v. 2, n. 22, 2022

Journal of
Engineering Research

SYSTEMS THEORY: METHODOLOGY FOR UNIFORMIZED MODELING OF DIFFERENT PHYSICAL REALITIES

João M. Brito Lima
University of Algarve

Journal of Engineering Research ISSN 2764-1317 DOI 10.22533/at.ed.3172222203106

Abstract: Automatic Control Systems are far and wide used in all modern and industrialized societies. Devices designed to control automatized tasks are each time more present from small plants to large industrial buildings. The development of mathematical models is a compulsory task for whom aim at analyzing or design any control systems. These mathematical models must reproduce some performance measures as accurate as possible. So, no matter the physical nature of the process we aim at control, an accurate mathematical model must be evaluated. So, the development of mathematical models can be considered an hi-level step over the physical nature of the system that we aim at analyze or design. For this reason the study of Systems Theory and Control Systems are considered transversal areas of the knowledge and them studies are compulsory in many branches of sciences and technologies in many universities all over the world. In spite of normal systems are non-linear the linearization procedure simplify the analysis and design of control systems and, depending on the accuracy of the model can give us good results.

Keywords: Physical Systems, mathematical models, differential equations.

INTRODUCTION

Systems Theory is composed of formal methods for the study, design, analytical and homogenized interpretation of physical systems in our daily lives, regardless of their nature (Ribeiro, M. (2002)).

Automatic control systems are increasingly present in all industrialized societies, so it is easy to understand that Systems Theory includes topics and methodologies that are useful in different branches of science and technology.

A more palpable framework of Systems Theory can be presented in the context of designing a control system (Ribeiro, M. (2002)). Thus, regardless of the size or physical nature of the system to be controlled, the design of a controller must take into account the following phases:

- Specification
- Modeling
- Analysis
- Specification verification
- Synthesis

The controller to be synthesized (or designed) must take into account a list of specifications that must be met by the system once it is controlled. A system (or process) to be controlled will be more or less complex. The design of controllers is sometimes an iterative procedure such that, until reaching the final solution, intermediate solutions are tried, which may result in operations that are more or less harmful to the process to be controlled. Therefore, the direct use of the process that is intended to be controlled during the controller synthesis procedure can cause serious damage, this way, the modeling phase is assumed with special importance. The use of an adequate model of the process to be controlled has the advantage of preserving it during the controller synthesis phase.

In addition, in a laboratory environment, devices are generally available that somehow model everyday processes.

A model generally consists of a simplification of reality and must take into account the specifications to be met in the controller synthesis phase.

Once a model has been established for the process to be controlled, the analysis phase is carried out to verify whether the list of specifications is met or not. At that time, depending on the degree of verification of the specifications, it will be decided (or not) to design a controller. This project consists of the association of devices and determination of the respective parameters in order to verify the specifications.

Systems Theory is present, to a greater or lesser extent, in all phases of the design of an automatic control system, regardless of its physical nature. Although the initial approaches were about electrical or mechanical systems, currently System Theory finds applicability in a wide range of areas of knowledge, including the social sciences (Ribeiro, M. (2002)).

In this article, using examples, mathematical representations of systems that could be included in the modeling stage will be studied; in this context, this article consists of the sections described below. Section 2 focuses on the mathematical representation of systems, with emphasis on continuous linear and time-invariant SLIT systems. In section 3 it will be exemplified how different physical realities are modeled by the same mathematical reality. In section 4 a standardizing procedure for representing systems will be presented. The article ends with section 5 where it will be concluded that different physical realities can be described by the same mathematical reality.

MATHEMATICAL REPRESENTATION OF SYSTEMS

The systems that are usually found in everyday life are generally non-linear. Linear systems correspond to approximations of reality that, to a greater or lesser extent, can be considered quite satisfactory, resulting in models whose accuracy must be taken into account, in view of the purpose for which they are intended. It can thus be said that a model is an abstraction of physical reality, extracting from it the characteristics that are considered relevant for the purpose in view, taking into account simplifying hypotheses.

In the scope of Systems Theory, the model is called a system, and this constitutes its basic entity on which it (Systems Theory) focuses (Ribeiro, M. (2002)).

The simplified character of the model in relation to the physical system that originated it explains the fact that, from the same physical system, several models can be extracted depending on the issues related to the physical system that are intended to be resolved. For example, considering the physical (electrical) transistor system, it is known that the model for low frequencies is different from the model for high frequencies, so the model to be adopted must take into account the frequency range where it is intended to work.

The simplifying assumptions in the linearization of a system must take into account the operating point of the non-linear physical system for which the model is intended to be extracted. For example, the dynamics of the gravitational pendulum can be linearized by assuming that for small elongations the sine of an angle can be approximated by its amplitude.

The linearization of a model results in a considerable simplification in terms of design and use of the mathematical tools necessary for its analysis. For this reason, this section is developed assuming linear and time-invariant systems, SLIT.

In this context, consider a continuous linear and arbitrary time-invariant system, described by the linear differential equation and constant coefficients (1).

$$\begin{aligned} & y^{(n)}(t)+a_{n-1}y^{(n-1)}(t)+\ldots+a_2\ddot{y}(t)+a_1\dot{y}(t)+a_0y(t)=\ldots \\ & b_m\,\mathrm{u}^{(m)}(t)+b_{m-1}\,\mathrm{u}^{(m-1)}(t)+\ldots+b_2\ddot{u}(t)+b_1\dot{u}(t)+b_0\,\mathrm{u}(t), \quad n>m \end{aligned} \tag{1}$$

Journal of Engineering Research ISSN 2764-1317 DOI 10.22533/at.ed.3172222203106

It is a system of order equal to the order of the differential equation, (order n), where the input is the sign u(t) and the output is the signal y(t).

The linearity of the system is reflected in the linearity of the corresponding differential equation (1) and the invariance in time is reflected in the fact that the coefficients are constant: a_i and b_j.

The differential equation (1) completely describes the corresponding system, meaning that, from it, the input signal is known. u(t) and *n*, initial output values y(t) and of its *n-1*, first derivatives, it is possible to determine, in a unique way, the temporal evolution of the system output, y(t).

Starting from (1) one can obtain representation in terms of input-output (or external representation) and representation in terms of state (or internal representation).

Looking at (1) from an input-output perspective and taking into account that the Laplace transform can be used to solve linear differential equations, then, one can obtain an external representation of the system that results in the quotient between the Laplace transform from the exit, Y(s), and the Laplace transform of the input, U(s). This external representation is called the transfer function which, for the system represented by (1), results in (2), and at the initial instant the output is considered null y(t) as well as its: *n-1*, first derivatives (Dorf, R. and Bishop, R. (1995).

$$\frac{Y(s)}{U(s)} = \frac{b_m s^m + b_{m-1} s^{m-1} + \ldots + b_1 s + b_0}{s^n + a_{n-1} s^{n-1} + a_{n-2} s^{n-2} + \ldots + a_1 s + a_0} \quad (2)$$

The differential equation (1) allows to obtain an alternative internal representation to the one expressed in (2). This internal representation is called a state model, accommodating, in addition to the input and output, the definition of internal variables. These internal variables are functions of time and state vector coordinates. X(t), may have physical meaning or be abstract mathematical entities: $X(t)=\left[x_1(t) \;\; x_2(t) \;\; \ldots \;\; x_n(t)\right]^T$ (Dorf, R. and Bishop, R. (1995)).

The choice of a representation, internal or external, is made according to the analysis or design techniques that are intended to be used within a possible procedure for controlling the system. Namely, the representation (2) allows calculating the output y(t) known at the input u(t) as long as the system starts from rest, that is, the initial conditions are null. However, many everyday situations exist in which the system does not start from rest but rather presents a non-zero initial state, in these circumstances model (2) proves to be incomplete, and an internal representation, or state model, by to accommodate the existence of non-zero initial conditions, proves to be adequate.

From the external representation (2) one can calculate an infinity of internal representations, (one for each set of state variables $\left[x_1(t) \;\; x_2(t) \;\; \ldots \;\; x_n(t)\right]$ to choose), one of which is represented by the simulation diagram in Figure 1.

Direct reading of Figure 1 allows establishing the equation of state (3) and the output equation **(4)**.

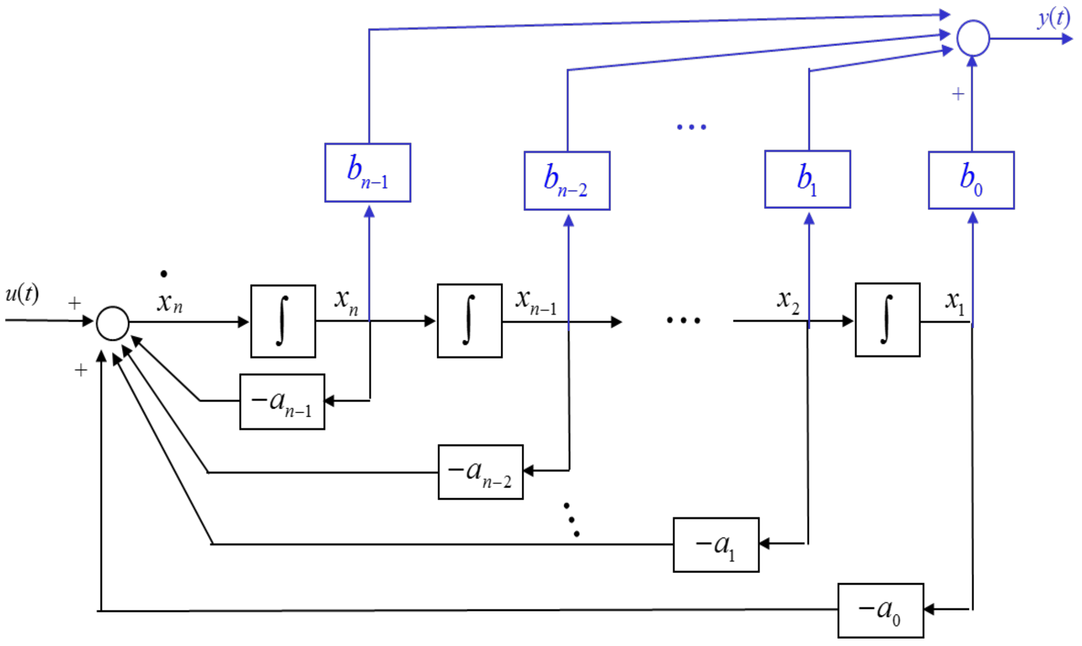


Figure 1 – Simulation diagram regarding the transfer function (2).

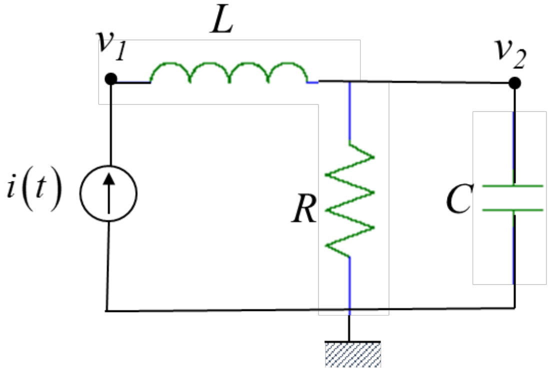


Figure 2 – Electric Circuit.

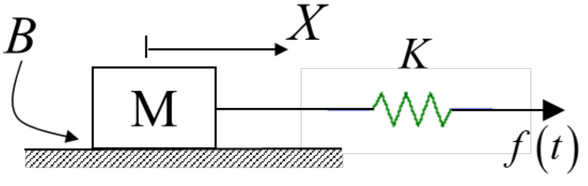


Figure 3 – Mechanical translational system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ -a_0 & -a_1 & \cdots & -a_{n-2} & -a_{n-1} \end{bmatrix} . \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u(t) \tag{3}$$

$$y(t) = \begin{bmatrix} b_0 & b_1 & \cdots & b_{m-1} & b_m & 0 & \cdots & 0 \end{bmatrix} . \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \\ x_{m+1} \\ x_{m+2} \\ \vdots \\ x_n \end{bmatrix} \tag{4}$$

The pair of equations (3) and (4) can be written in compact form **(5)**.

$$\begin{cases} \dot{X}(t) = AX(t) + BU(t) \\ Y(t) = CX(t) + DU(t) \end{cases} \tag{5}$$

Defining:

- A - dynamics matrix [n×n],
- B – entry matrix [n×q], q is the number of inputs,
- C – exit matrix [p×n], p is the number of outputs,
- D - matrix [p×q] matrix (D=0, for this case).

The SLIT representation developed in this section is generic, for continuous systems, and there was no need to specify its physical nature; similar study can be done for discrete systems. In these circumstances the linear differential equation with constant coefficients (1) would give way to an equation with differences, linear and with constant coefficients. The calculation of the Laplace transforms of (1) would be replaced by the calculation of the transform z of the equation with differences, giving rise to a discrete transfer function similar to the one presented in (2) but being a function of z. The simulation diagram (Figure 1) would give way to a simulation diagram in which the integrators would be replaced by discrete-time unit delay elements represented by Z^{-1}. The discrete-state model analogous to the continuum (5) is given by (6), with the matrices A, B, C and D having similar names.

$$\begin{cases} X[k+1] = AX[k] + BU[k] \\ Y[k] = CX[k] + DU[k] \end{cases} \tag{6}$$

MATHEMATICAL MODELS OF PHYSICAL SYSTEMS

The transfer function as an external representation of systems in terms of input-output and the state model where internal variables are defined (state variables) were presented in the previous section for continuous SLITs. It was clear in that section that the methodology exposed did not mention the physical nature of any particular system.

This section presents examples of physical systems for which the corresponding mathematical representations will be calculated (D'Azzo, J. and Houpis, C. (1988)). Although these are examples of physical systems of different natures, it will be clear that they will be representable by the same mathematical model.

ELECTRIC SYSTEM

In this section, the system in Figure 2 which consists of an electrical circuit.

This circuit consists of a coil of inductance L measured in Henry [H] connected to a parallel of a resistance R measured in Ohm [Ω] with a capacitor of capacity C measured in Farad [F]; the circuit is powered by a current source i(t). For this circuit, a transfer function and a state model will be determined, which will be confronted with the models represented by the expressions (2), (3) and (4).

In the external representation (transfer function), the voltage at the current source terminals is considered as input $v_1(t)$ and as output the voltage at the capacitor terminals $v_2(t)$. Since it is an electrical system, the fundamental laws of electrical circuit analysis (Dorf, R. (1993)) will be used to arrive at the transfer function **(7)**.

$$\frac{V_2(s)}{V_1(s)} = \frac{\frac{1}{LC}}{s^2 + \frac{1}{RC}s + \frac{1}{LC}} \qquad (7)$$

The transfer function given by (7) is formally identical to the model expressed by (2) taking into account that the output Y corresponds to the voltage: V_2, and the entry U corresponds to voltage: V_1. Furthermore, the rational function (7) is obtained from the one presented in **(2)** making m=0, n=2 and defining the coefficients as follows:

- $b_0 = \frac{1}{LC}$
- $a_0 = \frac{1}{LC}$
- $a_1 = \frac{1}{RC}$

Alternatively, an internal representation for the electrical circuit of Figure 2 can be established. For the values of the coefficients a_0, a_1 and b_0, taking into account the state model presented in (3) and (4), it is possible to establish the state model for the electrical circuit of Figure 2, represented by the equation of state (8) and the output equation **(9)**. The exit y(t) represents $v_2(t)$ and entry u(t) represents $v_1(t)$.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \frac{-1}{LC} & \frac{-1}{RC} \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \qquad (8)$$

$$y(t) = \begin{bmatrix} \frac{1}{LC} & 0 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad (9)$$

MECHANICAL SYSTEM

In this section, the translational mechanical system of Figure 3 will be studied.

It is a mass M measured in Kilogram [Kg] that moves on a horizontal surface along a straight line (X axis), by the action of a force f(t) measured in Newton [N] applied to a helical spring of elasticity constant K measured in $[^{N}/_{m}]$ connected to the mass. The contact of the mass with the surface generates a friction of constant B measured in $[^{Ns}/_{m}]$.

Analogously to what was done for the previous electrical example, a transfer function and a state model will be determined, which will be confronted with the models represented by the expressions **(2)**, **(3)** and **(4)**.

Given the physical nature of the system under study, the laws of mechanics will be used to establish the differential equations that describe its dynamics (Ribeiro, M.

(2002)). Assuming as input the speed of the end of the spring where the force is applied: *f*, v_F and as output the velocity of the mass, v_M, the transfer function (10) is determined from the differential equations obtained previously.

$$\frac{V_M(s)}{V_F(s)} = \frac{\frac{K}{M}}{s^2 + \frac{B}{M}s + \frac{K}{M}} \quad (10)$$

Similar to what happened with the electrical circuit, the transfer function now obtained corresponds to the one represented by (2) taking into account that the output Y corresponds to the speed V_M and the input U corresponds to the speed V_F. It is a 2nd order system with no zeros, so m=0 and n=2, thus, from (2) one obtains (10) by making the coefficients:

- $b_0 = \frac{K}{M}$
- $a_0 = \frac{K}{M}$
- $a_1 = \frac{B}{M}$

A state model for the system in Figure 3. Thus, particularizing the simulation diagram in Figure 1 for the coefficients a_0, a_1 and b_0 it is then possible to establish the state equations (11) and the output **(12)**.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \frac{-K}{M} & \frac{-B}{M} \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \quad (11)$$

$$y(t) = \begin{bmatrix} \frac{K}{M} & 0 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad (12)$$

The external representations in the form of transfer functions, respectively (7) and (10), as well as the internal representations in the form of state models, respectively (8) and (9), (11) and (12), serve basis for both analysis procedures and controller synthesis procedures. The choice of representation by transfer function or by state model depends on which techniques are intended to be used. Such representations allow studies both in the time domain and in the frequency domain.

UNIFORMIZED REPRESENTATION OF SYSTEMS

In the previous section, 2 physical systems of different natures were studied, electrical Figure 2, and mechanical Figure 3. Transfer functions and the respective state models were presented for such systems (Dorf, R. and Bishop, R. (1995)). In the case of SLIT, the transfer functions are in the form of (2) and the state models are in the form of (5). Given the similarity of the mathematical representations of these 2 different physical systems, the following question can be asked:

- Once a model has been obtained for a given system, are there other systems for which this model is suitable?

If the answer is affirmative, it may be asked whether the other systems may be of a different physical nature from the original system.

The examples presented in the previous section allow an affirmative answer to the question posed, insofar as the model arrived at for the electrical circuit is identical to the model arrived at for the mechanical system. Thus, it can be said that the other system for which the electrical system model is suitable is of a different (mechanical) nature.

In this context, focusing on electrical and mechanical systems, this section presents a methodology for, starting from an arbitrary mechanical system, to find an electrical system (circuit) whose mathematical model is identical to the mathematical model. of the mechanical system. Once the electrical circuit is found, it is said to be a system analogous to the mechanical system.

Let us consider the physical quantities

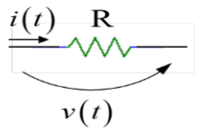
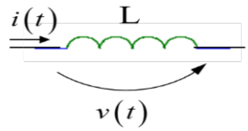
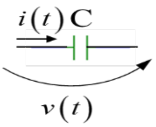
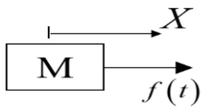
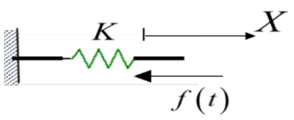
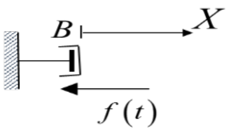
	Resistance	Coil	Condenser
Electric elements			
	$v(t) = Ri(t)$	$v(t) = L\frac{di(t)}{dt}$	$i(t) = C\frac{dv(t)}{dt}$
Mechanical elements	Mass	Spring	Friction
			
	$f(t) = M\frac{d^2x(t)}{dt^2}$	$f(t) = -Kx(t)$	$f(t) = -B\frac{dx(t)}{dt}$

Table 1 Basic elements of electrical circuits and mechanical systems, their representations and laws that govern them.

Mass – Condenser	$M=C$
Spring – Coil	$K = \frac{1}{L}$
Friction – Resistance	$B = \frac{1}{R}$

Table 2: Conditions to check for similar elements to be governed by the same model.

involved in each of the systems in the previous section; for the electrical circuit the disturbance of the system is done at the expense of a current source while for the mechanical system the disturbance is done by the application of a force. Thus, it can be considered that the electric current in a circuit has an analogous function to a force in a mechanical system. In fact, both force and current propagate through elements, mechanical and electrical respectively. The respective measuring devices, dynamometer for measuring force and ammeter for measuring current intensity, are placed in series. Thus, in the search for analogies, it can be stated that the following physical quantities:

<u>current intensity:</u> i $\sim$ <u>Power,</u> f

are analogous.

A similar reading can be taken for the electrical voltage in a circuit and the speed of a point in a mechanical system. Both quantities are measured in relation to a reference; the voltage at a circuit node is measured in relation to a reference node and the speed of a point is measured in relation to a reference that is considered stopped. Thus, in the search for analogies, it can be stated that the following physical quantities:

<u>Electric tension,</u> v $\sim$ <u>Speed,</u> v_e

are analogous.

Now consider the basic elements of Tab's electrical circuits and mechanical systems. 1 where the elementary laws that govern them are presented.

Based on the laws governing the mass M and the capacitor C, respectively, $f(t)=M\frac{dv_e(t)}{dt}$[1] and $i(t)=C\frac{dv(t)}{dt}$; then, assuming the previous analogies (force, electric current and velocity, voltage), it can be said that:

- the mass model M is the same as the capacitor model C as long as the capacity of the capacitor is C=M.

Now consider the laws that govern the spring constant of elasticity: K and the inductance coil: L, respectively $f(t)=Kx(t)$[2] and $v(t)=L\frac{di(t)}{dt}$. Deriving both members of the law that governs the spring and solving for speed, we have $v_e(t)=\frac{1}{K}\frac{df(t)}{dt}$. Comparing this expression with the law that governs the coil and taking into account the previous analogies, it can be said that:

- the spring model K is the same as the coil model L provided that the inductance of the coil is $L=\frac{1}{K}$.

Finally, notice the constant friction: B and in the resistance R, as well as the respective laws that govern them., $f(t)=Bv_e(t)$ and $i(t)=\frac{1}{R}v(t)$; so, taking into account the usual analogies it can be said that:

- the friction element model B is the same as the resistance model R as long as the resistance is $R=\frac{1}{B}$.

In short, in Tab. 2 the conditions that must be verified between each pair of analogous elements are registered so that the model of an electrical circuit is equal to the model of the mechanical system.

At this point, it can be seen that, both for the transfer function (10) and for the state model (11) and (12), obtained for the mechanical system Figure 3, substituting their M, K and B parameters according to Tab. 2, result in the transfer function (7) and the state model (8) and (9) of the electrical circuit Figure 2.

1 $v_e(t)=\frac{dx(t)}{dt}$

2 The absence of the negative sign in relation to what is shown in the table is due to the fact that now we are not considering the spring restitution force, but the external force that must be applied so that the springundergoes an elongation: x.

Journal of Engineering Research ISSN 2764-1317 DOI 10.22533/at.ed.3172222203106

Conversely, the models (transfer function and state model) arrived at for the electrical circuit result in the corresponding models for the mechanical system as long as the electrical parameters are replaced according to Tab. two.

CONCLUSIONS

We then come to the conclusion that we are dealing with distinct physical systems that are representable by the same mathematical model.

This conclusion can be extrapolated to systems of a nature other than electrical or mechanical, for example, systems as diverse as hydraulic or thermal are made up of basic elements that will have correspondence in terms of models with, for example, mechanical or electrical systems.

This conclusion allows us to say that:

- **different physical realities are represented by the same mathematical reality**

This unifying conclusion has implications for the entire methodology that integrates Systems Theory, whether in terms of systems representation, analysis or synthesis.

REFERENCES

Ribeiro, M. (2002). Análise de Sistemas Lineares. IST Press. ISBN: 972-8469-13-6.

D'Azzo, J. and Houpis, C. (1988). Linear Control System Analysis and Design Conventional and Modern. 3rd edition, McGraw-Hill International Editions. ISBN 0-07-016186-0.

Dorf, R. and Bishop, R. (1995). Modern Control Systems. 7th edition, Addison Wesley. ISBN 0-201-84559-8.

Dorf, R. (1993). Introduction to Electric Circuits. 2nd edition, John Wiley & Sons, Inc. ISBN 0-471-60011-3.