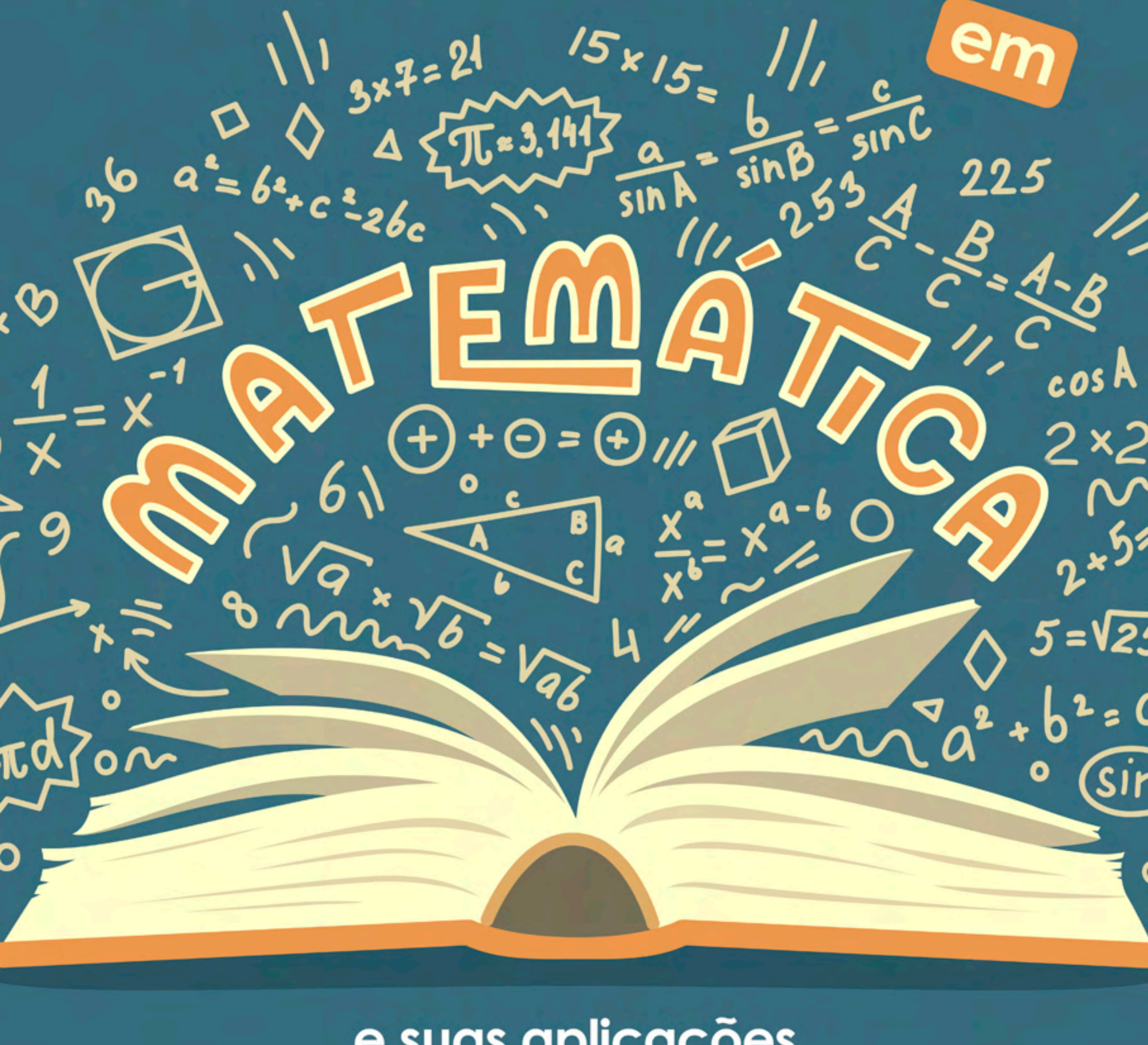
Américo Junior Nunes da Silva
André Ricardo Lucas Vieira
(Organizadores)

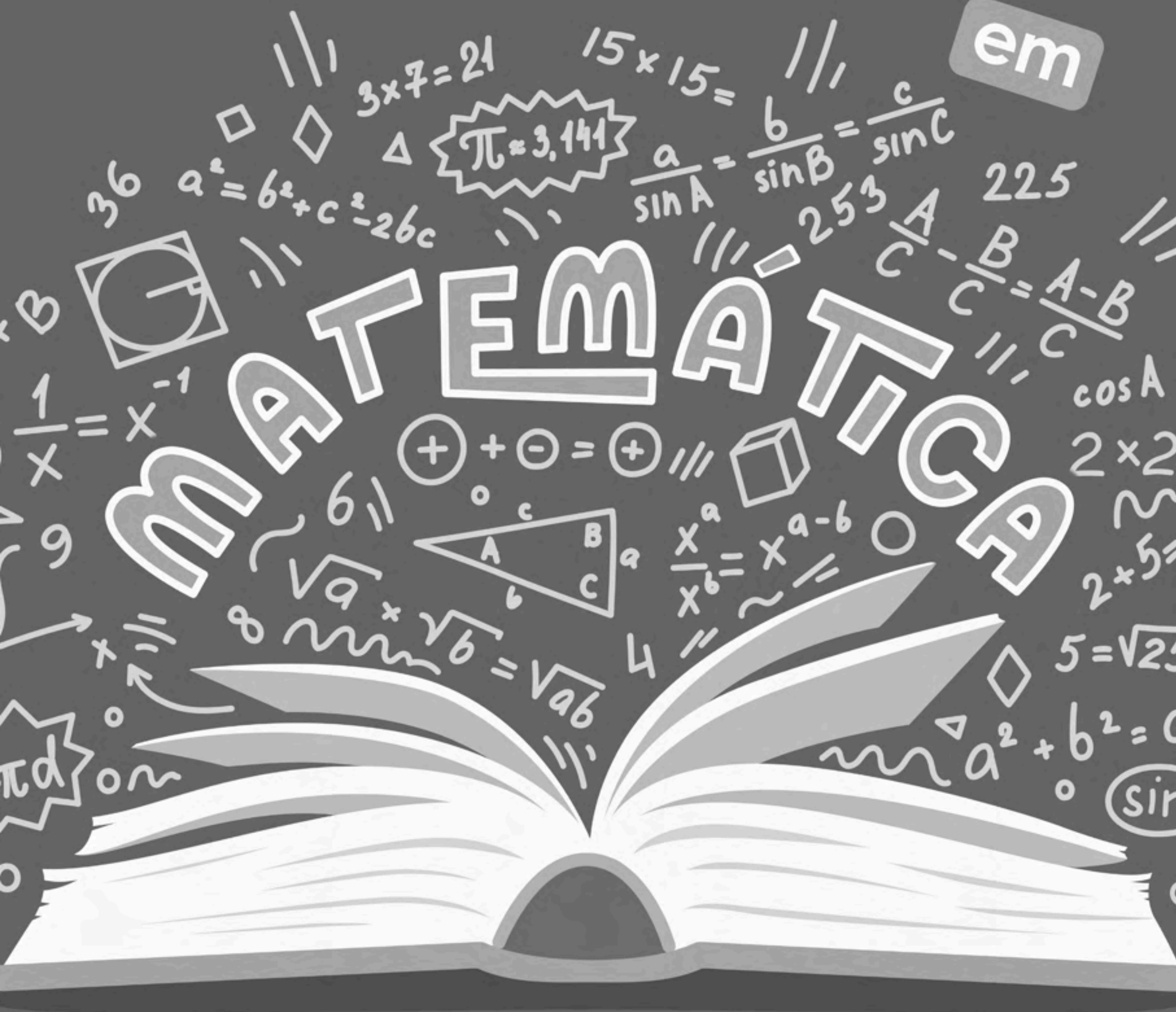
PESQUISAS DE VANGUARDA em MATEMÁTICA e suas aplicações



Atena Editora
Ano 2021

Américo Junior Nunes da Silva
André Ricardo Lucas Vieira
(Organizadores)

PESQUISAS DE VANGUARDA em MATEMÁTICA e suas aplicações



Atena Editora
Ano 2021

Editora chefe
Profª Drª Antonella Carvalho de Oliveira
Editora executiva
Natalia Oliveira
Assistente editorial
Flávia Roberta Barão
Bibliotecária
Janaina Ramos
Projeto gráfico
Camila Alves de Cremo
Luiza Alves Batista
Maria Alice Pinheiro
Natália Sandrini de Azevedo
Imagens da capa
iStock
Edição de arte
Luiza Alves Batista

Atena Editora
Ano 2021

Ciências Exatas e da Terra e Engenharias

Atena Editora
Ano 2021

Ciências Exatas e da Terra e Engenharias

Pesquisas de vanguarda em matemática e suas aplicações

Diagramação: Camila Alves de Cremo
Correção: Bruno Oliveira
Indexação: Gabriel Motomu Teshima
Revisão: Os autores
Organizadores: Américo Junior Nunes da Silva
André Ricardo Lucas Vieira

Dados Internacionais de Catalogação na Publicação (CIP)

P474 Pesquisas de vanguarda em matemática e suas aplicações / Organizadores Américo Junior Nunes da Silva, André Ricardo Lucas Vieira. – Ponta Grossa - PR: Atena, 2021.

Formato: PDF
Requisitos de sistema: Adobe Acrobat Reader
Modo de acesso: World Wide Web
Inclui bibliografia
ISBN 978-65-5983-440-2
DOI: https://doi.org/10.22533/at.ed.402212809

1. Matemática. I. Silva, Américo Junior Nunes da (Organizador). II. Vieira, André Ricardo Lucas (Organizador). III. Título.

CDD 510

Elaborado por Bibliotecária Janaina Ramos – CRB-8/9166

Atena Editora
Ponta Grossa – Paraná – Brasil
Telefone: +55 (42) 3323-5493
www.atenaeditora.com.br
contato@atenaeditora.com.br

Atena Editora

Ano 2021

DECLARAÇÃO DOS AUTORES

Os autores desta obra: 1. Atestam não possuir qualquer interesse comercial que constitua um conflito de interesses em relação ao artigo científico publicado; 2. Declaram que participaram ativamente da construção dos respectivos manuscritos, preferencialmente na: a) Concepção do estudo, e/ou aquisição de dados, e/ou análise e interpretação de dados; b) Elaboração do artigo ou revisão com vistas a tornar o material intelectualmente relevante; c) Aprovação final do manuscrito para submissão.; 3. Certificam que os artigos científicos publicados estão completamente isentos de dados e/ou resultados fraudulentos; 4. Confirmam a citação e a referência correta de todos os dados e de interpretações de dados de outras pesquisas; 5. Reconhecem terem informado todas as fontes de financiamento recebidas para a consecução da pesquisa; 6. Autorizam a edição da obra, que incluem os registros de ficha catalográfica, ISBN, DOI e demais indexadores, projeto visual e criação de capa, diagramação de miolo, assim como lançamento e divulgação da mesma conforme critérios da Atena Editora.

Atena Editora
Ano 2021

DECLARAÇÃO DA EDITORA

Atena Editora
Ano 2021

APRESENTAÇÃO

A Pandemia do novo coronavírus pegou todos de surpresa. De repente, ainda no início de 2020, tivemos que mudar as nossas rotinas de vida e profissional e nos adaptar a um “novo normal”, onde o distanciamento social foi posto enquanto a principal medida para barrar o contágio da doença. As escolas e universidades, por exemplo, na mão do que era posto pelas autoridades de saúde, precisaram repensar as suas atividades.

Da lida diária, no que tange as questões educacionais, e das dificuldades de inclusão de todos nesse “novo normal”, é que contexto pandêmico começa a escancarar um cenário de destrato que já existia antes mesmo da pandemia. Esse período pandêmico só desvelou, por exemplo, o quanto a Educação no Brasil acaba, muitas vezes, sendo uma reprodutora de Desigualdades.

O contexto social, político e cultural, como evidenciaram Silva, Nery e Nogueira (2020), tem demandado questões muito particulares para a escola e, sobretudo, para a formação, trabalho e prática docente. Isso, de certa forma, tem levado os gestores educacionais a olharem para os cursos de licenciatura e para a Educação Básica com outros olhos. A sociedade mudou, nesse cenário de inclusão, tecnologia e de um “novo normal”; com isso, é importante olhar mais atentamente para os espaços formativos, em um movimento dialógico e pendular de (re)pensar as diversas formas de se fazer ciências no país. A pesquisa, nesse interim, tem se constituído como um importante lugar de ampliar o olhar acerca das inúmeras problemáticas, sobretudo no que tange ao conhecimento matemático (SILVA; OLIVEIRA, 2020).

É nessa sociedade complexa e plural que a Matemática subsidia as bases do raciocínio e as ferramentas para se trabalhar em outras áreas; é percebida enquanto parte de um movimento de construção humana e histórica e constitui-se importante e auxiliar na compreensão das diversas situações que nos cerca e das inúmeras problemáticas que se desencadeiam diuturnamente. É importante refletir sobre tudo isso e entender como acontece o ensino desta ciência e o movimento humanístico possibilitado pelo seu trabalho.

Ensinar Matemática vai muito além de aplicar fórmulas e regras. Existe uma dinâmica em sua construção que precisa ser percebida. Importante, nos processos de ensino e aprendizagem da Matemática, priorizar e não perder de vista o prazer da descoberta, algo peculiar e importante no processo de matematizar. Isso, a que nos referimos anteriormente, configura-se como um dos principais desafios do educador matemático, como assevera D’Ambrósio (1993), e sobre isso, de uma forma muito particular, abordaremos nesta obra.

É neste sentido, que o livro “***Pesquisas de Vanguarda em Matemática e suas Aplicações***” nasceu: como forma de permitir que as diferentes experiências do professor pesquisador que ensina Matemática e do pesquisador em Matemática aplicada sejam apresentadas e constituam-se enquanto canal de formação para educadores da Educação

Básica e outros sujeitos. Reunimos aqui trabalhos de pesquisa e relatos de experiências de diferentes práticas que surgiram no interior da universidade e escola, por estudantes e professores pesquisadores de diferentes instituições do país.

Esperamos que esta obra, da forma como a organizamos, desperte nos leitores provocações, inquietações, reflexões e o (re)pensar da própria prática docente, para quem já é docente, e das trajetórias de suas formações iniciais para quem encontra-se matriculado em algum curso de licenciatura. Que, após esta leitura, possamos olhar para a sala de aula e para o ensino de Matemática com outros olhos, contribuindo de forma mais significativa com todo o processo educativo. Desejamos, portanto, uma ótima leitura.

Américo Junior Nunes da Silva
André Ricardo Lucas Vieira

REFERÊNCIAS

D´AMBROSIO, Beatriz S. Formação de Professores de Matemática Para o Século XXI: O Grande Desafio. **Pro-Posições**. v. 4. n. 1 [10]. 1993.

SILVA, A. J. N. DA; NERY, ÉRICA S. S.; NOGUEIRA, C. A. Formação, tecnologia e inclusão: o professor que ensina matemática no “novo normal”. **Plurais Revista Multidisciplinar**, v. 5, n. 2, p. 97-118, 18 ago. 2020.

SILVA, A. J. N. da; OLIVEIRA, C. M. de. A pesquisa na formação do professor de matemática. **Revista Internacional de Formação de Professores**, [S. l.], v. 5, p. e020015, 2020. Disponível em: https://periodicoscientificos.itp.ifsp.edu.br/index.php/rifp/article/view/41. Acesso em: 18 maio. 2021.

SUMÁRIO

CAPÍTULO 18

APPROXIMATION OF A SYSTEM OF A NON-NEWTONIAN FLUID BY A SYSTEM OF CAUCHY-KOWALESKA TYPE

Data de aceite: 01/09/2021

Geraldo Mendes de Araujo
Departamento de Matemática, UFPA
Belém - Pa - Brasil
http://lattes.cnpq.br/7240238904153392

Elizardo Fabricio Lima Lucena
Faculdade de Matemática, UFPA
Bragança - Pa - Brasil
http://lattes.cnpq.br/0726671417445350

Michel Melo Arnaud
Instituto de Ciências sociais aplicadas, UNIFESSPA
Rondon do Pará - Pa - Brasil
http://lattes.cnpq.br/2305615345496836

ABSTRACT: In this paper we investigate a problem for a model of a non-newtonian fluid. The problem is considered in a bounded domain of $\mathbb{R}^d$ with Dirichlet boundary conditions. The operator stress tensor is given by $\tau(e(u)) = [(\nu + \nu_0 M(|e(u)|^2))e(u)]$. We proved existence of weak solutions when $d \leq 4$ by using the method of approximations by a system of Cauchy-Kowaleska type. Uniqueness and periodicity of solutions are also considered. *2010 Mathematics Subject Classification*: 35Q35, 76A05, 76DXX.
KEYWORDS: Cauchy-Kowaleska, quasi-Newtonian, Galerkin.

APROXIMAÇÕES PARA UM SISTEMA DE UM FLUIDO NÃO NEWTONIANO POR UM SISTEMA DO TIPO CAUCHY-KOWALESKA

RESUMO: Neste artigo investigamos um problema para uma modelagem de um fluido não newtoniano. O problema é considerado em um domínio limitado do espaço euclidiano d-dimensional com condições de Dirichlet na fronteira. Provamos existência de soluções fracas quando d não supera 4 utilizando o método de aproximações por um sistema do tipo Cachy-Kowaleska. Unicidade e periodicidade de soluções também são consideradas.
PALAVRAS-CHAVE: Cauchy-kowaleska, quasi-newtonian, galerkin.

1 | INTRODUCTION

Let Ω be a bounded domain in $\mathbb{R}^d$ with smooth boundary $\partial\Omega$, and let $T > 0$. We denote by Q_T the time space cylinder $I\times\Omega$, with lateral boundary $\Sigma = I \times \partial\Omega$, where $I = (0,T)$ is a time interval. The unsteady flows of incompressible fluids in a boundary domain $\Omega \subset \mathbb{R}^d$, $d > 1$ is described by the system of equations

$$
\left|\begin{array}{rcll}
\rho \dfrac{\partial u}{\partial t} - \nabla \cdot \tau(e(u)) + \rho(u \cdot \nabla)\, u & = & -\nabla p + \rho f & \text{in } Q_T, \\
\nabla \cdot u & = & 0 & \text{in } Q_T, \\
u & = & 0 & \text{on } \Sigma_T, \\
u(0) & = & u_0 & \text{in } \Omega,
\end{array}\right. \qquad (1.1)
$$

where $u = (u_1, u_2, ..., u_d)$ is the velocity, p represents the pressure, ρ is a positive constant determining the density of a material, $f = (f_1, f_2, ..., f_d)$ stands for the given external body forces, $\tau : \mathbb{R}^{d^2}_{sym} \to \mathbb{R}^{d^2}_{sym}$ denotes the extra stress tensor, $e = e(u) : \mathbb{R}^d \to \mathbb{R}^{d^2}_{sym}$ denotes the symmetric part of the velocity gradient, that is,

$$
e(u) = \frac{1}{2}\left[\nabla u + (\nabla u)^T\right], \qquad (1.2)
$$

whose components are defined as in [7] by

$$
2e_{ij}(u) = \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \qquad i, j = 1, 2, ..., d \qquad (1.3)
$$

and$\mathbb{R}^{d^2}_{sym}$ represents the set of all symmetric $d \times d$ matrices, that is,

$$
\mathbb{R}^{d^2}_{sym} = \{D \in \mathbb{R}^{d^2}; D_{ij} = D_{ji},\ i, j = 1, 2, ..., d\}.
$$

Note for example, that when $\tau(e(u))$ is of the form

$$
\tau(e(u)) = \mu_0(1 + |e(u)|^{p-2})e(u), \qquad (1.4)
$$

with $p = 2$, the problem (1.1) turns into the Navier-Stokes system, which is a model for Newtonian fluids. In the expression (1.4), $|e(u)|$ denotes the usual Euclidean matrix-norm. We observe that (1.4) can be write in the form

$$
\tau(e(u)) = \mu_0 M\left(|e(u)|^2\right) e(u), \qquad (1.5)
$$

where $M : \mathbb{R}^+_0 \to \mathbb{R}^+_0, M \in C^0(0, \infty)$ is the generalized viscosity function.

Fluids constituted by (1.5), with $p \neq 2$, are sometimes named fluids with shear-dependent viscosity. Models belonging to this class of non-Newtonian fluid mechanics are frequently used in several fields of chemistry, glaciology, biology and geology, as discussed in Málek, Rajagopal, Růžička [8].

The mathematical analysis of the Problem (1.1) when $\tau(e(u)) = \nu e(u)$ was done first time by Leray ([10]). After this, it was investigated in general case by Ladyzhenskaya in 1963, where she proposed, among others, to study the system (1.1) with (1.4) and $p = 4$. Combining monotone operator theory and compactness arguments, she proved the existence of weak solution to model (1.1), if $p \geq 1 + \dfrac{2d}{d+2}$ and their uniqueness if$p \geq \dfrac{d+2}{2}$.

More results are known about the Problem (1.1) obtained in a series of papers, include Málek, Rajagopal and Růžička [8], Málek, Nečas and Růžička [6], Frehse and Málek [2], Málek, Nečas, Rokyta and Růžička [7] and among other mathematicians.

The problem that we study in this work consists investigate following mixed problem: let Ω be a bounded domain in $\mathbb{R}^d$ with smooth boundary $\partial\Omega$, and let $T > 0$. We denote by Q_T the time space cylinder $I \times \Omega$, with lateral boundary $\Sigma = I \times \partial\Omega$, where $I = (0,T)$ is a time interval. We find $u : Q_T \rightarrow \mathbb{R}^d$ and $p : Q_T \rightarrow \mathbb{R}$ solving the following system of equations

$$\left|\begin{array}{rcl} u' - \nabla \cdot [(\nu + \nu_0 M(|e(u)|^2))e(u)] + (u.\nabla)u + \nabla p & = & f \text{ in } Q_T, \\ \nabla \cdot u & = & 0 \text{ in } Q_T, \\ u & = & 0 \text{ on } \Sigma_T, \\ u(0) & = & u_0 \text{ in } \Omega, \end{array}\right. \quad (1.6)$$

where the extra stress tensor is given by $\tau(e(u)) = (\nu + \nu_0 M(|e(u)|^2))e(u)$, $e(u)$ as in (1.2)-(1.3), v_0 and v_1 are positives constants. Let us consider $M : (0,\infty) \rightarrow (0,\infty)$ satisfying the following hypothesis

$$M \in C^1(0,\infty), \quad M > M_0 > 0, \quad M' > 0, \quad (1.7)$$

$$c_1|e(u)|^2 \leq M(|e(u)|^2) \leq c_2|e(u)|^2, \quad (1.8)$$

where M_0, c_1 and c_2 are positive constants. This paper is devoted to analyze the existence of weak solutions to system (1.6) by approximating it by a system of Cauchy-Kowaleska type as in Lion ([5]). The method consists in considering the following system of Cauchy-Kowaleska type

$$\left|\begin{array}{ll} u'_\epsilon - \nabla \cdot \tau\,(e(u_\epsilon)) + (u_\epsilon.\nabla)u_\epsilon + \frac{1}{2}(\nabla \cdot u_\epsilon)u_\epsilon + \nabla p_\epsilon = f & \text{in } Q_T, \\ \epsilon p'_\epsilon + \nabla \cdot u_\epsilon = 0 & \text{in } Q_T, \\ u_\epsilon = 0 & \text{on } \Sigma_T, \\ u_\epsilon(0) = u_{\epsilon 0} & \text{in } \Omega, \\ p_\epsilon(0) = p_{\epsilon 0} & \text{in } L^2(\Omega). \end{array}\right. \quad (1.9)$$

Employing the method of Faedo-Galerking, we proved that (1.9) has a weak solution $\{u_\epsilon, p_\epsilon\}$, for each $\epsilon > 0$, which convergences as $\epsilon \rightarrow 0$ to a weak solution to the problem (1.6). Lions (see [5]) studied the approximation by Cauchy-Kowaleska system for the Navier-Stokes system, when the viscosity is constant. After, Arau´jo, Menezes and Gu´zman ([3]) analyzed the system (1.6) with $\nabla \cdot \tau(e(u)) = (\nu_0 + \nu_1\|u(t)\|^2)\Delta u.$. This paper is devoted to study the case

$$\tau(e(u)) = (\nu + \nu_0 M(|e(u)|^2))e(u), \tag{1.10}$$

that is, a extress tensor model for a non-Newtonian fluid as proposed by Ladyzhenskaya ([4]).

2 I NOTATION AND MAIN RESULTS

In order to formulate problem (1.6) we need some notations about Sobolev spaces. We use standard notation of $L^p(\Omega)$, $W^{m,p}(\Omega)$ and $C^p(\Omega)$ for functions that are defined on Ω and range in $\mathbb{R}$, and the notation $\mathbf{L}^p(\Omega)$, $\mathbf{W}^{m,p}(\Omega)$ and $\mathbf{C}^p(\Omega)$ for functions that range in $\mathbb{R}^d$. We also work with the spaces $L^p(I;H^m(\Omega))$ or $L^p(Q_T)$. To complete this recall on functional spaces, see for instance, *Lions* [5]. By $\langle .,. \rangle$ we will represent the duality pairing between X and X', X' being the topological dual of the space X.

Remark 1 *$\boldsymbol{H}_0^1(\Omega)$ and $\boldsymbol{L}^2(\Omega)$ are Hilbert's spaces. We note that $\boldsymbol{H}_0^1(\Omega) \hookrightarrow \boldsymbol{L}^2(\Omega) \hookrightarrow \boldsymbol{H}^{-1}(\Omega)$ with embeddings dense and compact.*

We introduce the following bilinear and the trilinear forms. As well as the convention of summation of indices, that is, $\alpha_i\beta_j$ instead of X $\sum_{i,j=1}^{d} \alpha_i\beta_j$.

$$a(u,v) = \int_\Omega \frac{\partial u_i}{\partial x_j}(x)\frac{\partial v_i}{\partial x_j}(x)\,dx = ((u,v)) \quad \forall\, u,v \in \mathbf{H}_0^1(\Omega), \tag{2.11}$$

$$b(u,v,w) = \int_\Omega u_i(x)\frac{\partial v_j}{\partial x_i}(x) w_j(x)\,dx \qquad \forall\, u,v,w \in \mathbf{H}_0^1(\Omega), \tag{2.12}$$

$$b_1(u,v,w) = \frac{1}{2}\int_\Omega u_i(x)(\nabla\cdot v(x))w_i(x)\,dx \quad \forall\, u,v,w \in \mathbf{H}_0^1(\Omega). \tag{2.13}$$

$$\tilde{b}(u,v,w) = b(u,v,w) + b_1(u,v,w) \qquad \forall\, u,v,w \in \mathbf{H}_0^1(\Omega). \tag{2.14}$$

We also introduce the notations

$$Au = -\Delta u, \quad Bu = (u\cdot\nabla)u, \quad B_1 u = \frac{1}{2}(\nabla\cdot u)u, \quad \tilde{B}u = Bu + B_1 u \quad \forall u \in \mathbf{H}_0^1(\Omega),$$

and

$$\mathcal{K}u = -\nabla\cdot M(|e(u)|^2)e(u) \quad \forall u \in \mathbf{H}_0^1(\Omega). \tag{2.15}$$

According this, we have

$$\langle Au, v\rangle = a(u,v) \qquad \forall u,v \in \mathbf{H}_0^1(\Omega), \tag{2.16}$$

$$\langle Bu, v\rangle = b(u,u,v) \qquad \forall u,v,w \in \mathbf{H}_0^1(\Omega), \tag{2.17}$$

$$\langle \mathcal{K}u, v\rangle = \int_\Omega M(|e(u)|^2) e_{ij}(u) e_{ij}(v) dx \quad \forall u,v \in \mathbf{H}_0^1(\Omega). \tag{2.18}$$

Remark 2 *We observe that M > 0 imply for all* $u_1, u_2 \in \boldsymbol{H}_0^1(\Omega)$ *that*

$$\langle \mathcal{K}u_1 - \mathcal{K}u_2, u_1 - u_2\rangle$$

$$= \int_\Omega \left[M(|e(u_1)|^2)e_{ij}(u_1) - M(|e(u_2)|^2)e_{ij}(u_2)\right]\left[e_{ij}(u_1) - e_{ij}(u_2)\right] dx \geq 0.$$

Results $\mathcal{K} : \boldsymbol{H}_0^1(\Omega) \to \boldsymbol{H}^{-1}(\Omega)$ *is monotonous.*

Remark 3 *We note that* $\tilde{b}(u,u,u) = 0,\ \forall u \in \boldsymbol{H}_0^1(\Omega)$. *In fact,*

$$b(u,v,v) = \int_\Omega u_j \frac{\partial v_i}{\partial x_j} v_i dx = -\int_\Omega v_i^2 \frac{\partial u_i}{\partial x_j} dx - \int_\Omega v_i \frac{\partial v_i}{\partial x_j} u_j dx \quad \forall u,v \in \boldsymbol{H}_0^1(\Omega)$$

In other words, $b(u,v,v) = -b_1(v,u,v) \quad \forall u,v \in \boldsymbol{H}_0^1(\Omega)$. *It follows that*

$$\tilde{b}(u,u,u) = b(u,u,u) + b_1(u,u,u) = -b_1(u,u,u) + b_1(u,u,u) = 0$$

Definition 1 *Let* $u_0 \in L^2(\Omega)$ *and* $f \in L^{4/3}(I, \boldsymbol{H}^{-1}(\Omega))$. *A weak solution to (1.6) is a function u, such that*

$$u \in L^4(I; \boldsymbol{H}_0^1\Omega) \cap L^\infty(I; \boldsymbol{L}^2(\Omega)),$$

satisfying the following identity

$$\left| \begin{array}{l} (u'(t), v) + \nu a(u(t), v) + \nu_0 \langle \mathcal{K}u(t), v\rangle + \langle Bu(t), v\rangle = \langle f(t), v\rangle, \\ \nabla \cdot u(0) = u_0, \\ \forall v \in \mathcal{D}(\Omega). \end{array} \right. \tag{2.19}$$

Definition 2 *Let* $u_{\epsilon 0} \in \boldsymbol{L}^2(\Omega),\ p_{\epsilon 0} \in L^2(\Omega)$ *and* $f \in L^{4/3}(I, \boldsymbol{H}^{-1}(\Omega))$. *A weak solution to (1.9) is a pair of functions* u_ϵ, p_ϵ, *such that*

$$u_\epsilon \in L^4(I; \boldsymbol{H}_0^1\Omega) \cap L^\infty(I; \boldsymbol{L}^2(\Omega)),$$

$$p_\epsilon \in L^\infty(I; L^2(\Omega)),$$

satisfying the following identity

$$\left|\begin{array}{l}(u_\epsilon'(t),v)+\nu a(u_\epsilon(t),v)+\nu_0\langle \mathcal{K}u_\epsilon(t),v\rangle+\langle \tilde{B}u_\epsilon(t),v\rangle+(\nabla p_\epsilon(t),v)\\ =\langle f(t),v\rangle,\\ \epsilon(p_\epsilon'(t),q)+(\nabla\cdot u_\epsilon(t),q)=0,\\ u_\epsilon(0)=u_{\epsilon 0},\\ p_\epsilon(0)=p_{\epsilon 0},\\ \forall v\in\mathcal{D}(\Omega)\text{ and }\forall q\in L^2(\Omega).\end{array}\right. \tag{2.20}$$

Lemma 1 *(Korn's Inequality) Let* $1 < p < \infty$. *Then, there exists a constant* $K_p = K_p(\Omega)$, *such that the inequality*

$$K_p||v||_{W^{1,p}(\Omega)} \leq ||e(v)||_{L^p(\Omega)} \tag{2.21}$$

is fulfilled for all v satisfying either $v \in W_0^{1,p}(\Omega)$, *where* $\Omega \subset \mathbb{R}^3$ *is open and bounded with* $\partial\Omega \subset C^1$.

Proof. See [9]

Lemma 2 *(Vitali) Let* Ω *be a bounded domain in* $\mathbb{R}^n$ *and* $f^m : \Omega \to \mathbb{R}$ *integrable for every* $m \in \mathbb{N}$. *Assume that*

1. $\lim\limits_{m\to\infty} f^m(x)$ *exists and is finite for almost all* $x \in \Omega$;
2. *for every* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that*

$$\sup_{m\in\mathbb{N}} \int_H |f^m(x)|\,dx < \varepsilon, \quad \forall H \in \Omega,\ |H| < \delta$$

then

$$\lim_{m\to\infty}\int_\Omega f^m(x)dx = \int_\Omega \lim_{m\to\infty} f^m(x)dx.$$

Proof. See [1]

Lemma 3 *Consider* $d \geq 3$ *and* $s,r \in \mathbb{R}$, *with* $s > 2$, $r > d$, *verifying* $\frac{2}{s}+\frac{d}{r}=1$. *If* $u \in \boldsymbol{L}^r(\Omega)$ *then,*

$$|b(u,v,w)| \leq c\|u\|_{L^r(\Omega)}\|v\||w|^{2/s}\|w\|^{d/r}$$

$\forall v,w \in \boldsymbol{H}_0^1(\Omega)$. *Where* $c \geq 0$ *is a constant independent of u,v and w.*

Proof. See [5].

Remark 4 *We note that is possible to obtain, after appropriate modifification of a proof due for Lions [5], that lemma 3 holds to* $b_1(u,v,w)$ $\forall u \in \boldsymbol{L}^r(\Omega)$ *and* $\forall v,w \in \boldsymbol{H}_0^1(\Omega)$.

Theorem 2.1 *If* $d \leq 4$, $u_0 \in \boldsymbol{L}^2(\Omega)$ *and* $f \in L^{4/3}(I;\boldsymbol{H}^{-1}(\Omega))$ *then, there exists a function* $u : Q_T \to \mathbb{R}$, *solution to Problem (1.6) in the sense of definition 1.*

Theorem 2.2 *(Periodic Solutions) Under the assumptions of the Theorem 2.1 there exists a function* $u : Q_T \to \mathbb{R}$ *, solution to problem (1.6) in the sense of definition 1 such that* $u(0) = u(T)$.

Theorem 2.3 *If* $d \leq 4$, $u_{\epsilon 0}, p_{\epsilon 0} \in \boldsymbol{L}^2(\Omega)$ *and* $f \in L^{4/3}(I;\boldsymbol{H}^{-1}(\Omega))$ *then, for each* $\epsilon > 0$*, there exists a weak solution* $\{u_\epsilon, p_\epsilon\}$*, solution to Problem (1.9) in the sense of definition 2. Moreover, if* $d \leq 3$ *then the solution* $\{u_\epsilon, p_\epsilon\}$ *is unique.*

3 | PROOFS OF THE RESULTS

Proof of Theorem 2.3.

We employ the method of Faedo-Galerkin. Let $\{\varphi_\nu, \lambda_\nu\}$ and $\{q_\nu, \overline{\lambda_\nu}\}, \nu \in \mathbb{N}$ be the solution to the espectral problem

$$\left| \begin{array}{ll} (\varphi, v) = \lambda(\varphi, v) & \forall v \in \mathbf{H}_0^1(\Omega), \\ (q, \overline{v}) = \overline{\lambda}(q, \overline{v}) & \forall \overline{v} \in L^2(\Omega). \end{array} \right. \tag{3.22}$$

Consider $V_m = [\varphi_1, ..., \varphi_m] \subset \mathbf{H}_0^1(\Omega)$(Ω) the subspace generated by $\{\varphi_1,...,\varphi_m\}$ and $W_m = [q_1,...,q_m] \subset L^2(\Omega)$ the subspace generated by $\{q_1,...,q_m\}$. Let us also consider the pair $\{u_m, p_m\}$, such that

$$u_{\epsilon m}(x,t) = \sum_{r=1}^{m} g_{r\epsilon m}(t)\varphi_r(x) \quad \text{and} \quad q_{\epsilon m}(x,t) = \sum_{r=1}^{m} h_{r\epsilon m}(t) q_r(x), \tag{3.23}$$

solution of the approximate problem

$$\left| \begin{array}{ll} (u'_{\epsilon m}(t), \varphi_r) + \nu(A u_{\epsilon m}(t), \varphi_r) + \nu_0(\mathcal{K} u_{\epsilon m}(t), \varphi_r) & \\ +\langle \tilde{B} u_{\epsilon m}(t), \varphi_r \rangle + (\nabla p_{\epsilon m}, \varphi_r) = (f(t), \varphi_r) & r = 1, ..., m, \\ \epsilon(p'_{\epsilon m}(t), q_r) + (\nabla \cdot u_{\epsilon m}(t), q_r) = 0 & r = 1, ..., m, \\ u_{\epsilon m}(0) = u_{\epsilon 0 m}, \quad u_{\epsilon 0 m} \to u_{\epsilon 0}, \quad \text{strong in} \quad \mathbf{L}^2(\Omega), & \\ p_{\epsilon m}(0) = p_{\epsilon 0 m}, \quad p_{\epsilon 0 m} \to p_{\epsilon 0}, \quad \text{strong in} \quad L^2(\Omega). & \end{array} \right. \tag{3.24}$$

The system of ordinary differential equation (3.24) has a local solution on a interval $[0, t_m[$, $0 < t_m < T$. The first estimate permits us to extend this solution to the whole interval $[0, T]$.

FIRST ESTIMATE

We sometimes omit parameter *t*. Multiplying both sides of $(3.24)_1$ by $g_{r\epsilon m}$ and $(3.24)_2$

by $h_{r\epsilon m}$, next adding from $r = 1$ to $r = m$, we obtain

$$\frac{1}{2}\frac{d}{dt}|u_{\epsilon m}(t)|^2 + \nu\|u_{\epsilon m}(t)\|^2 + \nu_0\int_\Omega M(|e(u_{\epsilon m}(t))|)^2|e_{ij}(u_{\epsilon m}(t))|^2dx \\ -(p_{\epsilon m}(t), \nabla\cdot u_{\epsilon m}(t)) \leq \|f(t)\|_{H^{-1}(\Omega)}\|u_{\epsilon m}(t)\|, \tag{3.25}$$

$$\epsilon\frac{1}{2}\frac{d}{dt}|p_{\epsilon m}(t)|^2 + (\nabla\cdot u_{\epsilon m}(t), p_{\epsilon m}(t)) = 0, \tag{3.26}$$

because $\tilde{b}(u,u,u) = 0$ (see remark 3). Now using Young's inequality we obtain from (3.25) that

$$\frac{1}{2}\frac{d}{dt}|u_{\epsilon m}(t)|^2 + \nu_2\|u_{\epsilon m}(t)\|^4 - (p_{\epsilon m}(t), \nabla\cdot u_{\epsilon m}(t)) \\ \leq \frac{\nu_2}{2}\|u_{\epsilon m}(t)\|^4 + c_{\nu_2}\|f(t)\|^{4/3}_{H^{-1}(\Omega)}, \tag{3.27}$$

Because, from (2.21) (Korn's inequality) and (1.8) we can get

$$\nu_0\int_\Omega M(|e(u_{\epsilon m}(t))|)^2|e_{ij}(u_{\epsilon m}(t))|^2dx \geq \nu_0 c_1\|e(u_{\epsilon m})\|^4_{L^4(\Omega)} \geq \nu_2\|u_{\epsilon m}\|^4,$$

Adding inequalities (3.26) and (3.27) and integrating from 0 to t, with $0 \leq t \leq T$, we conclude

$$(|u_{\epsilon m}(t)|^2 + \epsilon|p_{\epsilon m}(t)|^2) + \nu_2\int_0^t \|u_m(s)\|^4ds \\ \leq C + C\int_0^t \left(|u_{\epsilon m}(s)|^2 + |p_{\epsilon m}(s)|^2\right)ds. \tag{3.28}$$

Where C is a positive constant independent of m and t. By using Gronwall's inequality, we can write

$$(u_{\epsilon m}) \text{ is bounded in } L^\infty(I;\mathbf{L}^2(\Omega)), \tag{3.29}$$
$$(u_{\epsilon m}) \text{ is bounded in } L^4(I;\mathbf{H}^1_0(\Omega)), \tag{3.30}$$
$$(\sqrt{\epsilon}p_{\epsilon m}) \text{ is bounded in } L^\infty(I;\mathbf{L}^2(\Omega)), \tag{3.31}$$

SECOND ESTIMATE

We consider $P_m : \mathbf{H}^1_0(\Omega) \to V_m$ the orthogonal projections from $\mathbf{H}^1_0(\Omega)$ to V_m, that is

$$P_m u = \sum_{j=1}^m (u,\varphi_j)\varphi_j \qquad \forall u \in \mathbf{H}^1_0(\Omega).$$

We note that $P_m^* u'_{\epsilon m} = u'_{\epsilon m}$. By the choice of the special basis (φ_ν), we obtain

$$\|P_m\|_{\mathcal{L}(H_0^1(\Omega),H_0^1(\Omega))} \leq 1 \quad \text{and} \quad \|P_m^*\|_{\mathcal{L}(H^{-1}(\Omega),H^{-1}(\Omega))} \leq 1. \tag{3.32}$$

We will sometimes omit the paremeter t. Its follows from $(3.24)_1$, (2.16), (2.17) and (2.18)

$$u'_{\epsilon m} = -\nu P_m^* A u_{\epsilon m} - \nu_0 P_m^* \mathcal{K} u_{\epsilon m} - P_m^* \tilde{B} u_{\epsilon m} - P_m^* \nabla p_{\epsilon m} + P_m^* f. \tag{3.33}$$

We have $|\langle A u_{\epsilon m}, v\rangle| \leq \|u_{\epsilon m}\|\|v\|, \ \forall u_{\epsilon m}, v \in \mathbf{H}_0^1(\Omega)$. Thus, from (3.30) we can derive

$$(A u_{\epsilon m}) \text{ is bounded in } L^4(I; \mathbf{H}^{-1}(\Omega)) \hookrightarrow L^{4/3}(I; \mathbf{H}^{-1}(\Omega)). \tag{3.34}$$

Let $u_{\epsilon m}, v \in \mathbf{H}_0^1(\Omega)$. From Schwuarz's inequality and (1.8) we take

$$| < \mathcal{K} u_{\epsilon m}, v > | \leq |\langle M(|e(u_{\epsilon m})|^2) e(u_{\epsilon m}), \nabla v\rangle| \leq c_2 |e(u_{\epsilon m})|^3 \|v\|$$

$$\leq c\|u_{\epsilon m}\|^3 \|v\|.$$

Therefore, (3.30) holds

$$(\mathcal{K} u_{\epsilon m}) \text{ is bounded in } L^{4/3}(I; \mathbf{H}^{-1}(\Omega)). \tag{3.35}$$

Let $u_{\epsilon m}, v \in \mathbf{H}_0^1(\Omega)$. Using (2.14) and Hölder's inequality we conclude

$$| < \tilde{B} u_{\epsilon m}, v > | \leq |b(u_{\epsilon m}, u_{\epsilon m}, v)| + |b_1(u_{\epsilon m}, u_{\epsilon m}, v)|$$

$$\leq 2\|u_{\epsilon m}\|_{L^4(\Omega)} \|u_{\epsilon m}\| \|v\|_{L^4(\Omega)} \leq c\|u_{\epsilon m}\|^2 \|v\|.$$

Thus, from (3.30) we have that

$$(B u_{\epsilon m}) \text{ is bounded in } L^2(I; \mathbf{H}^{-1}(\Omega)) \hookrightarrow L^{4/3}(I; \mathbf{H}^{-1}(\Omega)). \tag{3.36}$$

On the other hand, let $u_{\epsilon m}, v \in \mathbf{H}_0^1(\Omega)$ we can write

$$|\langle \nabla p_{\epsilon m}, v\rangle| = |\langle p_{\epsilon m}, \nabla \cdot v\rangle| \leq |p_{\epsilon m}| \|v\|$$

It follows from (3.31)

$$(\nabla p_{\epsilon m}) \text{ is bounded in } L^{4/3}(I; \mathbf{H}^{-1}(\Omega)). \tag{3.37}$$

It follows from (3.34)-(3.37), (3.32) and hypotesis about f that

$$(u'_{\epsilon m}) \text{ is bounded in } L^{4/3}(I; \mathbf{H}^{-1}(\Omega)). \tag{3.38}$$

Analogously we obtain from $(3.24)_2$

$$|(\epsilon p'_{\epsilon m}, v)| = |(\nabla \cdot u_{\epsilon m}, v)| \leq \|u_{\epsilon m}\| |v|,$$

$\forall v \in L^2(\Omega)$. Thus, (3.30) implies

$$(\epsilon p'_{\epsilon m}) \text{ is bounded in } L^4(I; L^2(\Omega)) \hookrightarrow L^2(I; L^2(\Omega)). \qquad (3.39)$$

The limitations (3.29)-(3.31), (3.38), (3.39) and the Aubin-Lions Lemma implies that there exists subsequences from $(u_{\epsilon m})$ and $(p_{\epsilon m})$, still denoted by $(u_{\epsilon m})$ and $(p_{\epsilon m})$, such that

$$u_{\epsilon m} \longrightarrow u_\epsilon \text{ strong in } L^2(I; \mathbf{L}^2(\Omega)) \text{ and a.e. } Q_T, \qquad (3.40)$$
$$u_{\epsilon m} \longrightarrow u_\epsilon \text{ weak star in } L^\infty(I; \mathbf{L}^2(\Omega)), \qquad (3.41)$$
$$u_{\epsilon m} \longrightarrow u_\epsilon \text{ weak in } L^4(I; \mathbf{H}_0^1(\Omega)), \qquad (3.42)$$
$$u'_{\epsilon m} \longrightarrow u'_\epsilon \text{ weak in } L^{4/3}(I; \mathbf{H}^{-1}(\Omega)), \qquad (3.43)$$
$$\mathcal{K}u_{\epsilon m} \longrightarrow \chi \text{ weak in } L^{4/3}(I; \mathbf{H}^{-1}(\Omega)). \qquad (3.44)$$
$$p_{\epsilon m} \longrightarrow p_\epsilon \text{ weak star in } L^\infty(I; L^2(\Omega)), \qquad (3.45)$$
$$p'_{\epsilon m} \longrightarrow p'_\epsilon \text{ weak in } L^2(I; L^2(\Omega)). \qquad (3.46)$$

Finally, we note that (3.30)and (3.38) implies $u_\epsilon \in C^0(I;\mathbf{L}^2(\Omega))$. Analagously (3.31) and (3.39) implies and $p_\epsilon \in C^0(I;L^2(\Omega))$. Thus, make sense $u_\epsilon(0) = u_{\epsilon 0}$ and $p_\epsilon(0) = p_{\epsilon 0}$.

To prove that

$$\int_0^T \tilde{b}(u_{\epsilon m}, u_{\epsilon m}, \varphi) \longrightarrow \int_0^T \tilde{b}(u_\epsilon, u_\epsilon, \varphi), \quad \forall \varphi \in \mathcal{D}(I; \mathcal{D}(\Omega)), \quad (3.47)$$

we use (3.40) (see [7], pp.210). To prove that

$$\int_{Q_T} M(|e(u_{\epsilon m})|^2) e_{ij}(u_{\epsilon m}) e_{ij}(\varphi) dxdt \rightarrow \int_{Q_T} M(|e(u_\epsilon)|^2) e_{ij}(u_\epsilon) e_{ij}(\varphi) dxdt, \qquad (3.48)$$

we use the fact $\nabla u_{\epsilon m} \rightarrow \nabla u_\epsilon$ a.e. in Q_T, (see [2] pp. 565-566). Therefore,

$$|\nabla u_{\epsilon m}|^2 \longrightarrow |\nabla u_\epsilon|^2 \text{ a. e. in } Q_T,$$

that is,

$$|e(u_{\epsilon m})|^2 \longrightarrow |e(u_\epsilon)|^2 \text{ a. e. in } Q_T. \qquad (3.49)$$

Since $M \in C^1(0,\infty)$ follows from (3.49)

$$M(|e(u_{\epsilon m})|^2) \longrightarrow M(|e(u_\epsilon)|^2) \text{ a.e. in } Q_T, \qquad (3.50)$$

Thus,

$$M(|e(u_{\epsilon m})|^2) e_{ij}(u_{\epsilon m}) e_{ij}(\varphi) \longrightarrow M(|e(u_\epsilon)|^2) e_{ij}(u_\epsilon) e_{ij}(\varphi) \qquad (3.51)$$

a.e. in Q_T and $\forall \varphi \in \mathcal{D}(I; \mathcal{D}(\Omega))$. Using (3.30) and (1.8) we obtain

$$\int_{Q_T} M\left(|e(u_{\epsilon m})|^2\right) e_{ij}(u_{\epsilon m})e_{ij}(\varphi)dxdt \leq C \int_{Q_T} |e(u_{\epsilon m})|^3 |e_{ij}(\varphi)|dxdt$$
$$\leq C \int_{Q_T} |e(u_{\epsilon m})|^3 dxdt C \int_{Q_T} |\nabla u_{\epsilon m}|^3 dxdt \leq C. \tag{3.52}$$

It follows that

$$M\left(|e(u_{\epsilon m})|^2\right) e_{ij}(u_{\epsilon m})e_{ij}(\varphi) \in L^1(Q_T). \tag{3.53}$$

Moreover, if $H \subset Q_T$ is measurable set, we have from (1.8), (3.30) and Hölder's inequality

$$\int_H M(|e(u_{\epsilon m})|^2)e_{ij}(u_{\epsilon m})e_{ij}(\varphi)dxdt \leq c \int_H |e(u_{\epsilon m})|^3 |e(\varphi)|dxdt$$
$$\leq c\left(\int_{Q_T} |e(u_{\epsilon m})|^4 dxdt\right)^{3/4} \left(\int_H |e(\varphi)|^4 dxdt\right)^{1/4}$$
$$\leq c\left(\int_{Q_T} |\nabla u_{\epsilon m}|^4 dxdt\right)^{3/4} |H|^{1/4} \leq c|H|^{1/4}.$$

Therefore,

$$\sup_{m\in\mathbb{N}} \int_H M(|e(u_{\epsilon m})|^2)e_{ij}(u_{\epsilon m})e_{ij}(\varphi)dxdt \leq c|H|^{1/4}.$$

Assuming that $|H|$ is sufficiently small, we obtain

$$\sup_{m\in\mathbb{N}} \int_H M(|e(u_{\epsilon m})|^2)e_{ij}(u_{\epsilon m})e_{ij}(\varphi)dxdt \leq \varepsilon, \tag{3.54}$$

$\forall \varepsilon \in \mathbb{R}$. Now using (3.51), (3.53), (3.54) and Vitali's lemma we can derive (3.48). Therefore, we can write $\chi = \mathcal{K}u_\epsilon$ in $L^{4/3}(I;\mathbf{H}^{-1}(\Omega))$. The convergences (3.40)-(3.48) allow us to pass the limit on system (3.24), with φ_r and q_r fixed to obtain

$$u'_\epsilon + \nu A u_\epsilon + \nu_0 \mathcal{K} u_\epsilon + \tilde{B} u_\epsilon = f \quad \text{in } L^{4/3}(I;\mathbf{H}^{-1}(\Omega)), \tag{3.55}$$

$$\epsilon p'_\epsilon + \nabla \cdot u_\epsilon = 0 \qquad \text{in } L^2(I;L^2(\Omega)). \tag{3.56}$$

To prove uniqueness of solutions to the problem (1.9), let $(u_{\epsilon1},p_{\epsilon1})$ and $(u_{\epsilon2},p_{\epsilon2})$ weak solutions to Problem (1.9). Then,

$$u_{\epsilon1}, u_{\epsilon2} \in L^\infty(I;\mathbf{L}^2(\Omega)) \cap L^4(I,\mathbf{H}_0^1(\Omega)),$$
$$p_{\epsilon1}, p_{\epsilon2} \in L^\infty(I;\mathbf{L}^2(\Omega)). \tag{3.57}$$

Consider $z = u_{\epsilon1} - u_{\epsilon2}$ and $q = p_{\epsilon1} - p_{\epsilon2}$. Then, (z,q) verifies

$$
\begin{cases}
z' + \nu A z + \nu_0(\mathcal{K}u_{\epsilon 1} - \mathcal{K}u_{\epsilon 2}) + (\tilde{B}u_{\epsilon 1} - \tilde{B}u_{\epsilon 2}) + \nabla q = 0, \\
\epsilon q' + \nabla \cdot z = 0, \\
z(0) = q(0) = 0.
\end{cases} \tag{3.58}
$$

where the first equality is consider in $L^{4/3}(I;\mathbf{H}^{-1}(\Omega))$, and the second in $L^2(I;L^2(\Omega))$. After, we take the duality in the equations $(3.58)_1$ and $(3.58)_2$ with z and q, respectively, to obtain

$$
\begin{cases}
\frac{1}{2}\frac{d}{dt}|z|^2 + \nu\|z\|^2 + \nu_0 < \mathcal{K}u_{\epsilon 1} - \mathcal{K}u_{\epsilon 2}, z > + < \tilde{B}u_{\epsilon 1} - \tilde{B}u_{\epsilon 2}, z > \\
+(\nabla q, z) = 0, \\
\epsilon \frac{1}{2}\frac{d}{dt}|q|^2 + (\nabla \cdot z, q) = 0, \\
z(0) = q(0) = 0.
\end{cases} \tag{3.59}
$$

We note that

$$
\langle \tilde{B}u_{\epsilon 1} - \tilde{B}u_{\epsilon 2}, z \rangle \;=\; \tilde{b}(z, u_{\epsilon 1}, z) + \tilde{b}(u_{\epsilon 2}, z, z). \tag{3.60}
$$

From the monotonicity of K we have $< \mathcal{K}u_1 - \mathcal{K}u_2, \widetilde{u} > \geq 0$. Thus, adding member to member the equalities$(3.59)_1$ and $(3.59)_2$, we obtaine

$$
\frac{1}{2}\frac{d}{dt}\left(|z|^2 + \epsilon|q|^2\right) + \nu\|z\|^2 \leq |\tilde{b}(z, u_{\epsilon 1}, z)| + |\tilde{b}(u_{\epsilon 2}, z, z)|. \tag{3.61}
$$

Because $(\nabla q, z) = -(q, \nabla \cdot z)$. Considering $d = 2$, we get $H_0^1(\Omega) \hookrightarrow L^4(\Omega)$. It follows that (see Lions [5])

$$
|u|_{L^4(\Omega)} \leq c|u|^{1/2}\|u\|^{1/2}. \tag{3.62}
$$

Thus, using (2.14), Hölder's inequality, (3.62) and Young's inequality we take

$$
|\tilde{b}(z, u_{\epsilon 1}, z)| + |\tilde{b}(u_{\epsilon 2}, z, z)| \leq |b(z, u_{\epsilon 1}, z)| + |b_1(z, u_{\epsilon 1}, z)|
$$

$$
+|b(u_{\epsilon 2}, z, z)| + |b_1(u_{\epsilon 2}, z, z)| \leq c\|z\|^2_{L^4(\Omega)}\|u_{\epsilon 1}\| + c\|u_{\epsilon 2}\|_{L^4(\Omega)}\|z\|\|z\|_{L^4(\Omega)}
$$

$$
\leq c|z|\|z\|\|u_{\epsilon 1}\| + c\|u_{\epsilon 2}\|\|z\||z|^{1/2}\|z\|^{1/2}
$$

$$
\leq \frac{\nu}{2}\|z\|^2 + c_\nu\|u_{\epsilon 1}\|^2|z|^2 + \frac{\nu}{2}\|z\|^2 + c_\nu\|u_{\epsilon 2}\|^4|z|^2.
$$

It follows from (3.61) that we can write

$$\frac{1}{2}\frac{d}{dt}(|z|^2 + \epsilon|q|^2) \le c(\|u_{\epsilon 1}\|^2 + \|u_{\epsilon 2}\|^4)(|z|^2 + \epsilon|q|^2).$$

Integrating from 0 to *t* we obtain

$$|z(t)|^2 + |q(t)|^2 \le c\int_0^t (\|u_{\epsilon 1}(s)\|^2 + \|u_{\epsilon 2}(s)\|^4)(|z(s)|^2 + \epsilon|q(s)|^2)ds. \quad (3.63)$$

Applying Gronwall's inequality in (3.64), we deduce by using (3.30) that

$$u_{\epsilon 1}(t) = u_{\epsilon 2}(t) \quad \text{and} \quad p_{\epsilon 1}(t) = p_{\epsilon 2}(t) \quad \forall t \in [0,T] \quad \text{and } d = 2.$$

Now supposing *d* = 3 we have $H_0^1 \hookrightarrow L^6(\Omega)$. Using lemma 3 with *s* = 4 and *r* = 6, remark 4 and Young's inequality we have

$$|\tilde{b}(z, u_{\epsilon 1}, z)| + |\tilde{b}(u_{\epsilon 2}, z, z)| \le |b(z, u_{\epsilon 1}, z)| + |b_1(z, u_{\epsilon 1}, z)|$$

$$+|b(u_{\epsilon 2}, z, z)| + |b_1(u_{\epsilon 2}, z, z)| \le c\|z\|_{L^6(\Omega)}\|u_{\epsilon 1}\||z|^{1/2}\|z\|^{1/2}$$

$$+c\|u_{\epsilon 2}\|_{L^6(\Omega)}\|z\||z|^{1/2}\|z\|^{1/2}$$

$$\le \frac{\nu}{2}\|z\|^2 + c_\nu\|u_{\epsilon 1}\|^4|z|^2 + \frac{\nu}{2}\|z\|^2 + c_\nu\|u_{\epsilon 2}\|^4|z|^2.$$

It follows from (3.61) that

$$\frac{1}{2}\frac{d}{dt}(|z|^2 + \epsilon|q|^2) \le c(\|u_{\epsilon 1}\|^4 + \|u_{\epsilon 2}\|^4)(|z|^2 + \epsilon|q|^2).$$

Integrating from 0 to *t* we obtain

$$|z(t)|^2 + |q(t)|^2 \le c\int_0^t (\|u_{\epsilon 1}(s)\|^4 + \|u_{\epsilon 2}(s)\|^4)(|z(s)|^2 + \epsilon|q(s)|^2)ds. \quad (3.64)$$

Applying Gronwall's inequality in (3.64), we deduce again by using (3.30)

$$u_{\epsilon 1}(t) = u_{\epsilon 2}(t) \quad \text{and} \quad p_{\epsilon 1}(t) = p_{\epsilon 2}(t) \quad \forall t \in [0,T] \quad \text{and } d = 3.$$

Theorem 2.3 has been proved.

Proof of Theorem 2.1.

By a similar argument employed in the estimates of Theorem 2.3, we obtain that when $\epsilon \to 0$

$$(u_\epsilon) \text{ is bounded in } L^\infty(I;\mathbf{L}^2(\Omega)), \tag{3.65}$$
$$(u_\epsilon) \text{ is bounded in } L^4(I;\mathbf{H}_0^1(\Omega)), \tag{3.66}$$
$$(\sqrt{\epsilon}p_\epsilon) \text{ is bounded in } L^\infty(I;\mathbf{L}^2(\Omega)). \tag{3.67}$$

Now we will obtain some estimate to u_ϵ by using fractional derivates. We denote by $\hat{\psi}$ the Fourier transform of function ψ. Denoting $\tilde{u}_\epsilon$ and $\tilde{p}_\epsilon$ to extension of u_ϵ and p_ϵ respectively, by zero outside interval $[0, T]$ we obtain from the (3.24)

$$\left|\begin{array}{l} \dfrac{d}{dt}(\tilde{u}_\epsilon,\varphi_r)+\nu a(\tilde{u}_\epsilon,\varphi_r)+\nu_0\langle\mathcal{K}\tilde{u}_\epsilon,\varphi_r\rangle+\langle\tilde{B}\tilde{u}_\epsilon,\varphi_r\rangle-(\tilde{p}_\epsilon,\nabla\cdot\varphi_r) \\ =\langle\tilde{f},\varphi_r\rangle+(u_{\epsilon 0},\varphi_r)\delta_0-(u\epsilon(T),\varphi_r)\delta_T, \\ \epsilon\dfrac{d}{dt}(\tilde{p}_\epsilon,q_r)+(\nabla\cdot\tilde{u}_\epsilon,q_r)=\epsilon(p_{\epsilon 0},q_r)\delta_0-\epsilon(p_\epsilon(T),q_r)\delta_T. \end{array}\right. \tag{3.68}$$

Talking the Fourier transform in (3.68) and next adding equations of (3.68) with $\varphi_r = \tilde{u}_\epsilon$ and $q_r = \tilde{p}_\epsilon$, we derive

$$\begin{aligned} 2\pi i\tau|\hat{u}_\epsilon|^2+2\pi i\tau\epsilon|\hat{p}_\epsilon|^2 &= \langle\hat{f},\hat{u}_\epsilon\rangle-\nu(A\hat{u}_\epsilon,\hat{u}_\epsilon)-\nu_0(\mathcal{K}\hat{u}_\epsilon,\hat{u}_\epsilon) \\ &\quad -(\tilde{B}\hat{u}_\epsilon,\hat{u}_\epsilon)+(u_{\epsilon 0},\hat{u}_\epsilon)-(u_\epsilon(T),\hat{u}_\epsilon)e^{-2\pi i\tau T} \\ &\quad +\epsilon(p_{\epsilon 0},\hat{p}_\epsilon)-\epsilon(p_\epsilon(T),\hat{p}_\epsilon)e^{-2\pi i\tau T}, \end{aligned} \tag{3.69}$$

where we denoted $\hat{\tilde{u}} = \hat{u}$ and $\hat{\delta}_t = e^{-2\pi i\tau t}$. It follows from (3.69) that

$$\begin{aligned} &|\tau||\hat{u}_\epsilon|^2+\epsilon|\tau||\hat{p}_\epsilon|^2 \\ &\leq c\left(\|\hat{f}\|_{H^{-1}}+\|A\hat{u}_\epsilon\|_{H^{-1}}+\|\mathcal{K}\hat{u}_\epsilon\|_{H^{-1}}+\|\tilde{B}\hat{u}_\epsilon\|_{H^{-1}}\right)\|\hat{u}_\epsilon\| \\ &+(u_{\epsilon 0},\hat{u}_\epsilon)-(u_\epsilon(T),\hat{u}_\epsilon)e^{-2\pi i\tau T}+\epsilon(p_{\epsilon 0},\hat{p}_\epsilon)-\epsilon(p_\epsilon(T),\hat{p}_\epsilon)e^{-2\pi i\tau T}. \end{aligned} \tag{3.70}$$

From the hypothesis about f we have

$$\int_0^T \|f(s)\|_{H^{-1}(\Omega)}ds \leq C.$$

Therefore, $\|\hat{f}\|_{H^{-1}(\Omega)} \leq C$. Besides, we have $|\langle Au_\epsilon, v\rangle| \leq \|u_\epsilon\|\|v\|$, $\forall u_\epsilon, v \in \mathbf{H}_0^1(\Omega)$. So, (3.66) holds

$$(Au_\epsilon) \text{ is bounded in } L^4(I;\mathbf{H}^{-1}(\Omega)) \hookrightarrow L^1(I;\mathbf{H}^{-1}(\Omega)). \tag{3.71}$$

Thus $\|A\hat{u}_\epsilon\|_{H^{-1}(\Omega)} \leq C$. We note that Schwarz's inequality, (1.8) implies that $\forall u_\epsilon, v \in \mathbf{H}_0^1(\Omega)$

$$|\langle\mathcal{K}u_\epsilon, v\rangle| \leq c_2|e(u_\epsilon)|^3\|v\| \leq c\|u_\epsilon\|^3\|v\|.$$

It follows from (3.66)

$$(\mathcal{K}u_\epsilon) \text{ is bounded in } L^{4/3}(I;\mathbf{H}^{-1}(\Omega)) \hookrightarrow L^1(I;\mathbf{H}^{-1}(\Omega)). \tag{3.72}$$

Thus, $\|\mathcal{K}\hat{u}_\epsilon\|_{H^{-1}} \leq C$. Analagously we obtain $\|\tilde{B}\hat{u}_\epsilon\|_{H^{-1}} \leq C$. It follows from (3.72)

$$|\tau||\hat{u}_\epsilon|^2 + \epsilon|\tau||\hat{p}_\epsilon|^2 \leq c(\|\hat{u}_\epsilon\| + \epsilon|\hat{p}_\epsilon|). \tag{3.73}$$

Remark 5 *Let* $\gamma \in \mathbb{R}$ *with* $0 < \gamma < \frac{1}{4}$*. Then, there exists a positive constant C, such that*

$$|\tau|^{2\gamma} \leq C\frac{1+|\tau|}{1+|\tau|^{1-2\gamma}} \quad \forall \tau \in \mathbb{R}.$$

Remark 5 and (3.73) implies that

$$\begin{aligned}
\int_{\mathbb{R}} |\tau|^{2\gamma}|\hat{u}_\epsilon|^2 dt &\leq C\int_{\mathbb{R}} \frac{1+|\tau|}{1+|\tau|^{1-2\gamma}}|\hat{u}_\epsilon|^2 dt \\
&\leq C\int_{\mathbb{R}} \frac{|\hat{u}_\epsilon|^2}{1+|\tau|^{1-2\gamma}}dt + C\int_{\mathbb{R}} \frac{|\tau||\hat{u}_\epsilon|^2}{1+|\tau|^{1-2\gamma}}dt \qquad (3.74) \\
&\leq c\int_{\mathbb{R}} \|\hat{u}_\epsilon\|^2 dt + c\int_{\mathbb{R}} \frac{\|\hat{u}_\epsilon\|^2 + \epsilon|\hat{p}_\epsilon|^2}{1+|\tau|^{1-2\gamma}}dt.
\end{aligned}$$

From Plancherel identity we have

$$\int_{\mathbb{R}} \|\hat{u}_\epsilon\|^2 dt = \int_{\mathbb{R}} \|\tilde{u}_\epsilon\|^2 dt \leq c. \tag{3.75}$$

Besides, from Hölder's inequality and (3.66) we can write

$$\int_{\mathbb{R}} \frac{\|\hat{u}_\epsilon\|}{1+|\tau|^{1-2\gamma}}dt \leq \left(\int_{\mathbb{R}} \|\hat{u}_\epsilon\|^2 dt\right)^{1/2} \left[\int_{\mathbb{R}} \left(\frac{1}{1+|\tau|^{1-2\gamma}}\right)^2 dt\right]^{1/2} \leq c. \tag{3.76}$$

By using the same argument and (3.67) we obtain

$$\int_{\mathbb{R}} \frac{|\hat{p}_\epsilon|}{1+|\tau|^{1-2\gamma}}dt \leq c. \tag{3.77}$$

From (3.73), (3.76) and (3.77) we conclude that

$$\int_{\mathbb{R}} |\tau|^{2\gamma}|\hat{u}_\epsilon|^2 dt \leq c, \quad 0 < \gamma < \frac{1}{4}. \tag{3.78}$$

In other words,

$$|\tau|^\gamma \hat{u}_\epsilon \in L^2(\mathbb{R};\mathbf{L}^2(\Omega)), \quad 0 < \gamma < \frac{1}{4}. \tag{3.79}$$

Combining (3.66) and (3.79) we conclude that $\hat{u}_\epsilon$ is bounded in

$$\mathcal{H}\left(\mathbb{R}; \mathbf{H}_0^1(\Omega), \mathbf{L}^2(\Omega)\right) = \{\hat{u}_\epsilon; \hat{u}_\epsilon \in L^2(\mathbb{R}; \mathbf{H}_0^1(\Omega)), |\tau|^\gamma \hat{u}_\epsilon \in L^2(\mathbb{R}; \mathbf{L}^2(\Omega))\}.$$

This implies that $\tilde{u}_\epsilon$ is bounded in $\mathcal{H}\left(\mathbb{R}; \mathbf{H}_0^1(\Omega), \mathbf{L}^2(\Omega)\right)$. Thus, u_ϵ is bounded in $\mathcal{H}\left(I; \mathbf{H}_0^1(\Omega), \mathbf{L}^2(\Omega)\right)$. But the embedding $\mathcal{H}\left(I; \mathbf{H}_0^1(\Omega), \mathbf{L}^2(\Omega)\right) \hookrightarrow L^2(I; \mathbf{L}^2(\Omega)$ is compact (see for instance Lions [5]). It follows from this and from the limitations obtained that, there is a subsequence from (u_ϵ), still denoted by (u_ϵ), such that

$$\begin{aligned}
& u_\epsilon \longrightarrow u \text{ strong in } L^2\left(I; \mathbf{L}^2(\Omega)\right) \text{ and a.e. in } Q_T, \\
& u_\epsilon \longrightarrow u \text{ weak star in } L^\infty\left(I; \mathbf{L}^2(\Omega)\right), \\
& u_\epsilon \longrightarrow u \text{ weak in } L^4\left(I; \mathbf{H}_0^1(\Omega)\right), \\
& \mathcal{K}u_\epsilon \longrightarrow \chi \text{ weak in } L^{4/3}\left(I; \mathbf{H}^{-1}(\Omega)\right).
\end{aligned} \tag{3.80}$$

From (3.67) we obtain $\epsilon p'_\epsilon \to 0$ in $\mathcal{D}'(Q)$. Therefore $\nabla \cdot u_\epsilon = -\epsilon p'_\epsilon \to 0$ in $D'(Q)$. Combining this and (3.80) we derive $\nabla \cdot u = 0$. Thus, the convergences above permits us to write

$$\left|\begin{aligned}
& \langle u', v\rangle + \nu a(u, v) + \nu_0\langle \chi, v\rangle + \langle Bu, v\rangle = \langle f, v\rangle \quad \forall v \in \mathcal{D}(\Omega), \\
& \nabla \cdot u = 0,
\end{aligned}\right. \tag{3.81}$$

because $\nabla \cdot u = 0$ implies $\tilde{b}(u, u, v) = b(u, u, v)$. Finally, we can prove that $\mathcal{K}u = \chi$ by using the same argument used in the proof of Theorem 2.1.

Proof of Theorem 2.2.

We employ the method of Faedo-Galerkin again with the basis of eigenvectors from the Stoke's operator (φ_v). Let $V_m = [\varphi_1, ..., \varphi_m] \subset \mathbf{H}_0^1(\Omega)$ the subspace generated by $\{\varphi_1,...,\varphi_m\}$. So the approximate problem to (2.19) is given by

$$\left|\begin{aligned}
& (u'_m(t), v) + \nu(Au_m(t), v) + \nu_0(\mathcal{K}u_m(t), v) \\
& +\langle Bu_m(t), v\rangle = (f(t), v) \qquad \forall v \in V, \\
& u_m(0) = u_{0m}, \quad u_{0m} \to u_0, \quad \text{strong in} \quad \mathbf{L}^2(\Omega).
\end{aligned}\right. \tag{3.82}$$

We know that system (3.82) has a local solution defined on the interval $[0,T]$ and given by

$$u_m(x,t) = \sum_{r=1}^{m} g_{rm}(t)\varphi_r(x). \tag{3.83}$$

We first show the existence of periodic solutions to system (3.82). For this purpose, we make $v = u_m(t)$ in (3.82) to obtain, by using the same arguments used in the proof of Therem (2.3) to $\langle \mathrm{K}u_m, v\rangle$

$$\frac{1}{2}\frac{d}{dt}|u_m(t)|^2 + \nu\|u_m(t)\|^2 + \nu_2\|u_m\|^4 \leq \|f(t)\|_{H^{-1}}\|u_m(t)\|. \tag{3.84}$$

Considering the embedding $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$ and using Young' inequality we derive

$$\frac{1}{2}\frac{d}{dt}|u_m(t)|^2 + c|u_m(t)|^2 + \nu_2\|u_m\|^4 \leq \nu_2\|u_m(t)\|^4 + c_{\nu_2}\|f(t)\|_{H^{-1}}^{4/3}. \tag{3.85}$$

That is,

$$\frac{d}{dt}|u_m(t)|^2 + |u_m(t)|^2 \leq c\|f(t)\|_{H^{-1}}^{4/3}. \tag{3.86}$$

Multiplying both sides of (3.86) by e^t we get

$$e^t\frac{d}{dt}|u_m(t)|^2 + e^t|u_m(t)|^2 \leq c\|f(t)\|_{H^{-1}}^{4/3}e^t. \tag{3.87}$$

In other words,

$$\frac{d}{dt}\left(e^t|u_m(t)|^2\right) \leq c\|f(t)\|_{H^{-1}}^{4/3}e^t. \tag{3.88}$$

By integrating (3.88) on interval $[0,T]$ we obtain

$$e^T|u_m(T)|^2 \leq |u_m(0)|^2 + c\int_0^T \|f(t)\|_{H^{-1}}^{4/3}e^t dt. \tag{3.89}$$

Therefore

$$|u_m(T)|^2 \leq \theta(T)|u_m(0)|^2 + C, \tag{3.90}$$

where $\theta(T) = e^{-T}$. We have $0 < 1 - \theta(T) < 1$. Now let R be positive constant such that

$$\frac{C}{1-\theta(T)} < R^2. \tag{3.91}$$

Choosing $u_m(0) \in V_m$ such that $|u_m(0)| < R$ we obtain from (3.90) and (3.91)

$$|u_m(T)|^2 < \theta(T)R^2 + R^2\left(1 - \theta(T)\right) = R^2.$$

Let $\sigma : \mathcal{B}_R(0) \cap V_m \rightarrow \mathcal{B}_R(0) \cap V_m,$ be a nonlinear mapping such that $\sigma\left(u_m(0)\right) = u_m(T)$ and $\mathcal{B}_R(0) = \{u \in L^2(\Omega); |u| < R\}.$ We can establish the continuous dependence of the solution with respect initial data. Therefore, from the Brouwer Fixed-Point Theorem there is $u_{0m} \in V_m$ such that $\sigma(u_{0m}) = u_0m.$ In other words, $u_m(0) = u_m(T).$

Therefore, the system (3.82) has a periodic solution u_m. By using the initial data $u_m(0)$ in (3.82) we can obtain as in proof of Theorem 2.3 that there is subsequence of u_m, still denoted by u_m such that

$$u_m \to u \text{ weak star in } L^\infty(I; \mathbf{L}^2(\Omega)), \tag{3.92}$$

$$u_m \to u \text{ weak in } L^4(I; \mathbf{H}_0^1(\Omega)), \tag{3.93}$$

$$u_m' \to u' \text{ weak in } L^{4/3}(I; \mathbf{H}^{-1}(\Omega)). \tag{3.94}$$

We established in Theorem 2.1 that *u* is a solution to problem (1.6) in the sense of the definition (1). We will show now that *u*(0) = *u*(*T*). In fact, from (3.93) and (3.94)

$$\int_0^T \frac{d}{dt}(u_m(s), v)\,\theta(s)ds \to \int_0^T \frac{d}{dt}(u(s), v)\,\theta(s)ds, \tag{3.95}$$

$\forall v \in \mathbf{H}_0^1(\Omega)$ and $\theta \in \mathcal{D}(0, T)$, with $\theta(T) = 0$. In other words,

$$(u_m(0), v) \to (u(0), v) \quad \forall v \in \mathbf{H}_0^1(\Omega). \tag{3.96}$$

By using the same argument with $\theta \in \mathcal{D}(0, T)$ and *θ*(0) = 0 we derive

$$(u_m(T), v) \to (u(T), v) \quad \forall v \in \mathbf{H}_0^1(\Omega). \tag{3.97}$$

It follows from (3.96) and (3.97) that *u*(0) = *u*(*T*). This concludes the proof of the Theorem 2.2.

REFERENCES

1. *Dunford, N and Schwartz, J. (1958) Linear Operators*, Vol. 1-3. Interscience, New York.

2. Frehse J. and J. Málek, *Problems duo to the no-slip boundary in incompressible fluids dynamics*, Springer-Verlag Berlin Heidelberg New York, 2003 .

3. G. M. de Arau´jo, S. B. de Menezes and R. B. Gu´zman, *Approximation of the Navier-Stokes System With Variable Viscosity By a System of Cauchy-Kowaleska Type*, Mathematical Methods in The Applied Sciences, New York, 2008;**31**:1409-1425 .

4. Ladyzhenskaya O. A., *The Mathematical Theory of Viscous Incompressible Flow* (2nd edn), Gordon and Breach: London, New York, 1989.

5. J. L. Lions, *Quelques Méthodes de Resolution Des Problémes Aux Limites Non Linéaires*, Dunod, Paris, 1969.

6. J. Málek, J. Nečas, and M. Růžička, *On weak solutions to a class of nonNewtonian incompressible fluids in bounded three-dimensional domains: the case p* ≥ 2, Advances in Differential Equations, Vol. 6, N. 3, March 2001, pp. 257-302

7. J. Málek, J. Nečas, M. Rokyta and M. Růžička, *Weak and Measurevalued Solutions to Evolutionary PDEs*, Chapman & Hall, First Edition 1996.

8. J. Málek, K. R. Rajagopal, and M. Růžička, *Existence and regularity of solution and stability of the rest state for fluids with shear dependent viscosity*, Math. Models Methods Appl. Sci. Vol. 5 (6) (1995), 789-812.

9. J. Nečas, *Sur le normes équivalentes dans* $W_p^k(\Omega)$ *et sur la coercivité des formes formellement positives,* in Séminaire Equations aux Dérivées Partielles, Montreal (1966), 102-128

10. J. Leray, *Essai sur Les Mouvements Plans D'un Liquid Visqueux que Limitent des Parois*, Journal de Mathématiques Pures et Appliquées, 1934; **XIII**:331-418.

ÍNDICE REMISSIVO

F

G

H

I

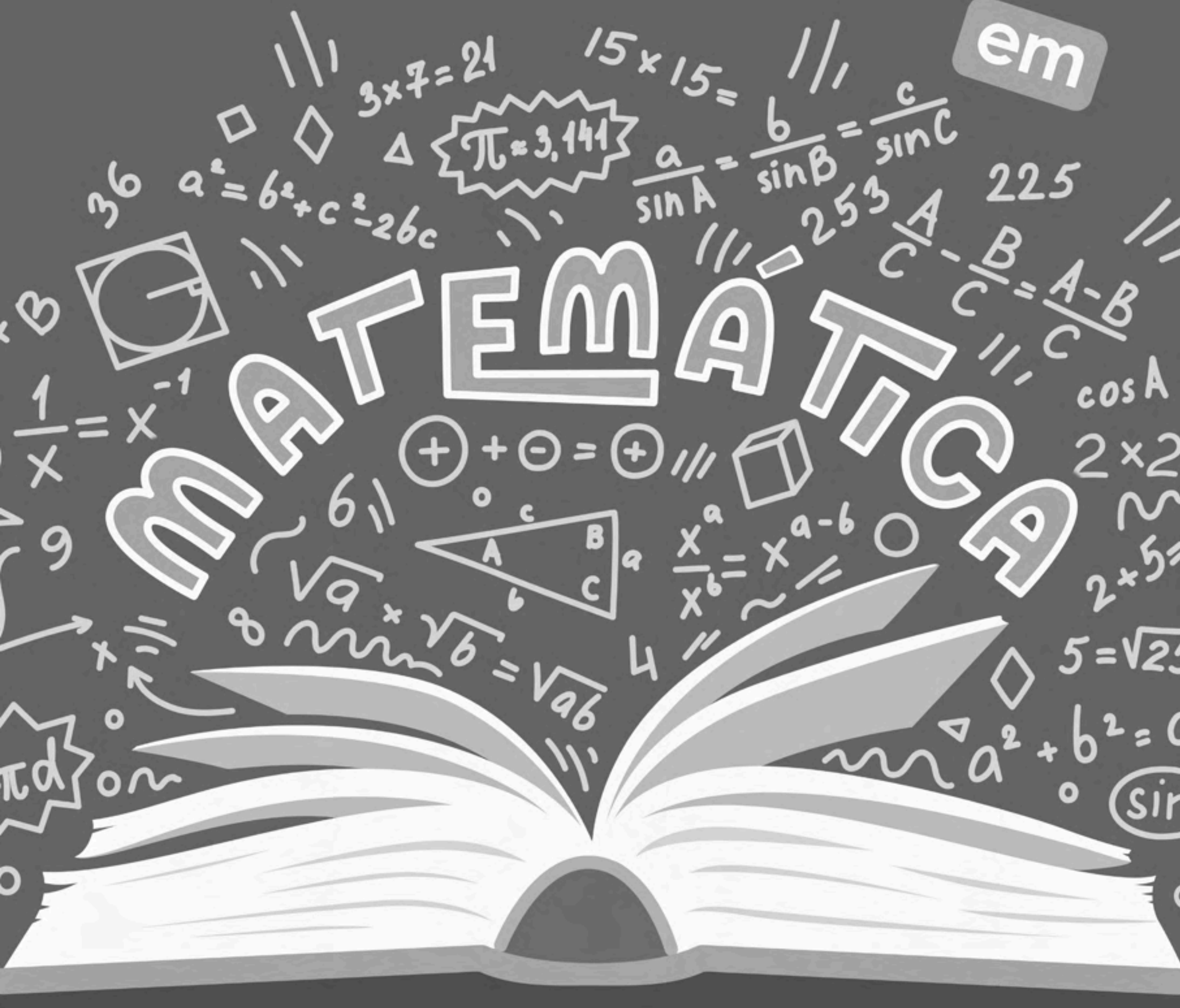
S

T

V

www.atenaeditora.com.br
contato@atenaeditora.com.br
@atenaeditora
www.facebook.com/atenaeditora.com.br

PESQUISAS DE VANGUARDA em MATEMÁTICA



e suas aplicações

Atena Editora
Ano 2021

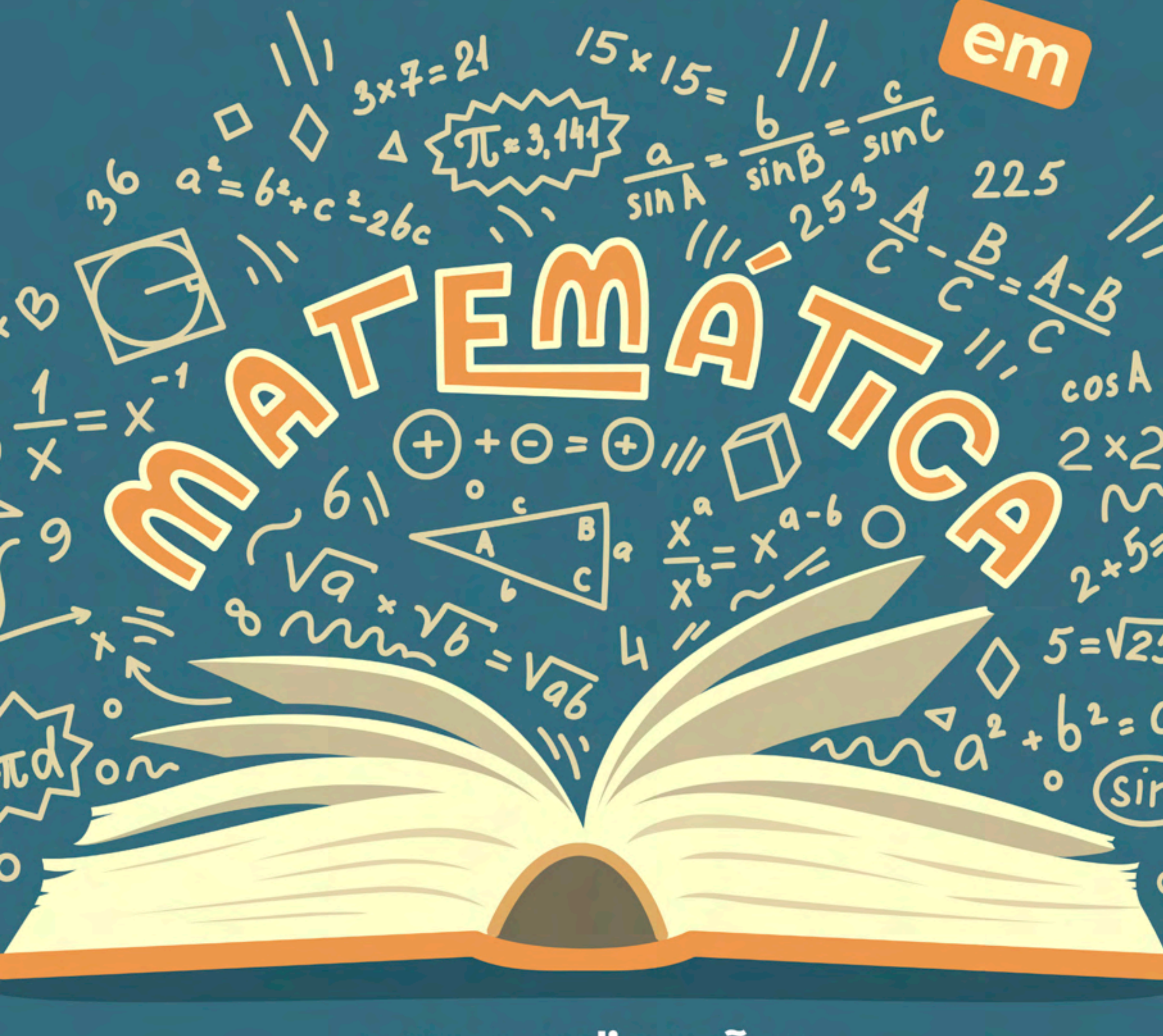
www.atenaeditora.com.br

contato@atenaeditora.com.br

@atenaeditora

www.facebook.com/atenaeditora.com.br

PESQUISAS DE VANGUARDA em MATEMÁTICA e suas aplicações



Atena Editora

Ano 2021