, we take the k-th term of the series and bound it with a convergent series, that is

$$\begin{aligned} I_{k,t,h} : &= 2\pi(1+k^2)^s|\hat{\phi}(k)|^2 \left| e^{(ik^3-\beta(k^2-1)k^2)(t+h)} - e^{(ik^3-\beta(k^2-1)k^2)t} \right|^2 \\ &\leq 8\pi(1+k^2)^s|\hat{\phi}(k)|^2 , \end{aligned}$$

where we have used the triangular inequality (property of the norm) and the inequality $e^{-\theta} \leq 1$ for $\theta \geq 0$.

Thus,

$$\sum_{k=-\infty}^{+\infty} I_{k,t,h} \leq 4\|\phi\|^2_{H^s_{per}} < \infty , \tag{3.37}$$

and using the M-Test Weierstrass Theorem we get the series in (3.37) converges uniformly. Then, interchanging limits is allowed, that is

$$\lim_{h\to 0} \|S(t+h)\phi - S(t)\phi\|^2_{H^s_{per}} = \sum_{k=-\infty}^{+\infty} \lim_{h\to 0} I_{k,t,h} = 0$$

and from here we conclude

$$\lim_{h\to 0} \|S(t+h)\phi - S(t)\phi\|_{H^s_{per}} = 0 \ .$$

Remark 3.5. It verifies

$$\lim_{t\to 0^+} \|S(t)\phi - \phi\|_{H^s_{per}} = 0 \ .$$

Therefore, $\{S(t)\}_{t\geq 0}$ is a semigroup of contraction of class C_o on $H_{per}{}^s$. Let ψ_1 and ψ_2 close data in $H_{per}{}^s$, then we will prove that their corresponding $S(\cdot)\psi_1$ and $S(\cdot)\psi_2$, respectively, are also close. Since $\{S(t)\}_{t\geq 0}$ is of contraction for $t \geq 0$, we have

$$\|S(t)\psi_1 - S(t)\psi_2\|_{H^s_{per}} = \|S(t)[\psi_1 - \psi_2]\|_{H^s_{per}} \leq \|\psi_1 - \psi_2\|_{H^s_{per}} \ .$$

Taking supremum over $(0, +\infty)$ we have

$$\sup_{t\in(0,+\infty)} \|S(t)\psi_1 - S(t)\psi_2\|_{H^s_{per}} \leq \|\psi_1 - \psi_2\|_{H^s_{per}} \ . \tag{3.38}$$

From here we have that if $\psi_1 \to \psi_2$ then $S(\cdot)\psi_1 \to S(\cdot)\psi_2$.

We will prove: If $\phi \in H_{per}{}^s$ then $\partial_t S(t)\phi \in H_{per}{}^{s-4}$ and $\| \partial_t S(t)\phi \|_{s-4} \leq (1 + 2\beta) \| \phi \|_s$.

In effect, using $|k^3 i - \beta(k^2 - 1)k^2|^2 \leq (1 + 2\beta)^2(|k|^4)^2 = (1 + 2\beta)^2(|k|^2)^4 \leq (1 + 2\beta)^2(1 + |k|^2)^4$, $\forall k \in Z$ and $e^{-\theta} \leq 1$, $\forall \theta \geq 0$, we have

$$\begin{aligned} &\|\partial_t S(t)\phi\|^2_{s-4} \\ &= 2\pi \sum_{k=-\infty}^{+\infty} (1+k^2)^{s-4} |(k^3 i - \beta(k^2-1)k^2) e^{[k^3 i - \beta(k^2-1)k^2]t} \widehat{\phi}(k)|^2 \\ &= 2\pi \sum_{k=-\infty}^{+\infty} (1+k^2)^{s-4} |k^3 i - \beta(k^2-1)k^2|^2 e^{[-2\beta(k^2-1)k^2]t} |\widehat{\phi}(k)|^2 \end{aligned}$$

$$\leq (1+2\beta)^2 2\pi \sum_{k=-\infty}^{+\infty} (1+k^2)^s e^{[-2\beta(k^2-1)k^2]t} |\widehat{\phi}(k)|^2$$
$$\leq (1+2\beta)^2 2\pi \sum_{k=-\infty}^{+\infty} (1+k^2)^s |\widehat{\phi}(k)|^2 < \infty$$
$$= (1+2\beta)^2 \|\phi\|_s^2 .$$

That is, $\| \partial_t S(t)\phi \|_{s-4} \leq (1 + 2\beta) \| \phi \|_s$. From this inequality we obtain

$$\|\partial_t S(t)\varphi_1 - \partial_t S(t)\varphi_2\|_{s-4} \leq (1+2\beta)\|\varphi_1 - \varphi_2\|_s ,$$

with $\varphi_i \in H_{per}^s$ for $i = 1, 2$.

So, taking supremum over $[0, \infty)$ we have

$$\sup_{t\in[0,\infty)} \|\partial_t S(t)\varphi_1 - \partial_t S(t)\varphi_2\|_{s-4} \leq (1+2\beta)\|\varphi_1 - \varphi_2\|_s .$$

Finally, if $\phi \in H_{per}^s$ we will prove the continuity of $t \to \partial_t S(t)\phi$. That is

$$\|\partial_t S(t+h)\phi - \partial_t S(t)\phi\|_{s-4} \to 0 \text{ when } h \to 0 .$$

In effect, as item 3 of the proof of the preceding theorem, we have

$$\begin{aligned}
&\|\partial_t S(t+h)\phi - \partial_t S(t)\phi\|^2_{H^{s-4}_{per}} \\
&= 2\pi \sum_{k=-\infty}^{+\infty} (1+k^2)^{s-4} \left| (e^{ik^3(t+h)} e^{-\beta(k^2-1)k^2(t+h)} - e^{ik^3 t} e^{-\beta(k^2-1)k^2 t}) \cdot \right. \\
&\qquad \left. (ik^3 - \beta(k^2-1)k^2)\hat{\phi}(k) \right|^2 \\
&= 2\pi \sum_{k=-\infty}^{+\infty} (1+k^2)^{s-4} \left| (e^{ik^3(h)} e^{-\beta(k^2-1)k^2(h)} - 1)(e^{ik^3 t} e^{-\beta(k^2-1)k^2 t}) \cdot \right. \\
&\qquad \left. (ik^3 - \beta(k^2-1)k^2)\hat{\phi}(k) \right|^2 \\
&= 2\pi \sum_{k=-\infty}^{+\infty} (1+k^2)^{s-4} \left| e^{ik^3(h)} e^{-\beta(k^2-1)k^2(h)} - 1 \right|^2 e^{-2\beta(k^2-1)k^2 t} \cdot \\
&\qquad \left| \hat{\phi}(k) \right|^2 |ik^3 - \beta(k^2-1)k^2|^2 \\
&= 2\pi \sum_{k=-\infty}^{+\infty} (1+k^2)^{s-4} |\hat{\phi}(k)|^2 |ik^3 - \beta(k^2-1)k^2|^2 e^{-2\beta(k^2-1)k^2 t} |H(h)|^2
\end{aligned} \tag{3.39}$$

where $H(h) := e^{(ik^3-\beta(k^2-1)k^2)(h)} - 1$.

We observe that $\lim_{h\to 0} H(h) = 0$.

Now, we again need the uniform convergence of the series in order to interchange the limits. For this, we take the k-th term of the series and bound it with a convergent series, that is

$$
\begin{aligned}
I_{k,t,h}: &= 2\pi(1+k^2)^{s-4}|\hat{\phi}(k)|^2|ik^3-\beta(k^2-1)k^2|^2e^{-2\beta(k^2-1)k^2t}|H(h)|^2 \\
&\leq (1+2\beta)^2 8\pi(1+k^2)^s|\hat{\phi}(k)|^2\,,
\end{aligned}
$$

where we have used the triangular inequality (property of the norm) and the inequality $e^{-\theta} \leq 1$ for $\theta \geq 0$.

Thus,

$$
\sum_{k=-\infty}^{+\infty} I_{k,t,h} \leq 4(1+2\beta)^2\|\phi\|^2_{H^s_{per}} < \infty\,, \tag{3.40}
$$

and using the M-Test Weierstrass Theorem we get the series in (3.40) converges uniformly. Then, interchanging limits is allowed, that is

$$
\lim_{h\to 0}\|\partial_t S(t+h)\phi - \partial_t S(t)\phi\|^2_{H^{s-4}_{per}} = \sum_{k=-\infty}^{+\infty} \underbrace{\lim_{h\to 0} I_{k,t,h}}_{=0} = 0
$$

hence we conclude

$$
\lim_{h\to 0}\|\partial_t S(t+h)\phi - \partial_t S(t)\phi\|_{H^{s-4}_{per}} = 0\ .
$$

We will give some additional properties of $\{S(t)\}_{t\geq 0}$.

Corollary 3.6. *With the hypothesis of preceding Theorem, the following assertions hold*

1. *If $\varphi \in H_{per}^s$ then $S(t)\varphi \in H_{per}^r$ and $\|S(t)\varphi\|_r \leq \|\varphi\|_s$, $\forall t \geq 0$, $\forall r \leq s$. That is $S(t) \in L(H_{per}^s, H_{per}^r)$, $\forall t \geq 0$, $\forall r \leq s$.*
2. *If $\varphi \in H_{per}^s$ then $S(\cdot)\varphi \in C([0,\infty), H_{per}^r)$, $\forall r \leq s$.*
3. *The application: $\varphi \to S(\cdot)\varphi$ is continuous and verifies:*

$$
\begin{aligned}
\|S(t)\varphi_1 - S(t)\varphi_2\|_r &\leq \|\varphi_1-\varphi_2\|_s\,,\ \forall t\geq 0\,,\ \forall r\leq s\,,\\
\sup_{t>0}\|S(t)\varphi_1 - S(t)\varphi_2\|_r &\leq \|\varphi_1-\varphi_2\|_s\,,\ \forall r\leq s
\end{aligned}
$$

with $\varphi_i \in H_{per}^s$ for $i = 1, 2$.

4. *If $\varphi \in H_{per}^s$ then $\partial_t S(t)\varphi \in H_{per}^{r-4}$ and $\|\partial_t S(t)\varphi\|_{r-4} \leq (1+2\beta)\|\varphi\|_s$, $\forall t \geq 0$, $\forall r \leq s$. That is $\partial_t S(t) \in L(H_{per}^s, H_{per}^{r-4})$, $\forall t \geq 0$, $\forall r \leq s$* where

$$
\partial_t S(t)\varphi = \left[\left((ik^3-\beta(k^2-1)k^2)e^{[ik^3-\beta(k^2-1)k^2]t}\widehat{\phi}(k)\right)_{k\in Z}\right]^{\vee} \in H^{r-4}_{per},
$$

$\forall\varphi \in H^s_{per}$.

5. If $\varphi \in H_{per}^s$ then $\partial_t S(\cdot)\varphi \in C([0,\infty), H_{per}^{r-4})$, $\forall r \le s$.

6. The application: $\varphi \to \partial_t S(\cdot)\varphi$ is continuous and verifies:

$$\begin{aligned} \|\partial_t S(t)\varphi_1 - \partial_t S(t)\varphi_2\|_{r-4} &\le (1+2\beta)\|\varphi_1 - \varphi_2\|_s\,,\ \forall t \ge 0\,, \\ \sup_{t\ge 0} \|\partial_t S(t)\varphi_1 - \partial_t S(t)\varphi_2\|_{r-4} &\le (1+2\beta)\|\varphi_1 - \varphi_2\|_s\,, \end{aligned}$$

$\forall r \le s$ with $\varphi_i \in H_{per}^s$ for $i = 1, 2$.

7. If $r > s$ then $\| S(t)\varphi \|_r \le \sqrt{C^*} \|\varphi\|_s$ and $\|\partial_t S(t)\varphi\|_{r-4} \le (1 + 2\beta)\sqrt{C^*}\|\varphi\|_s$, $\forall t > 0$, $\forall \varphi \in H_{per}^s$, where C^* is such that $|G(k, t)| \le C^*$, $\forall k \in Z$, $\forall t > 0$ with $G(k, t) := e^{-2\beta(k^2-1)k^2 t} \cdot (1 + k^2)^{r-s}$.

8. Finally,

$$\begin{aligned} \|S(t)\varphi\|_r &\le \min\{1, \sqrt{C^*}\}\|\varphi\|_s\,,\ \forall t > 0\,,\ \forall r \in \mathbb{R}\,, \\ \|\partial_t S(t)\varphi\|_{r-4} &\le (1+2\beta)\min\{1, \sqrt{C^*}\}\|\varphi\|_s\,,\ \forall t > 0\,,\ \forall r \in \mathbb{R}\,. \end{aligned}$$

Proof. Its proof is analogous to the proof of the second Corollary of The- orem 3.1, where we use Sobolev imbedding.

Following, we state the Theorem 3.1 in function of the semigroup $\{S(t)\}_{t\ge 0}$.

Theorem 3.7. *Let $\beta > 0$, $s \in \mathbb{R}$ and $\{S(t)\}_{t\ge 0}$ the semigroup of class C_0 from Theorem 3.4, then $S(\cdot)\phi$ is the unique solution of*

$$\left| \begin{array}{l} u \in C([0,\infty), H_{per}^s) \cap C^1([0,\infty), H_{per}^{s-4}) \\ u_t = Au \ in \ H_{per}^{s-4} \\ u(0) = \phi \in H_{per}^s \end{array} \right.$$

in the sense that

$$\lim_{h\to 0} \left\| \frac{S(t+h)\phi - S(t)\phi}{h} - AS(t)\phi \right\|_{H_{per}^{s-4}} = 0 \qquad (3.41)$$

where $A := -\partial_x^3 - \beta(\partial_x^4 + \partial_x^2)$, and if $\phi_1 \sim \phi_2$ then $S(\cdot)\phi_1 \sim S(\cdot)\phi_2$.

Moreover, the following regularity is satisfied: If $\phi \in H_{per}^s$ then $S(t)\phi \in H^\infty$, $\forall t > 0$ and there is a constant $C > 0$ such that $\| S(t)\phi \| H_{per}^r \le C \| \phi \| H_{per}^s$, $\forall t > 0$ and $\forall r \in \mathbb{R}$.

Also, $\| \partial_t S(t)\phi \|_r \le (1 + 2\beta) \| \phi \|_s$, $\forall t \ge 0$, $\forall \phi \in H_{per}^s$.

Proof. The proof of (3.41) is analogous to the item 4 of the proof of The- orem 3.1. And the proof of the remaining statement is also similar to the proof of Theorem 3.1 and as a consequence of Theorem 3.4.

4 | THE NON HOMOGENEOUS PROBLEM IS LOCALLY WELL POSED

Using the Fourier transform, we will prove that the non homogeneous problem has a unique solution and it continuously depends respect to the initial data and the non homogeneity in compact intervals.

Theorem 4.1. *Let* $s \in \mathbb{R}$, $\beta > 0$, $F \in C([0, T], Hper^s)$, *where* $T > 0$, $\{S(t)\}_{t \geq 0}$ *the semigroup of class* C_o *of homogeneous case* ($F = 0$)*, introduced in the Theorem 3.4, and*

$$u_p(t) := \int_0^t S(t-\tau)F(\tau)d\tau$$

then $u_p \in C([0, T], H_{per}^s) \cap C^1([0, T], H_{per}^{s-4})$ *and satisfies*

$$\left| \begin{array}{l} \partial_t u_p + \partial_x^3 u_p + \beta(\partial_x^4 u_p + \partial_x^2 u_p) = F(t) \in H_{per}^{s-4} \\ u_p(0) = 0 \end{array} \right. \tag{4.1}$$

with the derivative given by

$$\lim_{h \to 0} \left\| \frac{u_p(t+h) - u_p(t)}{h} + \partial_x^3 u_p + \beta(\partial_x^4 u_p + \partial_x^2 u_p) - F(t) \right\|_{s-4} = 0\,. \tag{4.2}$$

Proof. We remark that $S(t-\tau)F(\tau) \in H_{per}^s$ $\forall \tau \in (0, t)$ and $\tau \mapsto S(t-\tau)F(\tau)$ is continuous in [0, t] then $\exists \underbrace{\int_0^t S(t-\tau)F(\tau)d\tau}_{u_p(t):=} \in H_{per}^s.$

Now, we will prove $u_p \in C([0, T], H_{per}^s)$, that is, $\| u_p(t+h) - u_p(t) \|_s \to 0$ when $h \to 0$. Let $h > 0$

$$\begin{aligned} u_p(t+h) - u_p(t) &= \int_0^{t+h} S(t+h-\tau)F(\tau)d\tau - \int_0^t S(t-\tau)F(\tau)d\tau \\ &= \int_0^t \{S(t+h-\tau) - S(t-\tau)\}F(\tau)d\tau \\ &\quad + \int_t^{t+h} S(t+h-\tau)F(\tau)d\tau\,, \end{aligned}$$

taking the norm $\| \cdot \|_s$ we obtain

$$\begin{aligned} \|u_p(t+h) - u_p(t)\|_s &\leq \underbrace{\| \int_0^t \{S(t+h-\tau) - S(t-\tau)\}F(\tau)d\tau \|_s}_{I_1:=} \\ &\quad + \underbrace{\| \int_t^{t+h} S(t+h-\tau)F(\tau)d\tau \|_s}_{I_2:=}\,. \end{aligned}$$

Using the M-Test of Weierstrass we get

$$\begin{aligned} I_1 &\le \int_0^t \|S(t+h-\tau)F(\tau) - S(t-\tau)F(\tau)\|_s d\tau \\ &\le \frac{\epsilon}{2T}\int_0^t d\tau = \frac{\epsilon t}{2T} \le \frac{\epsilon T}{2T} = \frac{\epsilon}{2} \end{aligned}$$

since $\exists\delta > 0$ such that:

if $|h| < \delta$ then $\| S(t+h-\tau)F(\tau) - S(t-\tau)F(\tau)\|_s < \frac{\epsilon}{2T}$, $\forall \tau \in (0, t)$.

Using the mean value Theorem in H_{per}^{s} we obtain

$$\frac{1}{h}\int_t^{t+h} S(t+h-\tau)F(\tau)d\tau \longrightarrow S(0)F(t) = F(t)$$

when $h \to 0$. Then

$$\int_t^{t+h} S(t+h-\tau)F(\tau)d\tau = \underbrace{h}_{\to 0} \cdot \underbrace{\frac{1}{h}\int_t^{t+h} S(t+h-\tau)F(\tau)d\tau}_{\to F(t)} \longrightarrow 0$$

when $h \to 0$. So, we have

$$I_2 = \| \int_t^{t+h} S(t+h-\tau)F(\tau)d\tau\|_s \longrightarrow 0$$

when $h \to 0$. That is, $I_2 < \frac{\epsilon}{2}$ whenever $|h| < \delta^*$.

Therefore, $\| u_p(t+h) - u_p(t)\|_s \le I_1 + I_2 < \epsilon$ whenever $|h| < \min\{\delta, \delta^*\}$.

From definition of $u_p(t)$ we have $u_p(0) = 0$.

Now, we will prove that $\exists \partial_t u_p(t)$ in H_{per}^{s-4}. In effect,

$$\begin{aligned} &\frac{u_p(t+h) - u_p(t)}{h} \\ &= \frac{1}{h}\left\{\int_0^{t+h} S(t+h-\tau)F(\tau)d\tau - \int_0^t S(t-\tau)F(\tau)d\tau\right\} \\ &= \frac{1}{h}\int_0^t \{S(t+h-\tau)F(\tau) - S(t-\tau)F(\tau)\}d\tau \\ &\quad + \frac{1}{h}\int_t^{t+h} S(t+h-\tau)F(\tau)d\tau\,. \end{aligned}$$

Using the mean value Theorem in H_{per}^{s-4} we obtain

$$\frac{1}{h}\int_t^{t+h} S(t+h-\tau)F(\tau)d\tau \longrightarrow S(0)F(t) = F(t) \tag{4.3}$$

when $h \to 0$.

Since

$$\frac{S(t+h-\tau)F(\tau) - S(t-\tau)F(\tau)}{h}$$

converges uniformly to $\partial_t\{S(t-\tau)F(\tau)\}$ in H_{per}^{s-4} $\forall \tau \in [0, t]$, we obtain that

$$\int_0^t \frac{S(t+h-\tau)F(\tau) - S(t-\tau)F(\tau)}{h} d\tau$$

converges to $\int_0^t \partial_t\{S(t-\tau)F(\tau)\}d\tau$ when $h \to 0$.

Now, we remark that $\partial_t\{S(t-\tau)F(\tau)\} = (-\partial_x^3 - \beta(\partial_x^4 + \partial_x^2))S(t-\tau)F(\tau)$ in H_{per}^{s-4}.

Since $(-\partial_x^3 - \beta(\partial_x^4 + \partial_x^2))$ is a closed operator, then

$$\int_0^t (-\partial_x^3 - \beta(\partial_x^4 + \partial_x^2))S(t-\tau)F(\tau)d\tau = (-\partial_x^3 - \beta(\partial_x^4 + \partial_x^2)) \int_0^t S(t-\tau)F(\tau)d\tau .$$

Therefore,

$$\int_0^t \partial_t\{S(t-\tau)F(\tau)\}d\tau = (-\partial_x^3 - \beta(\partial_x^4 + \partial_x^2)) \underbrace{\int_0^t S(t-\tau)F(\tau)d\tau}_{u_p(t):=} .$$

That is,

$$\int_0^t \frac{S(t+h-\tau)F(\tau) - S(t-\tau)F(\tau)}{h} d\tau \longrightarrow (-\partial_x^3 - \beta(\partial_x^4 + \partial_x^2))u_p(t) \tag{4.4}$$

when $h \to 0$.

Finally, in H_{per}^{s-4}, using (4.3) and (4.4) we get

$$\underbrace{\exists \lim_{h \to 0} \frac{u_p(t+h) - u_p(t)}{h}}_{\partial_t u_p(t)=} = (-\partial_x^3 - \beta(\partial_x^4 + \partial_x^2))u_p(t) + F(t) .$$

Using that $\| \partial_x^m f \|_{s-m} \leq \| f \|_s$, $\forall m \in Z^+$, $\forall f \in H_{per}^s$, we obtain

$$\begin{aligned}
&\|\partial_t u_p(t+h) - \partial_t u_p(t)\|_{s-4} \\
&= \|(-\partial_x^3 - \beta(\partial_x^4 + \partial_x^2))u_p(t+h) + F(t+h) \\
&\qquad -\{(-\partial_x^3 - \beta(\partial_x^4 + \partial_x^2))u_p(t) + F(t)\}\|_{s-4} \\
&\leq \|F(t+h) - F(t)\|_{s-4} \\
&\qquad +\|(-\partial_x^3 - \beta(\partial_x^4 + \partial_x^2))\{u_p(t+h) - u_p(t)\}\|_{s-4} \\
&\leq \|F(t+h) - F(t)\|_{s-4} + \| - \partial_x^3\{u_p(t+h) - u_p(t)\}\|_{s-4} \\
&\qquad +\| - \beta\partial_x^4\{u_p(t+h) - u_p(t)\}\|_{s-4} \\
&\qquad +\| - \beta\partial_x^2\{u_p(t+h) - u_p(t)\}\|_{s-4} \\
&\leq \|F(t+h) - F(t)\|_{s-4} + \|u_p(t+h) - u_p(t)\|_{s-1} \\
&\qquad +\mu\|u_p(t+h) - u_p(t)\|_s + \beta\|u_p(t+h) - u_p(t)\|_{s-2} .
\end{aligned}$$

Since $u_p(t) \in H_{per}^s \subset H_{per}^{s-1} \subset H_{per}^{s-2}$ then

$$\|\partial_t u_p(t+h) - \partial_t u_p(t)\|_{s-4}$$
$$\leq \|F(t+h) - F(t)\|_{s-4} + (1+2\beta)\|u_p(t+h) - u_p(t)\|_s \, .$$

Now, since F: $[0, T] \to H_{per}^{s-4}$ and u_p: $[0, T] \to H_{per}^{s}$ are continuous, then $\partial_t u_p$: $[0, T] \to H_{per}^{s-4}$ is continuous for $t \in [0, T]$, that is, $\partial_t u_p \in C([0, T], H_{per}^{s-4})$.

Therefore, $u_p \in C([0, T], H_{per}^{s}) \cap C^1([0, T], H_{per}^{s-4})$.

Theorem 4.2. *Let $\phi \in H_{per}^{s}$, $s \in \mathbb{R}$, $\beta > 0$, $F \in C([0, T], H_{per}^{s})$, where $T > 0$, and $\{S(t)\}_{t \geq 0}$ the semigroup of class C_o of contraction in H_{per}^{s} as* in Theorem 4.1, then

1. The function:

$$u^F(t) := S(t)\phi + \underbrace{\int_0^t S(t-\tau)F(\tau)d\tau}_{u_p(t)=} \, , \; t \in [0, T] \tag{4.5}$$

belongs to $C([0, T], H_{per}^{s}) \cap C^1([0, T], H_{per}^{s-4})$ and

2. $u^F(t)$ is the unique solution of

$$(P_1^F) \left| \begin{array}{l} u_t + u_{xxx} + \beta(u_{xxxx} + u_{xx}) = F(t) \in H_{per}^{s-4} \\ u(0) = \phi \end{array} \right. \tag{4.6}$$

with the derivative given by

$$\lim_{h \to 0} \left\| \frac{u(t+h) - u(t)}{h} + u_{xxx} + \beta(u_{xxxx} + u_{xx}) - F(t) \right\|_{s-4} = 0 \, . \tag{4.7}$$

3. Let $\psi_j \in H_{per}^{s}$, $F_j \in C([0, T], H_{per}^{s})$, $j = 1, 2$. The map $\psi \to u$ is continuous in the following sense. Let u_1 and u_2 the corresponding solutions to initial data ψ_1 and ψ_2, and with non homogeneity F_1 and F_2 respectively. Then

$$\|u_1(t) - u_2(t)\|_s$$
$$\leq \|\psi_1 - \psi_2\|_s + T\|F_1 - F_2\|_{\infty,s} \, , \; t \in [0, T], \tag{4.8}$$

$$\underbrace{\sup_{t \in [0,T]} \|u_1(t) - u_2(t)\|_s}_{\|u_1 - u_2\|_{\infty,s} :=}$$
$$\leq \|\psi_1 - \psi_2\|_s + T\|F_1 - F_2\|_{\infty,s}, \tag{4.9}$$

$$\|\partial_t u_1(t) - \partial_t u_2(t)\|_{s-4}$$
$$\leq (1+2\beta)\|u_1(t) - u_2(t)\|_s + \|F_1 - F_2\|_{\infty,s-4} \, , \; t \in [0, T],$$
$$\leq (1+2\beta)\|u_1 - u_2\|_{\infty,s} + \|F_1 - F_2\|_{\infty,s}$$

$$\leq (1+2\beta)\|\psi_1-\psi_2\|_s + [(1+2\beta)T+1]\|F_1-F_2\|_{\infty,s} \tag{4.10}$$

where we have used the notation:

$$\|h\|_{\infty,r} = \sup_{t\in[0,T]} \|h(t)\|_r \ , \quad h \in C([0,T], H^r_{per})\,. \tag{4.11}$$

Proof. We work the following steps.

1. Let $u(t) := u^F(t) = S(t)\phi + u_p(t)$, we will prove that

$$u \in C([0,T], H^s_{per}) \cap C^1([0,T], H^{s-4}_{per}).$$

In effect, as $S(\cdot)\phi \in C([0, T], H_{per}{}^{s})$ and $u_p(\cdot) \in C([0, T], H_{per}{}^{s})$ then $u(\cdot) = S(\cdot)\phi + u_p(\cdot) \in C([0, T], H_{per}{}^{s})$.

Moreover, as $S(\cdot)\phi \in C^1([0, +\infty), H_{per}{}^{s-4})$ and $u_p(\cdot) \in C^1([0, T], H_{per}{}^{s-4})$, then $u(\cdot) = S(\cdot)\phi + u_p(\cdot) \in C^1([0, T], H_{per}{}^{s-4})$.

2. We will prove that u is the solution of $(P_1{}^F)$. In effect, we know that $\exists\partial_t S(t)\phi$ and $\exists\partial_t u_p(t)$ in $H_{per}{}^{s-4}$, then

$$\begin{aligned}
\partial_t u(t) &= \partial_t \underbrace{S(t)\phi}_{u_h(t)=} + \partial_t u_p(t) \\
&= -\partial_x^3 u_h(t) - \beta(\partial_x^4 u_h(t) + \partial_x^2 u_h(t)) \\
&\quad -\partial_x^3 u_p(t) - \beta(\partial_x^4 u_p(t) + \partial_x^2 u_p(t)) + F(t) \\
&= -\partial_x^3\{u_h(t)+u_p(t)\} - \beta\left(\partial_x^4\{u_h(t)+u_p(t)\}\right. \\
&\qquad \left. +\partial_x^2\{u_h(t)+u_p(t)\}\right) + F(t) \\
&= -\partial_x^3 u(t) - \beta\left(\partial_x^4 u(t) + \partial_x^2 u(t)\right) + F(t)
\end{aligned}$$

in $H_{per}{}^{s-4}$, where $u_h(\cdot)$ is solution of the homogeneous equation (P_1).

3. Also, $u(0) = u_h(0) + u_p(0) = \phi + 0 = \phi$.

4. Let $\psi_j \in H_{per}{}^{s}$ and $F_j \in C([0, T], H_{per}{}^{s})$ for $j = 1, 2$, then

$$u_j(t) = S(t)\psi_j + \int_0^t S(t-\tau)F_j(\tau)d\tau$$

is solution of $(P_1{}^{F_j})$ with initial data $u_j(0) = S(0)\psi_j = \psi_j$, for $j = 1, 2$.

Then

$$u_1(t) - u_2(t) = S(t)\{\psi_1-\psi_2\} + \int_0^t S(t-\tau)\{F_1(\tau)-F_2(\tau)\}d\tau\,.$$

From where we obtain, for $t < T$:

$$
\begin{aligned}
&\|u_1(t) - u_2(t)\|_s \\
&\leq \|S(t)\{\psi_1 - \psi_2\}\|_s + \int_0^t \|S(t-\tau)\{F_1(\tau) - F_2(\tau)\}\|_s d\tau \\
&\leq \|\psi_1 - \psi_2\|_s + \int_0^t \|F_1(\tau) - F_2(\tau)\|_s d\tau \\
&\leq \|\psi_1 - \psi_2\|_s + \sup_{\tau\in[0,T]} \|F_1(\tau) - F_2(\tau)\|_s \int_0^t d\tau \\
&\leq \|\psi_1 - \psi_2\|_s + T \cdot \sup_{\tau\in[0,T]} \|F_1(\tau) - F_2(\tau)\|_s .
\end{aligned}
$$

Therefore,

$$
\sup_{t\in[0,T]} \|u_1(t) - u_2(t)\|_s \leq \|\psi_1 - \psi_2\|_s + T \cdot \sup_{\tau\in[0,T]} \|F_1(\tau) - F_2(\tau)\|_s .
$$

5. On the other hand, in H_{per}^{s-4} we have

$$
\begin{aligned}
\partial_t u_j(t) &= \partial_t u_{h,j}(t) + \partial_t u_{p,j}(t) \\
&= [-\partial_x^3 - \beta(\partial_x^4 + \partial_x^2)]u_{h,j}(t) + [-\partial_x^3 - \beta(\partial_x^4 + \partial_x^2)]u_{p,j}(t) \\
&\quad + F_j(t) \\
&= [-\partial_x^3 - \beta(\partial_x^4 + \partial_x^2)]u_j + F_j(t)
\end{aligned}
$$

for $j = 1, 2$. So,

$$
\partial_t u_1(t) - \partial_t u_2(t) = [-\partial_x^3 - \beta(\partial_x^4 + \partial_x^2)]\{u_1(t) - u_2(t)\} + \{F_1(t) - F_2(t)\}.
$$

Taking norm, we obtain

$$
\begin{aligned}
&\|\partial_t u_1(t) - \partial_t u_2(t)\|_{s-4} \\
&\quad = \|[-\partial_x^3 - \beta(\partial_x^4 + \partial_x^2)]\{u_1(t) - u_2(t)\} + \{F_1(t) - F_2(t)\}\|_{s-4} \\
&\quad \leq \|\partial_x^3\{u_1(t) - u_2(t)\}\|_{s-4} + \beta\|\partial_x^4\{u_1(t) - u_2(t)\}\|_{s-4} \\
&\qquad + \beta\|\partial_x^2\{u_1(t) - u_2(t)\}\|_{s-4} + \|F_1(t) - F_2(t)\|_{s-4} .
\end{aligned}
$$

Using $\|\partial_x^m \psi\|_{s-m} \leq \|\psi\|_s$, for $m = 1, 2, 3$ and $H_{per}^{s} \subset H_{per}^{s-1} \subset H_{per}^{s-2}$, we get

$$
\begin{aligned}
&\|\partial_t u_1(t) - \partial_t u_2(t)\|_{s-4} \\
&\leq \|u_1(t) - u_2(t)\|_{s-1} + \beta\|u_1(t) - u_2(t)\|_s \\
&\quad + \beta\|u_1(t) - u_2(t)\|_{s-2} + \|F_1(t) - F_2(t)\|_{s-4} \\
&\leq (1 + 2\beta)\|u_1(t) - u_2(t)\|_s + \|F_1(t) - F_2(t)\|_s \\
&\leq (1 + 2\beta)\|u_1(t) - u_2(t)\|_s + \sup_{t\in[0,T]} \|F_1(t) - F_2(t)\|_{s-4} \\
&\leq (1 + 2\beta)\|u_1 - u_2\|_{\infty,s} + \|F_1 - F_2\|_{\infty,s} \\
&\leq (1 + 2\beta)\|\psi_1 - \psi_2\|_s + [(1 + 2\beta)T + 1]\|F_1 - F_2\|_{\infty,s} .
\end{aligned}
$$

Remark 4.3. Inequality (4.8) says that the solution of the non homoge- neous problem (P_1^F) continuously depends on the initial data and the non homogeneity F, in compact intervals.

Corollary 4.4. *The problem (P_1^F) has a unique solution.*

Proof. This follows by applying inequality (4.8) with $\psi_1 = \psi_2 = \phi$ and $F_1 = F_2 = F$.

Corollary 4.5. *The unique solution of (P_1^F) is*

$$\begin{aligned} u^F(t) &= \sum_{k=-\infty}^{+\infty} e^{-\{-ik^3+\beta(k^2-1)k^2\}t}\widehat{\phi}(k)\phi_k \\ &+ \int_0^t \sum_{k=-\infty}^{+\infty} e^{-\{-ik^3+\beta(k^2-1)k^2\}(t-\tau)}\widehat{F}(k,\tau)\phi_k d\tau\ , \end{aligned}$$

where $\phi_k(x) := e^{ikx}$ for $x \in \mathbb{R}$.

5 I DISSIPATIVE PROPERTY OF THE HOMOGENEOUS PROBLEM

We will study the uniqueness of the solution for homogeneous case using another technique that involves the dissipative property of the prob- lem.

Let $\beta > 0$, $s \in \mathbb{R}$ and the homogeneous problem

$$(P_1) \quad \left| \begin{array}{l} w \in C([0,\infty), H^s_{per}) \cap C^1([0,\infty), H^{s-4}_{per}) \\ \partial_t w + \partial_x^3 w + \beta(\partial_x^4 w + \partial_x^2 w) = 0 \ \in H^{s-4}_{per} \\ w(0) = \phi \in H^s_{per}\ . \end{array} \right.$$

Theorem 5.1. *Let w the solution of (P_1) with initial data $\phi \in H_{per}^s$, then we obtain the following results:*

1. $\partial_t \| w(t) \|^2_{s-4} = -2\beta < \partial_x^2 w(t) + \partial_x^4 w(t),\ w(t) >_{s-4} \leq 0$.

2. $\| w(t) \|_{s-4} \leq \| \phi \|_{s-4} \leq \| \phi \|_s,\ t \geq 0$.

Proof. As $H_{per}^s \subset H_{per}^{s-4}$ then the following expressions are well defined: $< \partial_t w,\ w >_{s-4}$ and $< w,\ \partial_t w >_{s-4}$.

So, we have

$$\begin{aligned} \partial_t \|w(t)\|^2_{s-4} &= \partial_t < w(t), w(t) >_{s-4} \\ &= < \partial_t w(t), w(t) >_{s-4} + < w(t), \partial_t w(t) >_{s-4} \\ &= 2Re < \partial_t w(t), w(t) >_{s-4}\ . \end{aligned} \tag{5.1}$$

For the other hand,

$$\begin{aligned} < \partial_x^2 w + \partial_x^4 w, w >_{s-4} &= 2\pi \sum_{k=-\infty}^{+\infty} (1+k^2)^{s-4} \{\partial_x^2 w + \partial_x^4 w\}\widehat{\ }(k)\overline{\widehat{w}(k)} \\ &= 2\pi \sum_{k=-\infty}^{+\infty} (1+k^2)^{s-4} \underbrace{k^2(k^2-1)}_{M(k):=} \widehat{w}(k)\overline{\widehat{w}(k)} \end{aligned}$$

$$= \quad 2\pi \sum_{k=-\infty}^{+\infty} (1+k^2)^{s-4} M(k) |\widehat{w}(k)|^2 \,. \qquad (5.2)$$

We remark that the series in (5.2) is convergent since $M(k) \leq k^4 \leq (k^2)^4 \leq (1+k^2)^4$, $\forall k \in Z$ and $w(t) \in H_{per}^{\ s}$.

Moreover, as

$$M(k) = \begin{cases} 0 & \text{if } k \in \{-1,0,1\} \\ k^2(k^2-1) > 0 & \text{if } k \in Z - \{-1,0,1\}\,, \end{cases}$$

and $M(k) \geq 12$, $\forall k \in Z - \{-1, 0, 1\}$, then the convergent series (5.2) is not negative. That is,

$$< \partial_x^2 w + \partial_x^4 w, w >_{s-4} \, \geq \, 0\,.$$

So, we have

$$- < \partial_x^2 w + \partial_x^4 w, w >_{s-4} \, \leq \, 0\,. \qquad (5.3)$$

Also, we obtain

$$\begin{aligned} < \partial_x^3 w, w >_{s-4} &= 2\pi \sum_{k=-\infty}^{+\infty} (1+k^2)^{s-4} \widehat{\partial_x^3 w}(k) \cdot \overline{\widehat{w}(k)} \\ &= 2\pi \sum_{k=-\infty}^{+\infty} (1+k^2)^{s-4} (ik)^3 \widehat{w}(k) \cdot \overline{\widehat{w}(k)} \\ &= -i\, 2\pi \underbrace{\sum_{k=-\infty}^{+\infty} (1+k^2)^{s-4} k^3 |\widehat{w}(k)|^2}_{b:=} \,. \qquad (5.4) \end{aligned}$$

Now, we will prove that the series of the equality (5.4) is convergent. In effect, using the inequality: $|k|^3 \leq |k|^8 = (|k|^2)^4 \leq (1+k^2)^4$ and $w(t) \in H_{per}^{\ s}$ we have

$$\begin{aligned} &\left| \sum_{k=-\infty}^{+\infty} (1+k^2)^{s-4} k^3 |\widehat{w}(k)|^2 \right| \\ &\leq \sum_{k=-\infty}^{+\infty} (1+k^2)^{s-4} |k|^3 |\widehat{w}(k)|^2 \\ &\leq \sum_{k=-\infty}^{+\infty} (1+k^2)^{s-4} (1+k^2)^4 |\widehat{w}(k)|^2 \\ &= \sum_{k=-\infty}^{+\infty} (1+k^2)^{s} |\widehat{w}(k)|^2 \, = \, \frac{1}{2\pi} \|w(t)\|_s^2 < \infty\,. \end{aligned}$$

Then the series (5.4) is convergent, that is,

$$< \partial_x^3 w, w >_{s-4} = -ib, \text{ with } b \in \mathbb{R}. \tag{5.5}$$

From (5.1), using $\partial_t w = -\beta(\partial_x^2 w + \partial_x^4 w) - \partial_x^3 w$, $\beta > 0$, the inequality (5.3) and the equality (5.5) we get

$$\begin{aligned}
\partial_t \|w(t)\|_{s-4}^2 &= 2Re < \partial_t w(t), w(t) >_{s-4} \\
&= 2Re < -\beta(\partial_x^2 w(t) + \partial_x^4 w(t)) - \partial_x^3 w(t), w(t) >_{s-4} \\
&= 2\beta Re \underbrace{\left\{- < \partial_x^2 w(t) + \partial_x^4 w(t), w(t) >_{s-4}\right\}}_{\leq 0} \\
&\quad -2 \underbrace{Re < \partial_x^3 w(t), w(t) >_{s-4}}_{=0} \\
&= -2\beta < \partial_x^2 w(t) + \partial_x^4 w(t), w(t) >_{s-4} \quad \leq \quad 0 .
\end{aligned}$$

Therefore, $\| w(t) \|^2_{s-4}$ is not increasing. Then, $\| w(t) \|^2_{s-4} \leq \| w(0) \|^2_{s-4}$, $\forall t \geq 0$.

As $(\| w(t) \|_{s-4} - \| w(0) \|_{s-4})(\| w(t) \|_{s-4} + \| w(0) \|_{s-4}) \leq 0$, we have

$$\| w(t) \|_{s-4} \leq \| w(0) \|_{s-4} \leq \| w(0) \|_s , \ \forall t \geq 0 .$$

That is, $\| w(t) \|_{s-4} \leq \| \phi \|_{s-4} \leq \| \phi \|_s$, $\forall t \geq 0$.

Corollary 5.2 (Continuous dependence of the solution of (P_1)). *Let u and v solutions of (P_1) with initial data ψ_1 and ψ_2 in H_{per}^s, respectively.*

Then the following assertions hold

$$\begin{aligned}
&\partial_t \|u(t) - v(t)\|_{s-4}^2 \\
&\quad = -2\beta < \partial_x^2\{u(t) - v(t)\} + \partial_x^4\{u(t) - v(t)\}, u(t) - v(t) >_{s-4} \\
&\quad \leq \ 0
\end{aligned}$$

and

$$\|u(t) - v(t)\|_{s-4} \leq \|\psi_1 - \psi_2\|_{s-4} \leq \|\psi_1 - \psi_2\|_s , \quad t \geq 0 . \tag{5.6}$$

Proof. Define $w := u - v$ then w satisfies:

$$\left| \begin{array}{l} \partial_t w + \partial_x^3 w + \beta\{\partial_x^2 w + \partial_x^4 w\} = 0 \\ w(0) = \psi_1 - \psi_2 . \end{array} \right.$$

We conclude using the Theorem 5.1.

Corollary 5.3 (Uniqueness of solution of (P_1))**.** *The problem (P_1) has a unique solution.*

Proof. In effect, let u and v solutions of (P_1) with the same initial data, that is $\psi_1 = \psi_2 = \psi$.

From (5.6) we obtain $\| u(t) - v(t) \|_{s-4} \leq \| 0 \|_s = 0$. Then, $\| u(t) - v(t) \|_{s-4} = 0$. So, $u(t) = v(t)$, $\forall t \geq 0$, that is $u = v$.

6 I CONCLUSIONS

From our study of the KdV-K-S equation in periodic Sobolev spaces, for the homogeneous (P_1) and non homogeneous (P_1^F) case we have obtained the following results:

1. By Fourier theory, we proved in detail the existence and uniqueness of solution to the model (P_1), as well as the continuous dependency of the solution respect to the initial data. Here we proved the regularity of the solution.

2. We introduced the semigroup theory to rewrite the solution of prob- lem (P_1) by this theory, making it much more fine.

3. In Theorem 4.2, using the Fourier transform, we proved that the non homogeneous problem (P_1^F) is locally well posed in compacts, obtaining continuous dependence with respect to the initial data and the non homogeneity.

4. We showed the dissipative property of the homogeneous problem, where an estimate for the norm of global solution was obtained, which allowed us to deduce the continuous dependence (with respect to the initial data) and uniqueness solution of (P_1).

5. In this paper we have obtained results that can be applied to the KdV-K-S models with two parameters, and these promote the anal- ysis of the corresponding convergence.

REFERENCES

[1] H.A. Biagioni, J.L. Bona, R. Iorio and M. Scialom, On the Korteweg de Vries Kuramoto Sivashinsky equation, *Adv. Diff. Eq.* **1**(1996), No. 1, 1–20.

[2] R. Iorio, V. Iorio, *Fourier Analysis and partial Differential Equa- tions*, Cambridge University, 2002.

[3] Z. Liu, S. Zheng, *Semigroups associated with dissipative system*, Chapman Hall/CRC, 1999.

[4] J. Muñoz Rivera, *Semigrupos e equações Diferenciais Parciais*, LNCC, Petrópolis, 2007.

[5] H. K. Pathak, *An introduction to Nonlinear Analysis and Fixed Point Theory*, Springer-Singapur, 2018.

[6] A. Pazy, *Semigroups of linear operator and applications to partial dif- ferential equations*, Applied Mathematical Sciences, Vol. 44, Springer- Verlag, Berlín, 1983.

[7] Y. Santiago Ayala, Sobre la analiticidad del semigrupo C_o asoci- ado a un sistema viscoelástico, *Pesquimat* **6**(2003), No. 2, 27–36.

[8] Y. Santiago Ayala, Global existence and exponential stability for a coupled wave system, *Journal of Mathematical Sciences: Advances and Applications* **16**(2012), No. 12, 29–46.

[9] Y. Santiago Ayala, *Tópicos de Análisis Funcional. Fundamentos y aplicaciones*, Ed. Acad. Española, Alemania, 2014.

[10] Y. Santiago Ayala, S. Rojas, Regularity and wellposedness of a problem to one parameter and its behavior at the limit, *Bulletin of the Allahabad Mathematical Society* **32**(2017), No. 2, 207–230.

[11] Y. Santiago Ayala, S. Rojas, Existencia y regularidad de solución de la ecuación del calor en espacios de Sobolev periódico, *Selecciones Matemáticas* **6**(2019), No. 1, 49–65.

[12] Y. Santiago Ayala, S. Rojas, Unicidad de solución de la ecuación del calor en espacios de Sobolev periódico, *Selecciones Matemáticas* **7**(2020), No. 1, 172–175.

[13] Y. Santiago Ayala, S. Rojas, Existence and continuous depen- dence of the local solution of non-homogeneous KdV-K-S equation in periodic Sobolev spaces, *Journal of Mathematical Sciences: Advances and Applications* **64**(2021), No. 1, 1–19.

[14] D. Stuart, *Partial Differential Equations Example sheet 2*, 2014. http://www.damtp.cam.ac.uk/user/examples/D15b.pdf

[15] J. Topper, T. Kawahara, Approximate equations for long nonlin- ear waves on a viscous fluid, *J. Phys. Soc. Japan*, **44**(1978), 663–666.

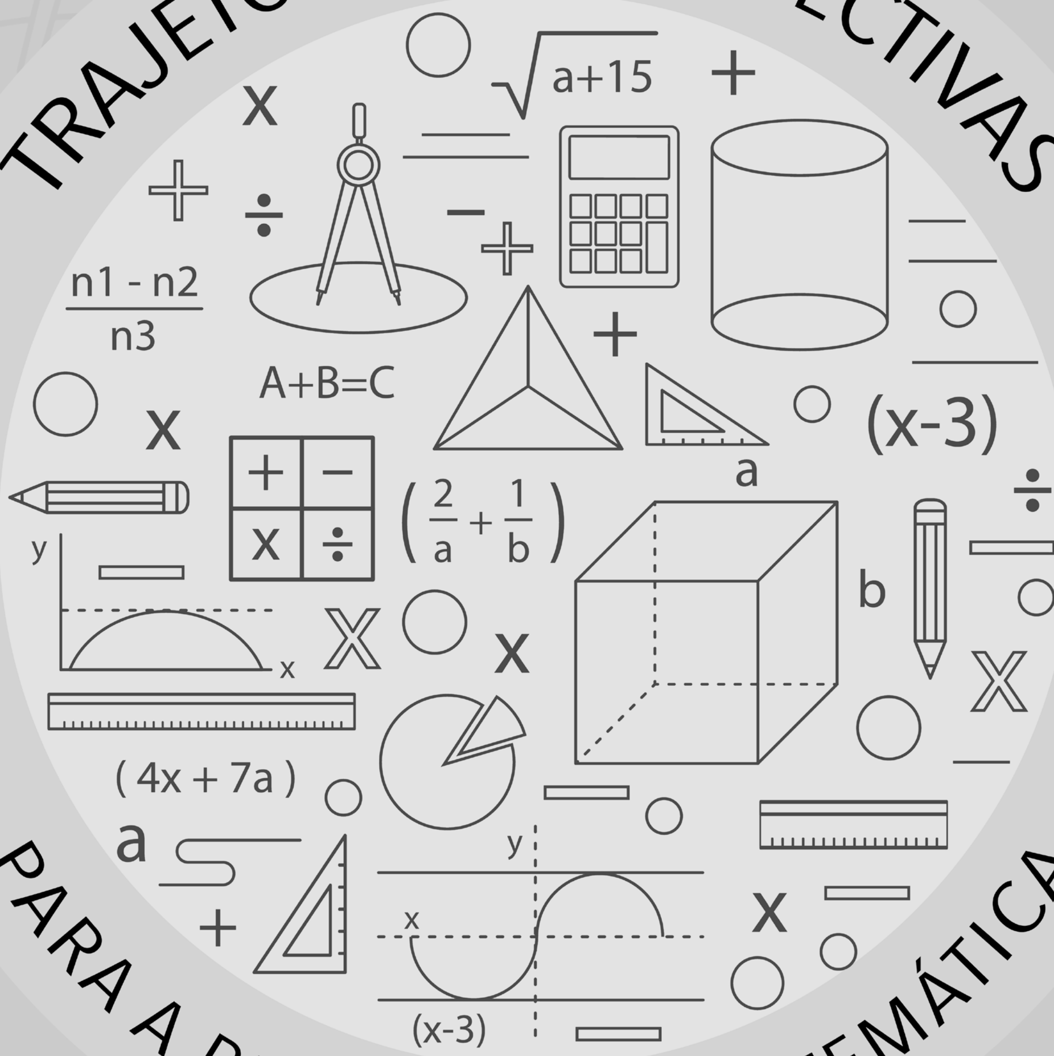
SOBRE O ORGANIZADOR

AMÉRICO JUNIOR NUNES DA SILVA - Professor do Departamento de Educação da Universidade do Estado da Bahia (Uneb - Campus VII) e docente permanente do Programa de Pós-Graduação em Educação, Cultura e Territórios Semiáridos - PPGESA (Uneb - Campus III). Doutor em Educação pela Universidade Federal de São Carlos (UFSCar), Mestre em Educação pela Universidade de Brasília (UnB), Especialista em Psicopedagogia Institucional e Clínica pela Faculdade Regional de Filosofia, Ciências e Letras de Candeias (IESCFAC), Especialista em Educação Matemática e Licenciado em Matemática pelo Centro de Ensino Superior do Vale do São Francisco (CESVASF). Foi professor e diretor escolar na Educação Básica. Coordenou o curso de Licenciatura em Matemática e o Programa Institucional de Bolsas de Iniciação à Docência (Pibid) no Campus IX da Uneb. Foi coordenador adjunto, no estado da Bahia, dos programas Pró-Letramento e PNAIC (Pacto Nacional pela Alfabetização na Idade Certa). Participou, como formador, do PNAIC/UFSCar, ocorrido no Estado de São Paulo. Pesquisa na área de formação de professores que ensinam Matemática, Ludicidade e Narrativas. Integra o Grupo de Estudos e Pesquisas em Educação Matemática (CNPq/UFSCar), na condição de pesquisador, o Grupo Educação, Desenvolvimento e Profissionalização do Educador (CNPq/PPGESA-Uneb), na condição de vice-líder e o Laboratório de Estudos e Pesquisas em Educação Matemática (CNPq/LEPEM-Uneb) na condição de líder. É editor-chefe da Revista Baiana de Educação Matemática (RBEM) e da Revista Multidisciplinar do Núcleo de Pesquisa e Extensão (RevNUPE); e coordenador do Encontro de Ludicidade e Educação Matemática (ELEM).

ÍNDICE REMISSIVO

Atena
Editora
Ano 2022

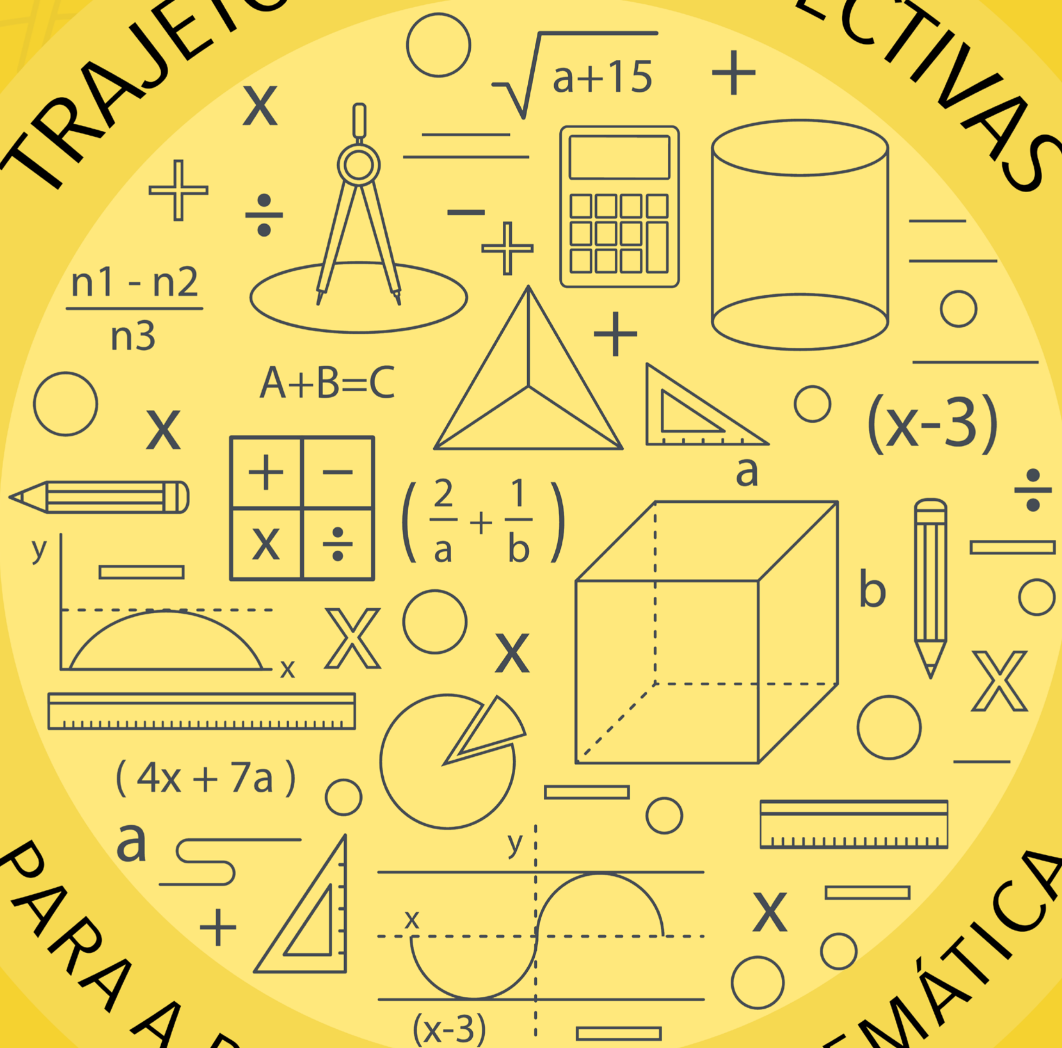
TRAJETÓRIAS E PERSPECTIVAS PARA A PESQUISA EM MATEMÁTICA



www.atenaeditora.com.br
contato@atenaeditora.com.br
@atenaeditora
www.facebook.com/atenaeditora.com.br

Atena Editora
Ano 2022

TRAJETÓRIAS E PERSPECTIVAS PARA A PESQUISA EM MATEMÁTICA



www.atenaeditora.com.br
contato@atenaeditora.com.br
@atenaeditora
www.facebook.com/atenaeditora.com.br