COLEÇÃO DESAFIOS DAS ENGENHARIAS:

ENGENHARIA DE COMPUTAÇÃO 3

LILIAN COELHO DE FREITAS
(ORGANIZADORA)

Atena Editora
Ano 2021

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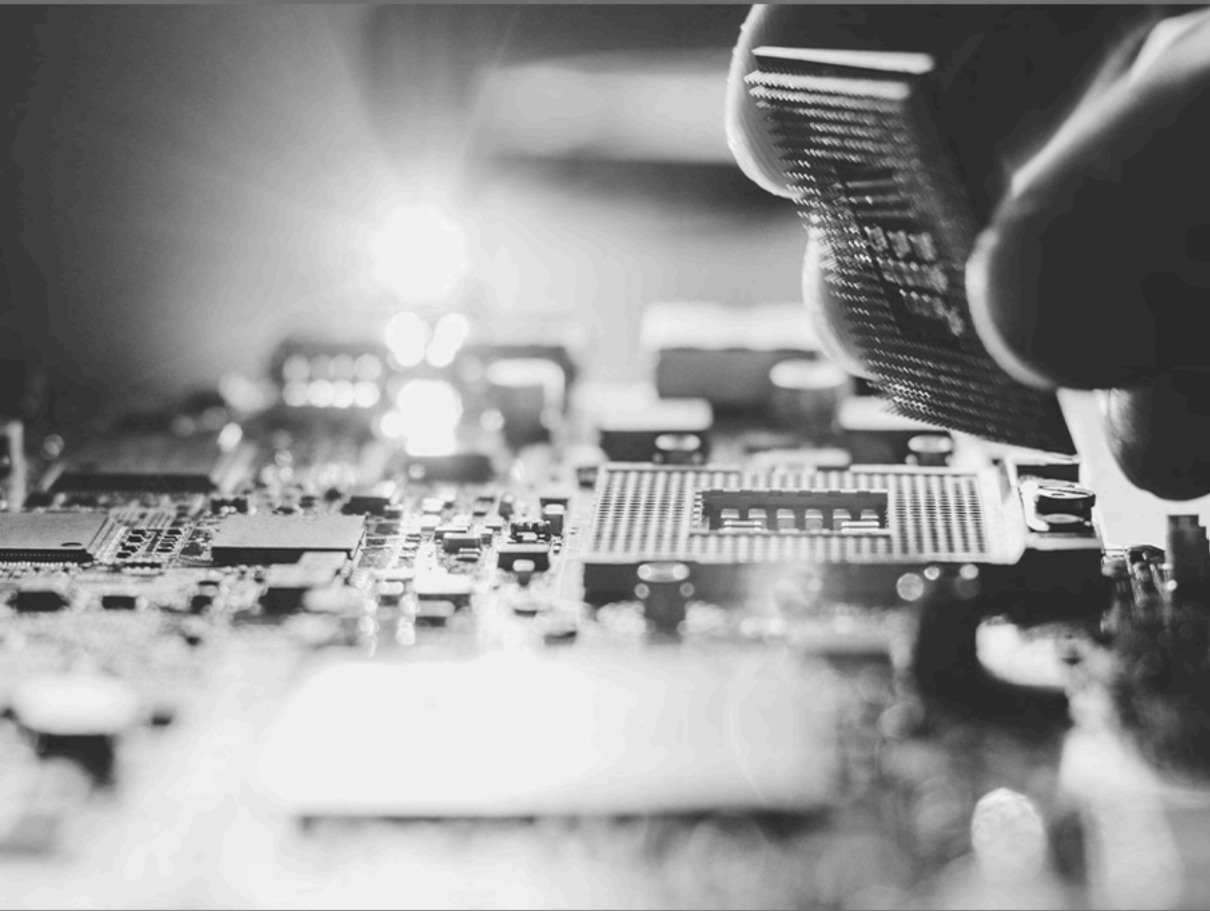

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Ano 2021

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Coleção desafios das engenharias: engenharia de computação 3

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Dados Internacionais de Catalogação na Publicação (CIP)

C691 Coleção desafios das engenharias: engenharia de computação 3 / Organizadora Lilian Coelho de Freitas. – Ponta Grossa - PR: Atena, 2021.

Formato: PDF
Requisitos de sistema: Adobe Acrobat Reader
Modo de acesso: World Wide Web
Inclui bibliografia
ISBN 978-65-5983-619-2
DOI: https://doi.org/10.22533/at.ed.192212911

1. Engenharia de computação. I. Freitas, Lilian Coelho de (Organizadora). II. Título.

CDD 621.39

Elaborado por Bibliotecária Janaina Ramos – CRB-8/9166

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APRESENTAÇÃO

A Atena Editora tem a honra de presentear o público em geral com a série de *e-books* intitulada "*Coleção desafios das engenharias: Engenharia de computação*". Em seu terceiro volume, esta obra tem o objetivo de divulgar aplicações tecnológicas da Engenharia de Computação na resolução de problemas atuais, com o intuito de facilitar a difusão do conhecimento científico produzido em várias instituições de ensino e pesquisa do país.

Organizado em 20 capítulos, este volume apresenta temas como utilização de aprendizagem de máquina na avaliação de riscos de infeção por COVID-19; dispositivos automatizados para administração de remédios; comunicação científica apoiada por realidade aumentada; métodos de elementos finitos aplicados na análise de materiais para indústria aeronáutica; aplicações de processamento digital de imagens e de algoritmos genéticos; entre diversas outras aplicações da automação e do desenvolvimento de *software,* combinados para melhorar as atividades do nosso dia-a-dia.

Dessa forma, esta obra contribuirá para aprimoramento do conhecimento de seus leitores e servirá de base referencial para futuras investigações.

Os organizadores da Atena Editora, agradecem especialmente os autores dos diversos capítulos apresentados, parabenizam a dedicação e esforço de cada um, os quais viabilizaram a construção deste trabalho.

Boa leitura.

Lilian Coelho de Freitas

SUMÁRIO

CAPÍTULO 4

COMPARATIVE ANALYSIS OF THE PERFORMANCE OF A ENRICHED MIXED FINITE ELEMENT METHOD WITH STATIC CONDENSATION FOR POISSON PROBLEMS

Data de aceite: 01/11/2021

Ricardo Javier Hancco Ancori
LabCC, Universidad Nacional de San Agustín de Arequipa Calle Santa Catalina Arequipa/Arequipa, Perú

Jose Diego Ayñayanque Pastor
LabCC, Universidad Nacional de San Agustín de Arequipa Calle Santa Catalina Arequipa/Arequipa, Perú

Rómulo Walter Condori Bustincio
LabCC, Universidad Nacional de San Agustín de Arequipa Calle Santa Catalina Arequipa/Arequipa, Perú

Eliseo Daniel Velasquez Condori
LabCC, Universidad Nacional de San Agustín de Arequipa Calle Santa Catalina Arequipa/Arequipa, Perú

Roger Edwar Mestas Chávez
LabCC, Universidad Nacional de San Agustín de Arequipa Calle Santa Catalina Arequipa/Arequipa, Perú

Fermín Flavio Mamani Condori
LabCC, Universidad Nacional de San Agustín de Arequipa Calle Santa Catalina Arequipa/Arequipa, Perú

Jorge Lizardo Díaz Calle
Dept. of Basic Sciences, FZEA, University of São Paulo, Brasil
Pirassununga, SP

ABSTRACT: The enriched mixed method is a variant of the mixed finite element method, obtained through a selection and appropriate configuration of the shape functions in the space of flux approximation, increasing the order of approximation just inside the elements, taking care of the balance with the space of approximation of the potential. The purpose of this paper is to analyze this method in the context of the Poisson equation. For spaces of approximation of various orders, we carry out numerical simulations considering two model problems: smooth (low oscillation and low gradient) and strongly oscillatory, in quadrilateral meshes. We conclude that the enriched mixed method of order p achieves a precision practically equivalent, with lower computational cost, to the mixed method of order .

KEYWORDS: Mixed finite elements, Poisson's equation, Hdiv spaces, Balanced spaces, Static condensation.

1 I INTRODUCTION

Various engineering problems are modeled by second-order elliptic equations with boundary conditions. The Poisson equation is a prototype that preserves essential characteristics of this class, which is why it is often used to validate new numerical methods [1]. The classical formulation according to is the most used to solve problem , however, in many engineering problems the flux is of greater interest than the primal variable (potential), so processes are usually applied indirect to find the flux by calculating the gradient of the number solution of the potential; with the consequence of obtaining an approximation of the flux with lower quality than the approximation of the potential, see for example the analysis carried out in [2]. The classical mixed finite element method reformulates equation 1 incorporating the flux as an additional variable, which allows to obtain simultaneously the numerical solution of the primal (potential) and dual (flux) variables. In this paper, in the field of the finite element method (FEM) with meshes of quadrilateral elements, we study the enriched mixed method, proposed by Devloo [3], comparing the errors generated with its use, with the errors obtained when applying the mixed method [4, 5] and the classical method of finite elements. In order to reduce the computational cost, the experiments are carried out applying adequate static condensation to each method one of the three methods. For the experimentation we use the NeoPZ computational environment [6, 7], which allows the implementation of algorithms in finite elements. In [1, 3, 8] characteristics of the order of approximation are presented, among others, we also validate some of their conclusions and increase an analysis from the point of view of the order . For the experiments we considered two model problems with known exact solutions and representative behaviors: smooth (low oscillation and low gradient) and strongly oscillatory. The paper is organized as follows. The description of computational tools and static condensation are set in section 2. The three finite elements methods used are described in section and the two model problems in section . The seccion contains the results of the numerical experiments. Seccion 6 concludes this paper.

2 I COMPUTATIONAL ASPECTS

2.1 NeoPZ environment

NeoPZ (originally PZ) (https://github.com/labmec/neopz) is a general-purpose finite element library organized in modules, open source. It uses advanced object-orientation techniques to implement a wide family of FEM technologies, with the purpose of carrying out numerical simulations of processes originating from various fields of engineering, based on in mathematical models represented by differential or integro-differential equations. During its continuous development, for approximately years, more FEM technologies were incorporated: approximation spaces, hp-adaptivity tools, new types of geometric elements, new variational formulations, among others, which allowed the incorporation of features such as multiscale and multiphysic [6, 7, 9, 10] which has increased their ability to manipulate increasingly complex mathematical models, to obtain more complex simulations close to reality. Some of the simulations carried out in NeoPZ are: flow in porous media, hydraulic fractures, oil reservoirs, dynamics of the grounding line in sea ice layers [11–14]. The bases of the higher order approximation spaces for the flow and potential are implemented in a hierarchical way and designed for the management of conforming or non-conforming meshes in dimensions , and [5, 15]. In dimension , it has implemented elements hexahedrons, tetrahedrons, prisms and even pyramids [16].

2.2 Static condensation

The need to carry out numerical simulations of increasingly complex phenomena has an effect on the increase in the complexity of the mathematical models used in engineering, causing a high computational cost, which persists in leaving lagging behind the continuous and rapid technological increase in processing speed and storage capacity of hardware. One way to reduce this effect is to apply clever degrees of freedom reduction maneuvers in the system of equations. Static condensation is a technique used, appropriately for the method, within the scope of the FEM. In this paper, for each method, we respectively use the static condensation techniques described in [1, 3, 17].

3 I METHODS

The Poisson equation to be studied is given by:

$$-\Delta u = f \, en \, \Omega \, y \, u = g \, en \, \Gamma = \partial\Omega \quad (1)$$

The approximation spaces that we consider are piecewise polynomial functions, more specifically, given a partition with quadrilateral elements $\tau_h = \{\mathcal{K}\}$ de Ω, the approximation space for u (potential) they are subspaces of

$$U_h = \left\{p \in L^2(\Omega, \mathbb{R}) : \; p|_K \in P_k(K, \mathbb{R}), \; K \in \tau_h \right\}$$

where P_k (K,R) is a space of polynomials of maximum degree in each coordinate.

3.1 Classical finite element method

Given an approximation space $\mathcal{U}_h \subset \mathcal{H}^1(\Omega)$, the classical variational formulation $\mathcal{H}^1(\Omega)$-as discretized is given by: find $u_h \in \mathcal{U}_h \cap \mathcal{H}^1(\Omega)$ such that $u_h|_\Gamma = g$ and

$$\int_\Omega \nabla u_h \cdot \nabla v_h \, d\Omega = \int_\Omega f v_h \, d\Omega, \quad \forall v_h \in U_h \cap H_0^1(\Omega)$$

The approximation spaces $\mathcal{U}_h \subset \mathcal{H}^1(\Omega)$ that we use are hierarchical, constructed in [15].

3.2 Mixed finite element method

Introducing the additional unknown in Poisson's equation , we obtain the system

$$\sigma = -\nabla u \quad \text{en } \Omega$$
$$\nabla \cdot \sigma = f \quad \text{en } \Omega \text{ y } u = g \quad \text{en } \Gamma = \partial\Omega$$

known as a mixed formulation of equation 1, which allows to obtain simultaneously the numerical solution of the primal (u) and dual (σ) variables. For a decomposition $\tau_h = \{\mathcal{K}\}$ of Ω, consider the approximation subspaces of finite dimension $V_h \subset \mathcal{H}(div, \Omega)$ and $\mathcal{U}_h \subset \mathcal{L}^2$

(Ω), the corresponding discrete mixed variational formulation is given by: find $(\sigma_h , u_h) \in V_h$ X $\mathcal{U}_h$ such that

$$\begin{aligned} \int_\Omega \sigma_h \cdot v_h \, dx + \int_\Omega u_h \nabla \cdot v_h \, dx &= \int_\Gamma g v_h \cdot \eta \, ds, \quad \forall v_h \in V_h \\ \int_\Omega w_h \nabla \cdot \sigma_h \, dx &= \int_\Omega f w_h \, dx, \quad \forall w_h \in U_h \end{aligned}$$

where η is the normal field to Γ, unitary and pointing outward.

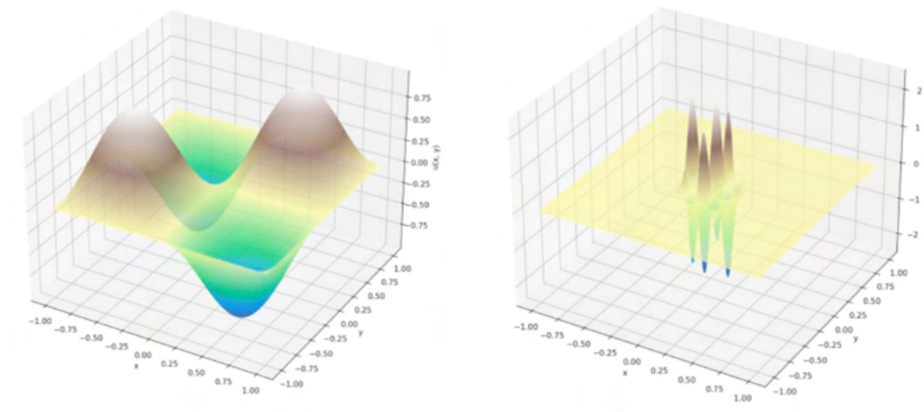
It is known that the selection of pairs of approximation spaces (V_h, U_h) must be carried out in a balanced way, to avoid instability or blocking phenomena [5, 18]. In this paper we use the construction of balanced approximation spaces of higher order and hierarchical proposed in [5], in it the functions of form vector $\sigma_h \in V_h$ and the scalar form functions $U_h \in U_h$ are constructed on each element K, from the corresponding polynomial spaces $\hat{V}$ y $\hat{U}$ defined on a master element $\hat{K}$. Furthermore, the flux approximation space has the structure $\hat{V} = \hat{V}^\partial \oplus \mathring{\hat{V}}$ where $\mathring{\hat{V}}$, is generated by functions of form polynomial vectors whose normal component vanishes at the sides of the element (vector functions of type interior) and $\hat{V}^\partial$ is generated by polynomial vector shape functions associated to the sides of the element, whose normal components do not vanish.

3.3 Enriched mixed finite element method

This method was proposed in [3, 19] and consolidated in [8], in which two new balanced pairs of approximation spaces are proposed for the potential and the flux, one for triangular meshes and the other for quadrilateral meshes. In the case of quadrilateral meshes, these spaces can be interpreted as enriched versions of Raviart-Thomas RTk spaces [4]. Enrichment procedures are applied by space increments using additional bubble terms. These bubble terms are scalar functions supported by a single element (in the case of $\mathcal{H}^1$ -conformal approximations) or vector functions whose normal components vanish at the edges of the elements (in the case of $\mathcal{H}(div)$ - compliant spaces). The advantage of using bubbles as stabilization correctors is based on the fact that all the corresponding degrees of freedom can be condensed, so that the number of equations to be solved and the structure of the matrix are not affected. by the enrichment process [8]. In this paper, we consider enriched approximation spaces for flux and potential structured respectively as follows: $\hat{V}_k^{1+} = \hat{V}_k^\partial \oplus \mathring{\hat{V}}_{k+1}$ y $\hat{U}_k^{1+} = \nabla \cdot \hat{V}_k^{1+}$.

4 I MODEL PROBLEMS

For numerical experiments, we consider two model problems of the Poisson equation 1, with exact solution, each one with a homogeneous Dirichlet boundary condition on the domain $\Omega = [-1,1] \times [-1,1]$. These problems respectively have the following representative characteristics: smooth (low oscillation and low gradient) and strongly oscillatory.



(a) Smooth solution (b) Strongly oscillating solution

Figure 1. Exact solutions to model problems.

For the Poisson equation 1, the problem with smooth solution that we consider has an exact solution: $u(x,y) = \sin(\pi x)\sin(\pi y)$. Figure 1a. The problem with strongly oscillatory solution that we take has the exact solution: $u(x,y) = 0.4\sin(9\pi x)(1+\cos(9\pi y))\left(\frac{\pi}{2} + \arctan(10 - 200(x^2+y^2))\right)$. Figure 1b.

5 I NUMERICAL RESULTS

To validate the ideas discussed in this paper the numerical experiments for the three methods were performed in a Macbookpro with a six-core Intel Core i7 processor (2.2 GHz), 16 GB of DDR4 MHz ram memory. We have used the NeoPZ library to implement a finite element numerical simulation program, In this section, we show results for three experiments.

5.1 The smooth model problem: h-refinement

The soft model problem was solved by applying the three methods considered in section 3; for orders p = 2; 3 and 4. The convergence rates for the variable u in $\mathcal{L}^2$ are shown in tables 1, 2 and 3. It is observed that the theoretical convergence rates are achieved. Table 1. Errors and convergence rates for u in the smooth model problem with h-refinement and $p = 2$

	Classic FEM		Mixed		Enriched Mixed	
h	Error in L^2	Convergence rate	Error in L^2	Convergence rate	Error in L^2	Convergence rate
0.25000	3.86E-03	-	2.14E-03	-	1.25E-04	-

0.12500	4.90E-04	2.9787	2.69E-04	2.9921	7.90E-06	3.9813
0.06250	6.15E-05	2.9950	3.37E-05	2.9980	4.95E-07	3.9951
0.03125	7.69E-06	2.9988	4.21E-06	2.9995	3.10E-08	3.9988

	Classic FEM			Mixed	Enriched Mixted	
h	Error in L²	Convergence rate	Error in L²	Convergence rate	Error in L²	Convergence rate
0.25000	1.76E-04	-	1.06E-04	-	4.52E-06	-
0.12500	1.11E-05	3.9854	6.66E-06	3.9933	1.41E-07	4.9994
0.06250	6.97E-07	3.9963	4,17E-07	3.9983	4.42E-09	4.9999
0.03125	4.36E-08	3.9991	2.61E-08	3,9996	1.38E-10	5.0000

Table 2. Errors and convergence rates for u in the smooth model problem with h-refinement y $p = 3$.

	Classic FEM			Mixed	Enriched Mixed	
h	Error in L²	Convergence rate	Error in L²	Convergence rate	Error in L²	Convergence rate
0.25000	6.70E-06	-	4.19E-06	-	1.48E-07	-
0.12500	2.11E-07	4.9906	1.32E-07	4.9942	2.33E-09	5.9911
0,06250	6,60E-09	4.9976	4.11E-09	4.9985	3.64E-11	5.9977
0,03125	2,06E-10	4.9993	1.29E-10	4.9996	5.70E-13	5.9994

Table 3. Errors and convergence rates for u in the smooth model problem with h-refinement and $p = 4$.

5.2 The strongly oscillatory model problem: p-refinement

Considering a refined mesh with $h = 0.0625$, the strongly oscillatory model problem was solved by applying the three methods described in section 3. For p-refinement, in table 4 the numerical results are shown, respectively, of the errors in $\mathcal{L}^2$ for u and $\mathcal{H}^1$. In addition, for comparison purposes, we select the respective p orders in each method; as highlighted in dark gray. Table 5 shows the respective condensed degrees of freedom. In Figures 2 and 3 we show a comparison of the convergence curves as a function of p-refinement.

	Error for u in $\mathcal{L}^2$			Error in $\mathcal{H}^1$		
p	Enriched Mixed	Mixed	Classic FEM	Enriched Mixed	Mixed	FEM
1	4.25E-02	2.26E-01	2.44E-01	2.196	3.234	7.663
2	3.65E-02	4.15E-02	2.17E-01	1.455	1.965	3.981
3	2.86E-02	3.65E-02	3.68E-02	0.942	1.435	2.026
4	6.73E-03	2.86E-02	3.65E-02	0.609	0.939	1.687
5	5.32E-03	6.73E-03	2.84E-02	0.410	0.608	1.044
6	4.99E-03	5.32E-03	6.65E-03	0.271	0.409	0.717
7	3.43E-03	4.99E-03	5.30E-03	0.185	0.271	0.461
8	-	3.43E-03	4.87E-03	-	0.185	0.322
9	-	-	3.41E-03	-	-	0.210

Table 4. p-refinement for the strongly oscillatory model with $h = 0.0625$.

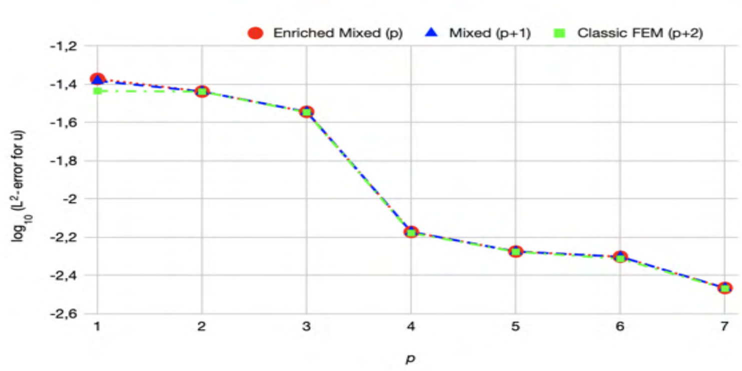


Figure 2. Error of u in $\mathcal{L}^2$ norm for the strongly oscillatory model with $h = 0.0625$.

	Degrees of Freedom Condensed		
p	Enriched Mixed	Mixed	Classic FEM
1	5248	5248	1089
2	7360	7360	3201
3	9472	9472	5313

4	11584	11584	7425
5	13696	13696	9537
6	15808	15808	11649
7	17920	17920	13761
8	-	20032	15873
9	-	-	17985

Table 5. Degrees of freedom through p-refinement for the strongly oscillatory model with $h = 0.0625$.

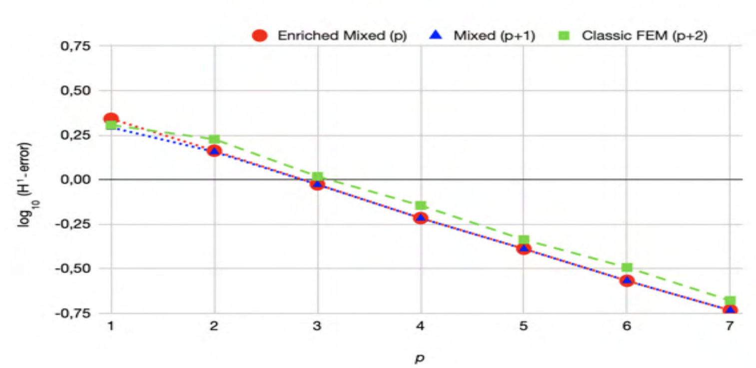


Figure 3. Error in norm $\mathcal{H}^1$ for the strongly oscillatory model with $h = 0.0625$.

6 I CONCLUSIONS

This paper has used an empirical approach to compare the performance of three methods in finite element methods. This contribution is important to establish a benchmark when using a method and obtain low computational costs without affecting the precession in a high way.

For the problem with a strongly oscillatory solution approached, the enriched mixed method of order p has a lower computational cost than the mixed method of order $p + 1$, since, at the master element level, the former has fewer polynomial vectors associated to the sides of the element. However, in the potential variables, their errors in the L^2 norms are practically the same. On the other hand, for the $\mathcal{H}^1$ error, the p order rich mixed method is slightly less accurate, in thousandths, than the $p + 1$ order mixed method; which is reasonable, because, on the master element, the first one has fewer vector polynomials in the flow.

Likewise, in the case of the classical finite element method od order $p + 2$, for the highly oscillatory model problem, we observe that the approximation error for the variable u in the norm $\mathcal{L}^2$ is practically the same as that the enriched mixed method of order , being relegated in the case of the error in norm $\mathcal{H}^1$ because, in this case, the flux calculation is carried out indirectly.

ACKNOWLEDGEMENTS

The authors are grateful for the financial support subsidized, with contract number IBAIB03-2019-UNSA, by the ”Universidad Nacional de San Agust´ın de Arequipa”.

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REFERENCES

1. Forti, T. L., Farias, A. M., Devloo, P. R., & Gomes, S. M., 2016. A comparative numerical study of different finite element formulations for 2d model elliptic problems: continuous and discontinuous galerkin, mixed and hybrid methods. *Finite Elements in Analysis and Design*, vol. 115, pp. 9–20.

2. Mose, R., Siegel, P., Ackerer, P., & Chavent, G., 1994. Application of the mixed hybrid finite element approximation in a groundwater flow model: Luxury or necessity? *Water resources research*, vol. 30, n. 11, pp. 3001–3012.

3. Farias, A. M., Devloo, P. R., Gomes, S. M., de Siqueira, D., & Castro, D. A., 2017. Two dimensional mixed finite element approximations for elliptic problems with enhanced accuracy for the potential and flux divergence. *Computers & Mathematics with Applications*, vol. 74, n. 12, pp. 3283–3295.

4. Raviart, P.-A. & Thomas, J.-M., 1977. A mixed finite element method for 2-nd order elliptic problems. In *Mathematical aspects of finite element methods*, pp. 292–315. Springer.

5. De Siqueira, D., Devloo, P. R., & Gomes, S. M., 2013. A new procedure for the construction of hierarchical high order hdiv and hcurl finite element spaces. *Journal of Computational and Applied Mathematics*, vol. 240, pp. 204–214.

6. Devloo, P. R. & Longhin, G. C., 2002. Object oriented design philosophy for scientific computing. *ESAIM: Mathematical Modelling and Numerical Analysis*, vol. 36, n. 5, pp. 793–807.

7. Calle, J. L. D., Devloo, P. R., & Gomes, S. M., 2015. Implementation of continuous hp-adaptive finite element spaces without limitations on hanging sides and distribution of approximation orders. *Computers & Mathematics with Applications*, vol. 70, n. 5, pp. 1051–1069.

8. Devloo, P. R., Gomes, S. M., Quinelato, T. O., & Tian, S., 2020. Enriched two dimensional mixed finite element models for linear elasticity with weak stress symmetry. *Computers & Mathematics with Applications*, vol. 79, n. 9, pp. 2678–2700.

9. Devloo, P. R. B., 1997. Pz: An object oriented environment for scientific programming. *Computer methods in applied mechanics and engineering*, vol. 150, n. 1-4, pp. 133–153.

10. Farias, A. M., Devloo, P. R., Gomes, S. M., & Duran, O., 2018. An object-oriented framework for multiphysics problems combining different approximation spaces. *Finite Elements in Analysis and Design*, vol. 151, pp. 34–49.

11. Devloo, P. R., Fernandes, P. D., Gomes, S. M., Bravo, C. M. A. A., & Damas, R. G., 2006. A finite element model for three dimensional hydraulic fracturing. *Mathematics and Computers in Simulation*, vol. 73, n. 1-4, pp. 142–155.

12. Triana, O. Y. D., 2017. *Development of a Surrogate Multiscale Reservoir Simulator Coupled with Geomechanics*. PhD thesis, PhD Thesis, FEM-Universidade Estadual de Campinas, 2017.

13. dos Santos, T. D., Devloo, P. R. B., Simoes, J. C., Morlighem, M., & Seroussi, H., 2018. h-adaptivity applied to ice sheet simulation. *Proceeding Series of the Brazilian Society of Computational and Applied Mathematics*, vol. 6, n. 2.

14. Devloo, P., Teng, W., & Zhang, C., 2019. Multiscale hybrid-mixed finite element method for flow simulation in fractured porous media. *Computer Modeling in Engineering & Sciences*, vol. 119, pp. 145–163.

15. Devloo, P. R. B., Bravo, C. M. A. A., & Rylo, E. C., 2009. Systematic and generic construction of shape functions for p-adaptive meshes of multidimensional finite elements. *Computer Methods in Applied Mechanics and Engineering*, vol. 198, n. 21-26, pp. 1716–1725.

16. Bravo, C. M. A., Pavanello, R., Devloo, P. R., & Calle, J. L., 2014. Definition of a p-interpolating space of hierarchical bases of finite elements on the pyramid. *Linear Algebra and its Applications*, vol. 460, pp. 174–204.

17. Devloo, P., Faria, C., Farias, A., Gomes, S., Loula, A., & Malta, S., 2018. On continuous, discontinuous, mixed, and primal hybrid finite element methods for second-order elliptic problems. *International Journal for Numerical Methods in Engineering*, vol. 115, n. 9, pp. 1083–1107.

18. Arnold, D. N., Falk, R. S., & Winther, R., 2006. Differential complexes and stability of finite element methods

19. i. the de rham complex. In *Compatible spatial discretizations*, pp. 23–46. Springer.

20. [19] Castro, D. A., Devloo, P. R., Farias, A. M., Gomes, S. M., de Siqueira, D., & Duran, O., 2016. Three dimensional hierarchical mixed finite element approximations with enhanced primal variable accuracy. *Computer Methods in Applied Mechanics and Engineering*, vol. 306, pp. 479–502.

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