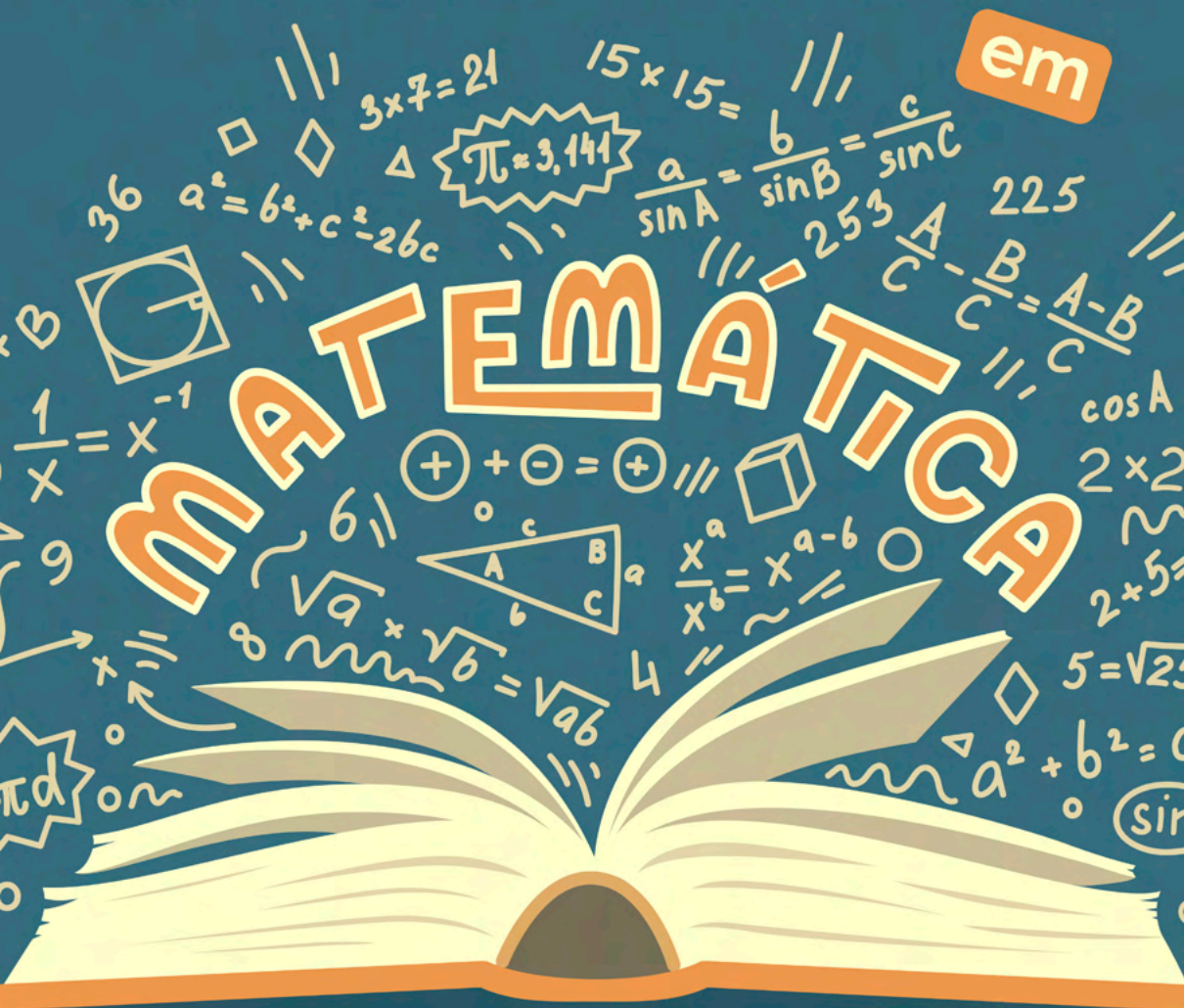


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PESQUISAS DE VANGUARDA

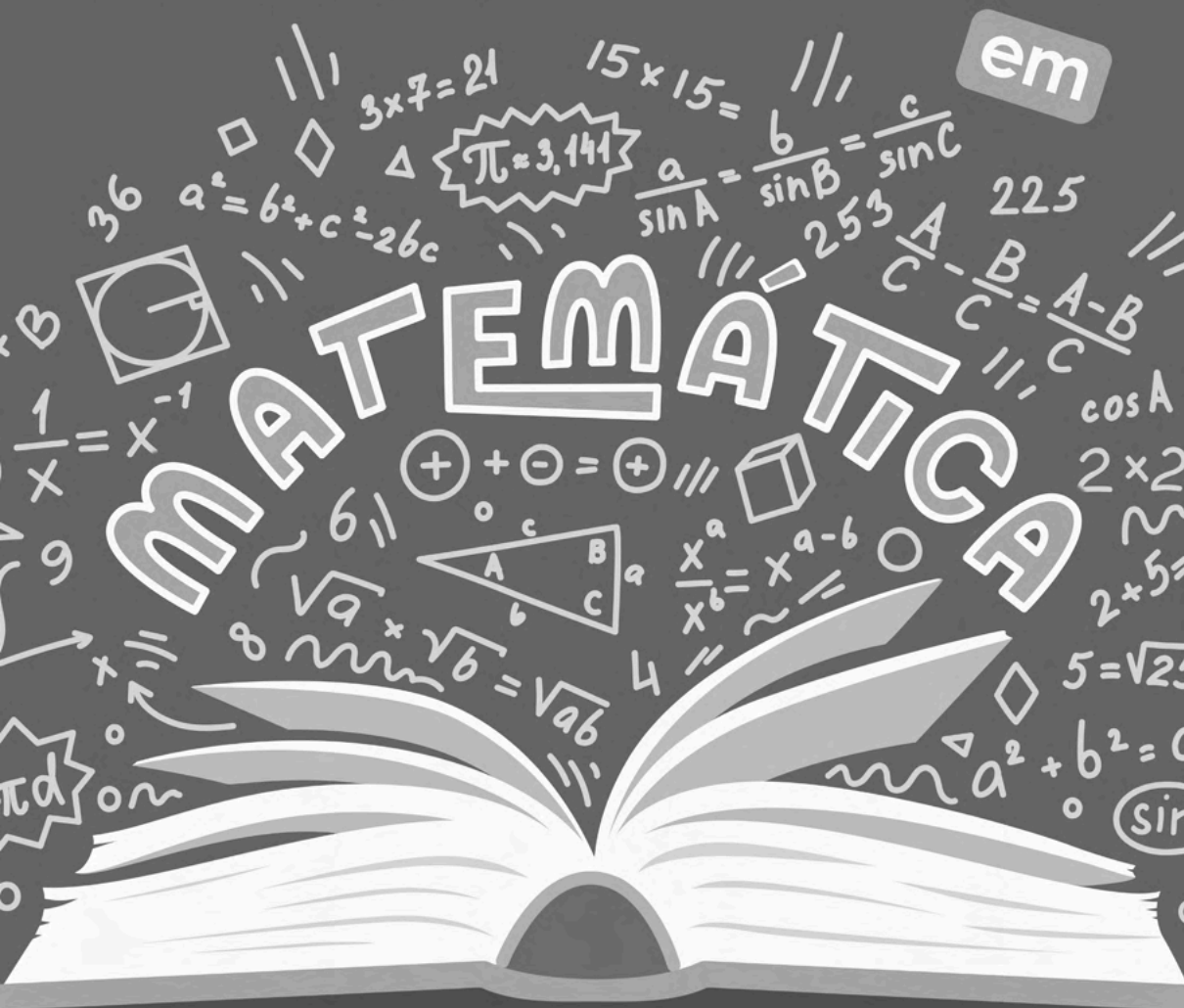


e suas aplicações

Atena
Editora
Ano 2021

Américo Junior Nunes da Silva
André Ricardo Lucas Vieira
(Organizadores)

PESQUISAS DE VANGUARDA



e suas aplicações

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APRESENTAÇÃO

A Pandemia do novo coronavírus pegou todos de surpresa. De repente, ainda no início de 2020, tivemos que mudar as nossas rotinas de vida e profissional e nos adaptar a um “novo normal”, onde o distanciamento social foi posto enquanto a principal medida para barrar o contágio da doença. As escolas e universidades, por exemplo, na mão do que era posto pelas autoridades de saúde, precisaram repensar as suas atividades.

Da lida diária, no que tange as questões educacionais, e das dificuldades de inclusão de todos nesse “novo normal”, é que contexto pandêmico começa a escancarar um cenário de destrato que já existia antes mesmo da pandemia. Esse período pandêmico só desvelou, por exemplo, o quanto a Educação no Brasil acaba, muitas vezes, sendo uma reprodutora de Desigualdades.

O contexto social, político e cultural, como evidenciaram Silva, Nery e Nogueira (2020), tem demandado questões muito particulares para a escola e, sobretudo, para a formação, trabalho e prática docente. Isso, de certa forma, tem levado os gestores educacionais a olharem para os cursos de licenciatura e para a Educação Básica com outros olhos. A sociedade mudou, nesse cenário de inclusão, tecnologia e de um “novo normal”; com isso, é importante olhar mais atentamente para os espaços formativos, em um movimento dialógico e pendular de (re)pensar as diversas formas de se fazer ciências no país. A pesquisa, nesse interim, tem se constituído como um importante lugar de ampliar o olhar acerca das inúmeras problemáticas, sobretudo no que tange ao conhecimento matemático (SILVA; OLIVEIRA, 2020).

É nessa sociedade complexa e plural que a Matemática subsidia as bases do raciocínio e as ferramentas para se trabalhar em outras áreas; é percebida enquanto parte de um movimento de construção humana e histórica e constitui-se importante e auxiliar na compreensão das diversas situações que nos cerca e das inúmeras problemáticas que se desencadeiam diuturnamente. É importante refletir sobre tudo isso e entender como acontece o ensino desta ciência e o movimento humanístico possibilitado pelo seu trabalho.

Ensinar Matemática vai muito além de aplicar fórmulas e regras. Existe uma dinâmica em sua construção que precisa ser percebida. Importante, nos processos de ensino e aprendizagem da Matemática, priorizar e não perder de vista o prazer da descoberta, algo peculiar e importante no processo de matematizar. Isso, a que nos referimos anteriormente, configura-se como um dos principais desafios do educador matemático, como assevera D’Ambrósio (1993), e sobre isso, de uma forma muito particular, abordaremos nesta obra.

É neste sentido, que o livro **“Pesquisas de Vanguarda em Matemática e suas Aplicações”** nasceu: como forma de permitir que as diferentes experiências do professor pesquisador que ensina Matemática e do pesquisador em Matemática aplicada sejam apresentadas e constituam-se enquanto canal de formação para educadores da Educação

Básica e outros sujeitos. Reunimos aqui trabalhos de pesquisa e relatos de experiências de diferentes práticas que surgiram no interior da universidade e escola, por estudantes e professores pesquisadores de diferentes instituições do país.

Esperamos que esta obra, da forma como a organizamos, desperte nos leitores provocações, inquietações, reflexões e o (re)pensar da própria prática docente, para quem já é docente, e das trajetórias de suas formações iniciais para quem encontra-se matriculado em algum curso de licenciatura. Que, após esta leitura, possamos olhar para a sala de aula e para o ensino de Matemática com outros olhos, contribuindo de forma mais significativa com todo o processo educativo. Desejamos, portanto, uma ótima leitura.

Américo Junior Nunes da Silva

André Ricardo Lucas Vieira

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SUMÁRIO

CAPÍTULO 1..... 1

O USO DA ROBÓTICA EDUCACIONAL COMO FERRAMENTA NO ENSINO E APRENDIZAGEM DE FUNÇÃO AFIM E QUADRÁTICA

Bruna Nogueira Simões Cobuci

Rigoberto Gregório Sanabria Castro


 <https://doi.org/10.22533/at.ed.4022128091>

CAPÍTULO 2..... 12

BANCO IMOBILIÁRIO MATEMÁTICO: UMA PROPOSTA DE ENSINO EM AULAS DE MATEMÁTICA

Thayná Schleider de Matos

Joyce Jaquelinne Caetano

 <https://doi.org/10.22533/at.ed.4022128092>

CAPÍTULO 3..... 18

APLICAÇÃO DE MONITORIAS ON-LINES DE CÁLCULO COMO FERRAMENTA DE NIVELAMENTO E INICIAÇÃO A DOCÊNCIA

Tamires Ester Peixoto Bravo

Pedro Lucas Moreira Rodrigues

Matheus Alencar de Freitas

Enrique Dias de Matos

Pedro Augusto Araújo Sant'Ana

Ivano Alessandro Devilla

 <https://doi.org/10.22533/at.ed.4022128093>


CAPÍTULO 4..... 24

A PSICOLOGIA EDUCACIONAL, A EDUCAÇÃO MATEMÁTICA E A PSICOLOGIA DA EDUCAÇÃO MATEMÁTICA: DISCUSSÕES SOBRE ASPECTOS RELACIONADOS À APRENDIZAGEM

André de Lima Pereira Gomes

Gyliane Ornela Barbosa

Márcia Santos Melo

 <https://doi.org/10.22533/at.ed.4022128094>

CAPÍTULO 5..... 34


DA INFORMALIDADE A SALA DE AULA: A MATEMÁTICA DO MEU ALUNO

Evren Ney da Silva Jean

Meiry Jane Cavalcante Rattes

Márcio Laranjeira Anselmo

Reginaldo Nascimento da Silva


 <https://doi.org/10.22533/at.ed.4022128095>

CAPÍTULO 6..... 42

A METODOLOGIA DO SISTEMA *NODET* E SUAS POSSIBILIDADES DE PESQUISA

SOBRE O USO DO ORIGAMI NA EDUCAÇÃO MATEMÁTICA EM TEMPOS DE USO DE NOVAS TECNOLOGIAS NA EDUCAÇÃO


Daniel Albernaz de Paiva Brito

 <https://doi.org/10.22533/at.ed.4022128096>

CAPÍTULO 7..... 57

A MATEMÁTICA DO AGRONEGÓCIO: CONTRIBUIÇÕES PARA UMA APRENDIZAGEM SIGNIFIC(ATIVA)

Luiz Carlos dos Santos Filho

 <https://doi.org/10.22533/at.ed.4022128097>

CAPÍTULO 8..... 63


DESIGUALDADE DE CAFFARELLI-KOHN-NIRENBERG EM VARIEDADES RIEMANNIANAS

Willian Isao Tokura

Levi Rosa Adriano

Priscila Marques Kai


Elismar Dias Batista

 <https://doi.org/10.22533/at.ed.4022128098>

CAPÍTULO 9..... 71

O ENSINO DE FUNÇÃO DO 1º GRAU NA EDUCAÇÃO INCLUSIVA: TRANSPOSIÇÃO DIDÁTICA E O SABER MATEMÁTICO PARA ALUNOS CEGOS

Camila Ferreira e Silva

 <https://doi.org/10.22533/at.ed.4022128099>

CAPÍTULO 10..... 85

OPORTUNIDADES PARA ARTICULAÇÃO DE ENSINO, PESQUISA E EXTENSÃO NAS AULAS DE MATEMÁTICA A PARTIR DO USO DE *SOFTWARES* MATEMÁTICOS

José Cirqueira Martins Júnior

 <https://doi.org/10.22533/at.ed.40221280910>

CAPÍTULO 11..... 100

ENSINANDO MATEMÁTICA POR MEIO DA RESOLUÇÃO DE EQUAÇÕES COM MATERIAL CONCRETO

Graciela Sieglloch Lins

Marcos Lübeck

Jocinéia Medeiros

Fernando Luiz Andretti


 <https://doi.org/10.22533/at.ed.40221280911>

CAPÍTULO 12..... 108

A UTILIZAÇÃO DO EXCEL COM ATIVIDADES EXPLORATÓRIAS PARA O TRATAMENTO DE INFORMAÇÕES EM CONTEÚDOS DE ESTATÍSTICA

José Cirqueira Martins Júnior

Leandro Vieira dos Santos

 <https://doi.org/10.22533/at.ed.40221280912>

CAPÍTULO 13..... 119

NARRATIVAS SOBRE UM LUGAR COMUM: SALA DE RECURSOS

Rozana Morais Lopes Feitosa


 <https://doi.org/10.22533/at.ed.40221280913>

CAPÍTULO 14..... 128

MODELO EPIDÊMICO SIR, COM E SEM VACINAÇÃO E MODELO EPIDÊMICO SEIR

Lívia de Carvalho Faria

Mehran Sabeti


 <https://doi.org/10.22533/at.ed.40221280914>

CAPÍTULO 15..... 139

GROUNDRED THEORY COMO METODOLOGIA DE PESQUISA EM EDUCAÇÃO MATEMÁTICA: CONTRIBUIÇÕES, RACIOCÍNIO E PROCEDIMENTOS

Eliandra Moraes Pires

Everaldo Silveira

 <https://doi.org/10.22533/at.ed.40221280915>

CAPÍTULO 16..... 154

STOMACHION: UMA ABORDAGEM SOBRE A HISTÓRIA DA ANÁLISE COMBINATÓRIA

Paula Francisca Gomes Rodrigues

 <https://doi.org/10.22533/at.ed.40221280916>

CAPÍTULO 17..... 160

RESOLUÇÃO DE PROBLEMAS ALÉM DA SALA DE AULA: EM CENA A SEMELHANÇA DE TRIÂNGULOS

Fábio Vieira Abrão

Luciano Soares Gabriel

Norma S. Gomes Allevato

 <https://doi.org/10.22533/at.ed.40221280917>


CAPÍTULO 18..... 172

APPROXIMATION OF A SYSTEM OF A NON-NEWTONIAN FLUID BY A SYSTEM OF CAUCHY-KOWALESKA TYPE

Geraldo Mendes de Araujo

Elizardo Fabricio Lima Lucena

Michel Melo Arnaud



 <https://doi.org/10.22533/at.ed.40221280918>

CAPÍTULO 19..... 191

INTERPOLAÇÃO PELO MÉTODO DE HERMITE USANDO DIFERENÇAS DIVIDIDAS

João Socorro Pinheiro Ferreira

 <https://doi.org/10.22533/at.ed.40221280919>

CAPÍTULO 20	208
APRENDIZAGEM DAS OPERAÇÕES COM FRAÇÕES NO 7º ANO DO ENSINO FUNDAMENTAL: UMA INVESTIGAÇÃO À LUZ DA TEORIA DAS SITUAÇÕES DIDÁTICAS	
Bruno José de Sá Ferraz Lemerton Matos Nogueira	
 https://doi.org/10.22533/at.ed.40221280920	
CAPÍTULO 21	219
AS POTENCIALIDADES DE UMA AULA DO CAMPO NO ENSINO FUNDAMENTAL II	
Marco André Dantas Leonardo Sturion	
 https://doi.org/10.22533/at.ed.40221280921	
SOBRE OS ORGANIZADORES	230
ÍNDICE REMISSIVO	231

APPROXIMATION OF A SYSTEM OF A NON-NEWTONIAN FLUID BY A SYSTEM OF CAUCHY-KOWALESKA TYPE

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ABSTRACT: In this paper we investigate a problem for a model of a non-newtonian fluid. The problem is considered in a bounded domain of \mathbb{R}^d with Dirichlet boundary conditions. The operator stress tensor is given by $\tau(e(u)) = [(v + v_0 M(|e(u)|^2))e(u)]$. We proved existence of weak solutions when $d \leq 4$ by using the method of approximations by a system of Cauchy-Kowaleska type. Uniqueness and periodicity of solutions are also considered. *2010 Mathematics Subject Classification:* 35Q35, 76A05, 76DXX.

KEYWORDS: Cauchy-Kowaleska, quasi-Newtonian, Galerkin.

APROXIMAÇÕES PARA UM SISTEMA DE UM FLUIDO NÃO NEWTONIANO POR UM SISTEMA DO TIPO CAUCHY-KOWALESKA

RESUMO: Neste artigo investigamos um problema para uma modelagem de um fluido não newtoniano. O problema é considerado em um domínio limitado do espaço euclidiano d -dimensional com condições de Dirichlet na fronteira. Provamos existência de soluções fracas quando d não supera 4 utilizando o método de aproximações por um sistema do tipo Cauchy-Kowaleska. Unicidade e periodicidade de soluções também são consideradas.

PALAVRAS-CHAVE: Cauchy-kowaleska, quasi-newtonian, galerkin.

1 | INTRODUCTION

Let Ω be a bounded domain in \mathbb{R}^d with smooth boundary $\partial\Omega$, and let $T > 0$. We denote by Q_T the time space cylinder $I \times \Omega$, with lateral boundary $\Sigma = I \times \partial\Omega$, where $I = (0, T)$ is a time interval. The unsteady flows of incompressible fluids in a boundary domain $\Omega \subset \mathbb{R}^d$, $d > 1$ is described by the system of equations

$$\left\{ \begin{array}{ll} \rho \frac{\partial u}{\partial t} - \nabla \cdot \tau(e(u)) + \rho(u \cdot \nabla) u & = -\nabla p + \rho f & \text{in } Q_T, \\ \nabla \cdot u & = 0 & \text{in } Q_T, \\ u & = 0 & \text{on } \Sigma_T, \\ u(0) & = u_0 & \text{in } \Omega, \end{array} \right. \quad (1.1)$$

where $u = (u_1, u_2, \dots, u_d)$ is the velocity, p represents the pressure, ρ is a positive constant determining the density of a material, $f = (f_1, f_2, \dots, f_d)$ stands for the given external body forces, $\tau : \mathbb{R}_{sym}^{d^2} \rightarrow \mathbb{R}_{sym}^{d^2}$ denotes the extra stress tensor, $e = e(u) : \mathbb{R}^d \rightarrow \mathbb{R}_{sym}^{d^2}$ denotes the symmetric part of the velocity gradient, that is,

$$e(u) = \frac{1}{2} [\nabla u + (\nabla u)^T], \quad (1.2)$$

whose components are defined as in [7] by

$$2e_{ij}(u) = \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \quad i, j = 1, 2, \dots, d \quad (1.3)$$

and $\mathbb{R}_{sym}^{d^2}$ represents the set of all symmetric $d \times d$ matrices, that is,

$$\mathbb{R}_{sym}^{d^2} = \{D \in \mathbb{R}^{d^2}; D_{ij} = D_{ji}, i, j = 1, 2, \dots, d\}.$$

Note for example, that when $\tau(e(u))$ is of the form

$$\tau(e(u)) = \mu_0(1 + |e(u)|^{p-2})e(u), \quad (1.4)$$

with $p = 2$, the problem (1.1) turns into the Navier-Stokes system, which is a model for Newtonian fluids. In the expression (1.4), $|e(u)|$ denotes the usual Euclidean matrix-norm. We observe that (1.4) can be write in the form

$$\tau(e(u)) = \mu_0 M (|e(u)|^2) e(u), \quad (1.5)$$

where $M : \mathbb{R}_0^+ \rightarrow \mathbb{R}_0^+$, $M \in C^0(0, \infty)$ is the generalized viscosity function.

Fluids constituted by (1.5), with $p \neq 2$, are sometimes named fluids with shear-dependent viscosity. Models belonging to this class of non-Newtonian fluid mechanics are frequently used in several fields of chemistry, glaciology, biology and geology, as discussed in Málek, Rajagopal, Růžička [8].

The mathematical analysis of the Problem (1.1) when $\tau(e(u)) = \nu e(u)$ was done first time by Leray ([10]). After this, it was investigated in general case by Ladyzhenskaya in 1963, where she proposed, among others, to study the system (1.1) with (1.4) and $p = 4$. Combining monotone operator theory and compactness arguments, she proved the existence of weak solution to model (1.1), if $p \geq 1 + \frac{2d}{d+2}$ and their uniqueness if $p \geq \frac{d+2}{2}$.

More results are known about the Problem (1.1) obtained in a series of papers, include Málek, Rajagopal and Růžička [8], Málek, Nečas and Růžička [6], Frehse and Málek [2], Málek, Nečas, Rokyta and Růžička [7] and among other mathematicians.

The problem that we study in this work consists investigate following mixed problem: let Ω be a bounded domain in \mathbb{R}^d with smooth boundary $\partial\Omega$, and let $T > 0$. We denote by Q_T the time space cylinder $I \times \Omega$, with lateral boundary $\Sigma = I \times \partial\Omega$, where $I = (0, T)$ is a time interval. We find $u : Q_T \rightarrow \mathbb{R}^d$ and $p : Q_T \rightarrow \mathbb{R}$ solving the following system of equations

$$\left\{ \begin{array}{l} u' - \nabla \cdot [(\nu + \nu_0 M(|e(u)|^2))e(u)] + (u \cdot \nabla)u + \nabla p = f \text{ in } Q_T, \\ \nabla \cdot u = 0 \text{ in } Q_T, \\ u = 0 \text{ on } \Sigma_T, \\ u(0) = u_0 \text{ in } \Omega, \end{array} \right. \quad (1.6)$$

where the extra stress tensor is given by $\tau(e(u)) = (\nu + \nu_0 M(|e(u)|^2))e(u)$, $e(u)$ as in (1.2)-(1.3), ν_0 and ν_1 are positives constants. Let us consider $M : (0, \infty) \rightarrow (0, \infty)$ satisfying the following hypothesis

$$M \in C^1(0, \infty), \quad M > M_0 > 0, \quad M' > 0, \quad (1.7)$$

$$c_1 |e(u)|^2 \leq M(|e(u)|^2) \leq c_2 |e(u)|^2, \quad (1.8)$$

where M_0 , c_1 and c_2 are positive constants. This paper is devoted to analyze the existence of weak solutions to system (1.6) by approximating it by a system of Cauchy-Kowaleska type as in Lion ([5]). The method consists in considering the following system of Cauchy-Kowaleska type

$$\left\{ \begin{array}{l} u'_\epsilon - \nabla \cdot \tau(e(u_\epsilon)) + (u_\epsilon \cdot \nabla)u_\epsilon + \frac{1}{2}(\nabla \cdot u_\epsilon)u_\epsilon + \nabla p_\epsilon = f \quad \text{in } Q_T, \\ \epsilon p'_\epsilon + \nabla \cdot u_\epsilon = 0 \quad \text{in } Q_T, \\ u_\epsilon = 0 \quad \text{on } \Sigma_T, \\ u_\epsilon(0) = u_{\epsilon 0} \quad \text{in } \Omega, \\ p_\epsilon(0) = p_{\epsilon 0} \quad \text{in } L^2(\Omega). \end{array} \right. \quad (1.9)$$

Employing the method of Faedo-Galerking, we proved that (1.9) has a weak solution $\{u_\epsilon, p_\epsilon\}$, for each $\epsilon > 0$, which convergences as $\epsilon \rightarrow 0$ to a weak solution to the problem (1.6). Lions (see [5]) studied the approximation by Cauchy-Kowaleska system for the Navier-Stokes system, when the viscosity is constant. After, Araujo, Menezes and Guzman ([3]) analyzed the system (1.6) with $\nabla \cdot \tau(e(u)) = (\nu_0 + \nu_1 \|u(t)\|^2)\Delta u$. This paper is devoted to study the case

$$\tau(e(u)) = (\nu + \nu_0 M(|e(u)|^2))e(u), \quad (1.10)$$

that is, a stress tensor model for a non-Newtonian fluid as proposed by Ladyzhenskaya ([4]).

2 | NOTATION AND MAIN RESULTS

In order to formulate problem (1.6) we need some notations about Sobolev spaces. We use standard notation of $L^p(\Omega)$, $W^{m,p}(\Omega)$ and $C^p(\Omega)$ for functions that are defined on Ω and range in \mathbb{R} , and the notation $\mathbf{L}^p(\Omega)$, $\mathbf{W}^{m,p}(\Omega)$ and $\mathbf{C}^p(\Omega)$ for functions that range in \mathbb{R}^d . We also work with the spaces $L^p(I; H^m(\Omega))$ or $L^p(Q_T)$. To complete this recall on functional spaces, see for instance, Lions [5]. By $\langle \cdot, \cdot \rangle$ we will represent the duality pairing between X and X' , X' being the topological dual of the space X .

Remark 1 $\mathbf{H}_0^1(\Omega)$ and $\mathbf{L}^2(\Omega)$ are Hilbert's spaces. We note that $\mathbf{H}_0^1(\Omega) \hookrightarrow \mathbf{L}^2(\Omega) \hookrightarrow \mathbf{H}^{-1}(\Omega)$ with embeddings dense and compact.

We introduce the following bilinear and the trilinear forms. As well as the convention of summation of indices, that is, $\alpha_i \beta_j$ instead of $\sum_{i,j=1}^d \alpha_i \beta_j$.

$$a(u, v) = \int_{\Omega} \frac{\partial u_i}{\partial x_j}(x) \frac{\partial v_i}{\partial x_j}(x) dx = \langle (u, v) \rangle \quad \forall u, v \in \mathbf{H}_0^1(\Omega), \quad (2.11)$$

$$b(u, v, w) = \int_{\Omega} u_i(x) \frac{\partial v_j}{\partial x_i}(x) w_j(x) dx \quad \forall u, v, w \in \mathbf{H}_0^1(\Omega), \quad (2.12)$$

$$b_1(u, v, w) = \frac{1}{2} \int_{\Omega} u_i(x) (\nabla \cdot v(x)) w_i(x) dx \quad \forall u, v, w \in \mathbf{H}_0^1(\Omega). \quad (2.13)$$

$$\tilde{b}(u, v, w) = b(u, v, w) + b_1(u, v, w) \quad \forall u, v, w \in \mathbf{H}_0^1(\Omega). \quad (2.14)$$

We also introduce the notations

$$Au = -\Delta u, \quad Bu = (u \cdot \nabla)u, \quad B_1 u = \frac{1}{2}(\nabla \cdot u)u, \quad \tilde{B}u = Bu + B_1 u \quad \forall u \in \mathbf{H}_0^1(\Omega),$$

and

$$\mathcal{K}u = -\nabla \cdot M(|e(u)|^2)e(u) \quad \forall u \in \mathbf{H}_0^1(\Omega). \quad (2.15)$$

According this, we have

$$\langle Au, v \rangle = a(u, v) \quad \forall u, v \in \mathbf{H}_0^1(\Omega), \quad (2.16)$$

$$\langle Bu, v \rangle = b(u, u, v) \quad \forall u, v, w \in \mathbf{H}_0^1(\Omega), \quad (2.17)$$

$$\langle \mathcal{K}u, v \rangle = \int_{\Omega} M(|e(u)|^2) e_{ij}(u) e_{ij}(v) dx \quad \forall u, v \in \mathbf{H}_0^1(\Omega). \quad (2.18)$$

Remark 2 We observe that $M > 0$ imply for all $u_1, u_2 \in \mathbf{H}_0^1(\Omega)$ that

$$\begin{aligned} & \langle \mathcal{K}u_1 - \mathcal{K}u_2, u_1 - u_2 \rangle \\ &= \int_{\Omega} [M(|e(u_1)|^2) e_{ij}(u_1) - M(|e(u_2)|^2) e_{ij}(u_2)] [e_{ij}(u_1) - e_{ij}(u_2)] dx \geq 0. \end{aligned}$$

Results $\mathcal{K} : \mathbf{H}_0^1(\Omega) \rightarrow \mathbf{H}^{-1}(\Omega)$ is monotonous.

Remark 3 We note that $\tilde{b}(u, u, u) = 0, \forall u \in \mathbf{H}_0^1(\Omega)$. In fact,

$$b(u, v, v) = \int_{\Omega} u_j \frac{\partial v_i}{\partial x_j} v_i dx = - \int_{\Omega} v_i^2 \frac{\partial u_i}{\partial x_j} dx - \int_{\Omega} v_i \frac{\partial v_i}{\partial x_j} u_j dx \quad \forall u, v \in \mathbf{H}_0^1(\Omega)$$

In other words, $b(u, v, v) = -b_1(v, u, v) \quad \forall u, v \in \mathbf{H}_0^1(\Omega)$. It follows that

$$\tilde{b}(u, u, u) = b(u, u, u) + b_1(u, u, u) = -b_1(u, u, u) + b_1(u, u, u) = 0$$

Definition 1 Let $u_0 \in L^2(\Omega)$ and $f \in L^{4/3}(I, \mathbf{H}^{-1}(\Omega))$. A weak solution to (1.6) is a function u , such that

$$u \in L^4(I; \mathbf{H}_0^1(\Omega)) \cap L^\infty(I; L^2(\Omega)),$$

satisfying the following identity

$$\left\{ \begin{array}{l} (u'(t), v) + \nu a(u(t), v) + \nu_0 \langle \mathcal{K}u(t), v \rangle + \langle Bu(t), v \rangle = \langle f(t), v \rangle, \\ \nabla \cdot u(0) = u_0, \\ \forall v \in \mathcal{D}(\Omega). \end{array} \right. \quad (2.19)$$

Definition 2 Let $u_{e0} \in L^2(\Omega)$, $p_{e0} \in L^2(\Omega)$ and $f \in L^{4/3}(I, \mathbf{H}^{-1}(\Omega))$.

A weak solution to (1.9) is a pair of functions u_ϵ, p_ϵ , such that

$$u_\epsilon \in L^4(I; \mathbf{H}_0^1(\Omega)) \cap L^\infty(I; L^2(\Omega)),$$

$$p_\epsilon \in L^\infty(I; L^2(\Omega)),$$

satisfying the following identity

$$\begin{cases}
(u'_\epsilon(t), v) + \nu a(u_\epsilon(t), v) + \nu_0 \langle \mathcal{K}u_\epsilon(t), v \rangle + \langle \tilde{B}u_\epsilon(t), v \rangle + (\nabla p_\epsilon(t), v) \\
= \langle f(t), v \rangle, \\
\epsilon(p'_\epsilon(t), q) + (\nabla \cdot u_\epsilon(t), q) = 0, \\
u_\epsilon(0) = u_{\epsilon 0}, \\
p_\epsilon(0) = p_{\epsilon 0}, \\
\forall v \in \mathcal{D}(\Omega) \text{ and } \forall q \in L^2(\Omega).
\end{cases} \tag{2.20}$$

Lemma 1 (Korn's Inequality) Let $1 < p < \infty$. Then, there exists a constant $K_p = K_p(\Omega)$, such that the inequality

$$K_p \|v\|_{W^{1,p}(\Omega)} \leq \|e(v)\|_{L^p(\Omega)} \tag{2.21}$$

is fulfilled for all v satisfying either $v \in W_0^{1,p}(\Omega)$, where $\Omega \subset \mathbb{R}^3$ is open and bounded with $\partial\Omega \subset C^1$.

Proof. See [9]

Lemma 2 (Vitali) Let Ω be a bounded domain in \mathbb{R}^n and $f^m: \Omega \rightarrow \mathbb{R}$ integrable for every $m \in \mathbb{N}$. Assume that

1. $\lim_{m \rightarrow \infty} f^m(x)$ exists and is finite for almost all $x \in \Omega$;
2. for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\sup_{m \in \mathbb{N}} \int_H |f^m(x)| dx < \varepsilon, \quad \forall H \in \Omega, |H| < \delta$$

then

$$\lim_{m \rightarrow \infty} \int_\Omega f^m(x) dx = \int_\Omega \lim_{m \rightarrow \infty} f^m(x) dx.$$

Proof. See [1]

Lemma 3 Consider $d \geq 3$ and $s, r \in \mathbb{R}$, with $s > 2, r > d$, verifying $\frac{2}{s} + \frac{d}{r} = 1$. If $u \in L^r(\Omega)$ then,

$$|b(u, v, w)| \leq c \|u\|_{L^r(\Omega)} \|v\| \|w\|^{2/s} \|w\|^{d/r}$$

$\forall v, w \in \mathbf{H}_0^1(\Omega)$. Where $c \geq 0$ is a constant independent of u, v and w .

Proof. See [5].

Remark 4 We note that is possible to obtain, after appropriate modification of a proof due for Lions [5], that lemma 3 holds to $b_1(u, v, w) \forall u \in L^r(\Omega)$ and $\forall v, w \in \mathbf{H}_0^1(\Omega)$.

Theorem 2.1 If $d \leq 4, u_0 \in L^2(\Omega)$ and $f \in L^{4/3}(I; \mathbf{H}^{-1}(\Omega))$ then, there exists a function $u: Q_T \rightarrow \mathbb{R}$, solution to Problem (1.6) in the sense of definition 1.

Theorem 2.2 (Periodic Solutions) Under the assumptions of the Theorem 2.1 there exists a function $u : Q_T \rightarrow \mathbb{R}$, solution to problem (1.6) in the sense of definition 1 such that $u(0) = u(T)$.

Theorem 2.3 If $d \leq 4$, $u_{\epsilon_0}, p_{\epsilon_0} \in L^2(\Omega)$ and $f \in L^{4/3}(I; \mathbf{H}^1(\Omega))$ then, for each $\epsilon > 0$, there exists a weak solution $\{u_{\epsilon}, p_{\epsilon}\}$, solution to Problem (1.9) in the sense of definition 2. Moreover, if $d \leq 3$ then the solution $\{u_{\epsilon}, p_{\epsilon}\}$ is unique.

3 | PROOFS OF THE RESULTS

Proof of Theorem 2.3.

We employ the method of Faedo-Galerkin. Let $\{\varphi_{\nu}, \lambda_{\nu}\}$ and $\{q_{\nu}, \bar{\lambda}_{\nu}\}, \nu \in \mathbb{N}$ be the solution to the espectral problem

$$\begin{cases} (\varphi, v) = \lambda(\varphi, v) & \forall v \in \mathbf{H}_0^1(\Omega), \\ (q, \bar{v}) = \bar{\lambda}(q, \bar{v}) & \forall \bar{v} \in L^2(\Omega). \end{cases} \quad (3.22)$$

Consider $V_m = [\varphi_1, \dots, \varphi_m] \subset \mathbf{H}_0^1(\Omega)$ the subspace generated by $\{\varphi_1, \dots, \varphi_m\}$ and $W_m = [q_1, \dots, q_m] \subset L^2(\Omega)$ the subspace generated by $\{q_1, \dots, q_m\}$. Let us also consider the pair $\{u_m, p_m\}$, such that

$$u_{em}(x, t) = \sum_{r=1}^m g_{rem}(t) \varphi_r(x) \quad \text{and} \quad q_{em}(x, t) = \sum_{r=1}^m h_{rem}(t) q_r(x), \quad (3.23)$$

solution of the approximate problem

$$\begin{cases} (u'_{em}(t), \varphi_r) + \nu(Au_{em}(t), \varphi_r) + \nu_0(\mathcal{K}u_{em}(t), \varphi_r) \\ + \langle \tilde{B}u_{em}(t), \varphi_r \rangle + (\nabla p_{em}, \varphi_r) = (f(t), \varphi_r) & r = 1, \dots, m, \\ \epsilon(p'_{em}(t), q_r) + (\nabla \cdot u_{em}(t), q_r) = 0 & r = 1, \dots, m, \\ u_{em}(0) = u_{\epsilon 0m}, \quad u_{\epsilon 0m} \rightarrow u_{\epsilon 0}, \quad \text{strong in } \mathbf{L}^2(\Omega), \\ p_{em}(0) = p_{\epsilon 0m}, \quad p_{\epsilon 0m} \rightarrow p_{\epsilon 0}, \quad \text{strong in } L^2(\Omega). \end{cases} \quad (3.24)$$

The system of ordinary differential equation (3.24) has a local solution on a interval $[0, t_m]$, $0 < t_m < T$. The first estimate permits us to extend this solution to the whole interval $[0, T]$.

FIRST ESTIMATE

We sometimes omit parameter t . Multiplying both sides of (3.24)₁ by g_{rem} and (3.24)₂

by $h_{\epsilon m}$, next adding from $r = 1$ to $r = m$, we obtain

$$\frac{1}{2} \frac{d}{dt} |u_{\epsilon m}(t)|^2 + \nu \|u_{\epsilon m}(t)\|^2 + \nu_0 \int_{\Omega} M(|e(u_{\epsilon m}(t))|)^2 |e_{ij}(u_{\epsilon m}(t))|^2 dx \quad (3.25)$$

$$- (p_{\epsilon m}(t), \nabla \cdot u_{\epsilon m}(t)) \leq \|f(t)\|_{H^{-1}(\Omega)} \|u_{\epsilon m}(t)\|,$$

$$\epsilon \frac{1}{2} \frac{d}{dt} |p_{\epsilon m}(t)|^2 + (\nabla \cdot u_{\epsilon m}(t), p_{\epsilon m}(t)) = 0, \quad (3.26)$$

because $\tilde{b}(u, u, u) = 0$ (see remark 3). Now using Young's inequality we obtain from (3.25) that

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} |u_{\epsilon m}(t)|^2 + \nu_2 \|u_{\epsilon m}(t)\|^4 - (p_{\epsilon m}(t), \nabla \cdot u_{\epsilon m}(t)) \\ \leq \frac{\nu_2}{2} \|u_{\epsilon m}(t)\|^4 + c_{\nu_2} \|f(t)\|_{H^{-1}(\Omega)}^{4/3}, \end{aligned} \quad (3.27)$$

Because, from (2.21) (Korn's inequality) and (1.8) we can get

$$\nu_0 \int_{\Omega} M(|e(u_{\epsilon m}(t))|)^2 |e_{ij}(u_{\epsilon m}(t))|^2 dx \geq \nu_0 c_1 \|e(u_{\epsilon m})\|_{L^4(\Omega)}^4 \geq \nu_2 \|u_{\epsilon m}\|^4,$$

Adding inequalities (3.26) and (3.27) and integrating from 0 to t , with $0 \leq t \leq T$, we conclude

$$\begin{aligned} (|u_{\epsilon m}(t)|^2 + \epsilon |p_{\epsilon m}(t)|^2) + \nu_2 \int_0^t \|u_{\epsilon m}(s)\|^4 ds \\ \leq C + C \int_0^t (|u_{\epsilon m}(s)|^2 + |p_{\epsilon m}(s)|^2) ds. \end{aligned} \quad (3.28)$$

Where C is a positive constant independent of m and t . By using Gronwall's inequality, we can write

$$(u_{\epsilon m}) \text{ is bounded in } L^\infty(I; \mathbf{L}^2(\Omega)), \quad (3.29)$$

$$(u_{\epsilon m}) \text{ is bounded in } L^4(I; \mathbf{H}_0^1(\Omega)), \quad (3.30)$$

$$(\sqrt{\epsilon} p_{\epsilon m}) \text{ is bounded in } L^\infty(I; \mathbf{L}^2(\Omega)), \quad (3.31)$$

SECOND ESTIMATE

We consider $P_m : \mathbf{H}_0^1(\Omega) \rightarrow V_m$ the orthogonal projections from $\mathbf{H}_0^1(\Omega)$ to V_m , that is

$$P_m u = \sum_{j=1}^m (u, \varphi_j) \varphi_j \quad \forall u \in \mathbf{H}_0^1(\Omega).$$

We note that $P_m^* u'_{em} = u'_{em}$. By the choice of the special basis (φ_j) , we obtain

$$\|P_m\|_{\mathcal{L}(H_0^1(\Omega), H_0^1(\Omega))} \leq 1 \quad \text{and} \quad \|P_m^*\|_{\mathcal{L}(H^{-1}(\Omega), H^{-1}(\Omega))} \leq 1. \quad (3.32)$$

We will sometimes omit the parameter t . It follows from (3.24)₁, (2.16), (2.17) and (2.18)

$$u'_{em} = -\nu P_m^* A u_{em} - \nu_0 P_m^* \mathcal{K} u_{em} - P_m^* \tilde{B} u_{em} - P_m^* \nabla p_{em} + P_m^* f. \quad (3.33)$$

We have $|\langle A u_{em}, v \rangle| \leq \|u_{em}\| \|v\|$, $\forall u_{em}, v \in \mathbf{H}_0^1(\Omega)$. Thus, from (3.30) we can derive

$$(A u_{em}) \text{ is bounded in } L^4(I; \mathbf{H}^{-1}(\Omega)) \hookrightarrow L^{4/3}(I; \mathbf{H}^{-1}(\Omega)). \quad (3.34)$$

Let $u_{em}, v \in \mathbf{H}_0^1(\Omega)$. From Schwarz's inequality and (1.8) we take

$$\begin{aligned} |\langle \mathcal{K} u_{em}, v \rangle| &\leq |\langle M(|e(u_{em})|^2) e(u_{em}), \nabla v \rangle| \leq c_2 |e(u_{em})|^3 \|v\| \\ &\leq c \|u_{em}\|^3 \|v\|. \end{aligned}$$

Therefore, (3.30) holds

$$(\mathcal{K} u_{em}) \text{ is bounded in } L^{4/3}(I; \mathbf{H}^{-1}(\Omega)). \quad (3.35)$$

Let $u_{em}, v \in \mathbf{H}_0^1(\Omega)$. Using (2.14) and Hölder's inequality we conclude

$$\begin{aligned} |\langle \tilde{B} u_{em}, v \rangle| &\leq |b(u_{em}, u_{em}, v)| + |b_1(u_{em}, u_{em}, v)| \\ &\leq 2 \|u_{em}\|_{L^4(\Omega)} \|u_{em}\| \|v\|_{L^4(\Omega)} \leq c \|u_{em}\|^2 \|v\|. \end{aligned}$$

Thus, from (3.30) we have that

$$(B u_{em}) \text{ is bounded in } L^2(I; \mathbf{H}^{-1}(\Omega)) \hookrightarrow L^{4/3}(I; \mathbf{H}^{-1}(\Omega)). \quad (3.36)$$

On the other hand, let $u_{em}, v \in \mathbf{H}_0^1(\Omega)$ we can write

$$|\langle \nabla p_{em}, v \rangle| = |\langle p_{em}, \nabla \cdot v \rangle| \leq |p_{em}| \|v\|$$

It follows from (3.31)

$$(\nabla p_{em}) \text{ is bounded in } L^{4/3}(I; \mathbf{H}^{-1}(\Omega)). \quad (3.37)$$

It follows from (3.34)-(3.37), (3.32) and hypothesis about f that

$$(u'_{em}) \text{ is bounded in } L^{4/3}(I; \mathbf{H}^{-1}(\Omega)). \quad (3.38)$$

Analogously we obtain from (3.24)₂

$$|\langle \epsilon p'_{em}, v \rangle| = |\langle \nabla \cdot u_{em}, v \rangle| \leq \|u_{em}\| \|v\|,$$

$\forall v \in L^2(\Omega)$. Thus, (3.30) implies

$$(\epsilon p'_{\epsilon m}) \text{ is bounded in } L^4(I; L^2(\Omega)) \hookrightarrow L^2(I; L^2(\Omega)). \quad (3.39)$$

The limitations (3.29)-(3.31), (3.38), (3.39) and the Aubin-Lions Lemma implies that there exists subsequences from $(u_{\epsilon m})$ and $(p_{\epsilon m})$, still denoted by $(u_{\epsilon m})$ and $(p_{\epsilon m})$, such that

$$u_{\epsilon m} \longrightarrow u_\epsilon \text{ strong in } L^2(I; \mathbf{L}^2(\Omega)) \text{ and a.e. } Q_T, \quad (3.40)$$

$$u_{\epsilon m} \longrightarrow u_\epsilon \text{ weak star in } L^\infty(I; \mathbf{L}^2(\Omega)), \quad (3.41)$$

$$u_{\epsilon m} \longrightarrow u_\epsilon \text{ weak in } L^4(I; \mathbf{H}_0^1(\Omega)), \quad (3.42)$$

$$u'_{\epsilon m} \longrightarrow u'_\epsilon \text{ weak in } L^{4/3}(I; \mathbf{H}^{-1}(\Omega)), \quad (3.43)$$

$$\mathcal{K}u_{\epsilon m} \longrightarrow \chi \text{ weak in } L^{4/3}(I; \mathbf{H}^{-1}(\Omega)). \quad (3.44)$$

$$p_{\epsilon m} \longrightarrow p_\epsilon \text{ weak star in } L^\infty(I; L^2(\Omega)), \quad (3.45)$$

$$p'_{\epsilon m} \longrightarrow p'_\epsilon \text{ weak in } L^2(I; L^2(\Omega)). \quad (3.46)$$

Finally, we note that (3.30) and (3.38) implies $u_\epsilon \in C^0(I; \mathbf{L}^2(\Omega))$. Analogously (3.31) and (3.39) implies and $p_\epsilon \in C^0(I; L^2(\Omega))$. Thus, make sense $u_\epsilon(0) = u_{\epsilon 0}$ and $p_\epsilon(0) = p_{\epsilon 0}$.

To prove that

$$\int_0^T \tilde{b}(u_{\epsilon m}, u_{\epsilon m}, \varphi) \longrightarrow \int_0^T \tilde{b}(u_\epsilon, u_\epsilon, \varphi), \quad \forall \varphi \in \mathcal{D}(I; \mathcal{D}(\Omega)), \quad (3.47)$$

we use (3.40) (see [7], pp.210). To prove that

$$\int_{Q_T} M(|e(u_{\epsilon m})|^2) e_{ij}(u_{\epsilon m}) e_{ij}(\varphi) dx dt \longrightarrow \int_{Q_T} M(|e(u_\epsilon)|^2) e_{ij}(u_\epsilon) e_{ij}(\varphi) dx dt, \quad (3.48)$$

we use the fact $\nabla u_{\epsilon m} \rightarrow \nabla u_\epsilon$ a.e. in Q_T , (see [2] pp. 565-566). Therefore,

$$|\nabla u_{\epsilon m}|^2 \longrightarrow |\nabla u_\epsilon|^2 \text{ a. e. in } Q_T,$$

that is,

$$|e(u_{\epsilon m})|^2 \longrightarrow |e(u_\epsilon)|^2 \text{ a. e. in } Q_T. \quad (3.49)$$

Since $M \in C(0, \infty)$ follows from (3.49)

$$M(|e(u_{\epsilon m})|^2) \longrightarrow M(|e(u_\epsilon)|^2) \text{ a.e. in } Q_T, \quad (3.50)$$

Thus,

$$M(|e(u_{\epsilon m})|^2) e_{ij}(u_{\epsilon m}) e_{ij}(\varphi) \longrightarrow M(|e(u_\epsilon)|^2) e_{ij}(u_\epsilon) e_{ij}(\varphi) \quad (3.51)$$

a.e. in Q_T and $\forall \varphi \in \mathcal{D}(I; \mathcal{D}(\Omega))$. Using (3.30) and (1.8) we obtain

$$\begin{aligned} \int_{Q_T} M(|e(u_{\epsilon m})|^2) e_{ij}(u_{\epsilon m}) e_{ij}(\varphi) dxdt &\leq C \int_{Q_T} |e(u_{\epsilon m})|^3 |e_{ij}(\varphi)| dxdt \\ &\leq C \int_{Q_T} |e(u_{\epsilon m})|^3 dxdt C \int_{Q_T} |\nabla u_{\epsilon m}|^3 dxdt \leq C. \end{aligned} \quad (3.52)$$

It follows that

$$M(|e(u_{\epsilon m})|^2) e_{ij}(u_{\epsilon m}) e_{ij}(\varphi) \in L^1(Q_T). \quad (3.53)$$

Moreover, if $H \subset Q_T$ is measurable set, we have from (1.8), (3.30) and Hölder's inequality

$$\begin{aligned} \int_H M(|e(u_{\epsilon m})|^2) e_{ij}(u_{\epsilon m}) e_{ij}(\varphi) dxdt &\leq c \int_H |e(u_{\epsilon m})|^3 |e(\varphi)| dxdt \\ &\leq c \left(\int_{Q_T} |e(u_{\epsilon m})|^4 dxdt \right)^{3/4} \left(\int_H |e(\varphi)|^4 dxdt \right)^{1/4} \\ &\leq c \left(\int_{Q_T} |\nabla u_{\epsilon m}|^4 dxdt \right)^{3/4} |H|^{1/4} \leq c|H|^{1/4}. \end{aligned}$$

Therefore,

$$\sup_{m \in \mathbb{N}} \int_H M(|e(u_{\epsilon m})|^2) e_{ij}(u_{\epsilon m}) e_{ij}(\varphi) dxdt \leq c|H|^{1/4}.$$

Assuming that $|H|$ is sufficiently small, we obtain

$$\sup_{m \in \mathbb{N}} \int_H M(|e(u_{\epsilon m})|^2) e_{ij}(u_{\epsilon m}) e_{ij}(\varphi) dxdt \leq \varepsilon, \quad (3.54)$$

$\forall \varepsilon \in \mathbb{R}$. Now using (3.51), (3.53), (3.54) and Vitali's lemma we can derive (3.48).

Therefore, we can write $\chi = Ku_\epsilon$ in $L^{4/3}(I; \mathbf{H}^{-1}(\Omega))$. The convergences (3.40)-(3.48) allow us to pass the limit on system (3.24), with φ and q fixed to obtain

$$u'_\epsilon + \nu Au_\epsilon + \nu_0 \mathcal{K}u_\epsilon + \tilde{B}u_\epsilon = f \quad \text{in } L^{4/3}(I; \mathbf{H}^{-1}(\Omega)), \quad (3.55)$$

$$\epsilon p'_\epsilon + \nabla \cdot u_\epsilon = 0 \quad \text{in } L^2(I; L^2(\Omega)). \quad (3.56)$$

To prove uniqueness of solutions to the problem (1.9), let $(u_{\epsilon_1}, p_{\epsilon_1})$ and $(u_{\epsilon_2}, p_{\epsilon_2})$ weak solutions to Problem (1.9). Then,

$$\begin{aligned} u_{\epsilon_1}, u_{\epsilon_2} &\in L^\infty(I; \mathbf{L}^2(\Omega)) \cap L^4(I; \mathbf{H}_0^1(\Omega)), \\ p_{\epsilon_1}, p_{\epsilon_2} &\in L^\infty(I; \mathbf{L}^2(\Omega)). \end{aligned} \quad (3.57)$$

Consider $z = u_{\epsilon_1} - u_{\epsilon_2}$ and $q = p_{\epsilon_1} - p_{\epsilon_2}$. Then, (z, q) verifies

$$\left\{ \begin{array}{l} z' + \nu Az + \nu_0(\mathcal{K}u_{\epsilon 1} - \mathcal{K}u_{\epsilon 2}) + (\tilde{B}u_{\epsilon 1} - \tilde{B}u_{\epsilon 2}) + \nabla q = 0, \\ \epsilon q' + \nabla \cdot z = 0, \\ z(0) = q(0) = 0. \end{array} \right. \quad (3.58)$$

where the first equality is consider in $L^{4/3}(t; \mathbf{H}^{-1}(\Omega))$, and the second in $L^2(t; L^2(\Omega))$. After, we take the duality in the equations (3.58)₁ and (3.58)₂ with z and q , respectively, to obtain

$$\left\{ \begin{array}{l} \frac{1}{2} \frac{d}{dt} |z|^2 + \nu \|z\|^2 + \nu_0 \langle \mathcal{K}u_{\epsilon 1} - \mathcal{K}u_{\epsilon 2}, z \rangle + \langle \tilde{B}u_{\epsilon 1} - \tilde{B}u_{\epsilon 2}, z \rangle \\ + (\nabla q, z) = 0, \\ \frac{1}{2} \frac{d}{dt} |q|^2 + (\nabla \cdot z, q) = 0, \\ z(0) = q(0) = 0. \end{array} \right. \quad (3.59)$$

We note that

$$\langle \tilde{B}u_{\epsilon 1} - \tilde{B}u_{\epsilon 2}, z \rangle = \tilde{b}(z, u_{\epsilon 1}, z) + \tilde{b}(u_{\epsilon 2}, z, z). \quad (3.60)$$

From the monotonicity of \mathcal{K} we have $\langle \mathcal{K}u_1 - \mathcal{K}u_2, \tilde{u} \rangle \geq 0$. Thus, adding member to member the equalities(3.59)₁ and (3.59)₂, we obtaine

$$\frac{1}{2} \frac{d}{dt} (|z|^2 + \epsilon |q|^2) + \nu \|z\|^2 \leq |\tilde{b}(z, u_{\epsilon 1}, z)| + |\tilde{b}(u_{\epsilon 2}, z, z)|. \quad (3.61)$$

Because $(\nabla q, z) = -(\mathcal{q}, \nabla \cdot z)$. Considering $d = 2$, we get $H_0^1(\Omega) \hookrightarrow L^4(\Omega)$. It follows that (see Lions [5])

$$|u|_{L^4(\Omega)} \leq c|u|^{1/2} \|u\|^{1/2}. \quad (3.62)$$

Thus, using (2.14), Hölder's inequality, (3.62) and Young's inequality we take

$$\begin{aligned} |\tilde{b}(z, u_{\epsilon 1}, z)| + |\tilde{b}(u_{\epsilon 2}, z, z)| &\leq |b(z, u_{\epsilon 1}, z)| + |b_1(z, u_{\epsilon 1}, z)| \\ + |b(u_{\epsilon 2}, z, z)| + |b_1(u_{\epsilon 2}, z, z)| &\leq c \|z\|_{L^4(\Omega)}^2 \|u_{\epsilon 1}\| + c \|u_{\epsilon 2}\|_{L^4(\Omega)} \|z\| \|z\|_{L^4(\Omega)} \\ &\leq c \|z\| \|z\| \|u_{\epsilon 1}\| + c \|u_{\epsilon 2}\| \|z\| \|z\|^{1/2} \|z\|^{1/2} \\ &\leq \frac{\nu}{2} \|z\|^2 + c_\nu \|u_{\epsilon 1}\|^2 \|z\|^2 + \frac{\nu}{2} \|z\|^2 + c_\nu \|u_{\epsilon 2}\|^4 \|z\|^2. \end{aligned}$$

It follows from (3.61) that we can write

$$\frac{1}{2} \frac{d}{dt} (|z|^2 + \epsilon |q|^2) \leq c(\|u_{\epsilon 1}\|^2 + \|u_{\epsilon 2}\|^4)(|z|^2 + \epsilon |q|^2).$$

Integrating from 0 to t we obtain

$$|z(t)|^2 + |q(t)|^2 \leq c \int_0^t (\|u_{\epsilon 1}(s)\|^2 + \|u_{\epsilon 2}(s)\|^4)(|z(s)|^2 + \epsilon |q(s)|^2) ds. \quad (3.63)$$

Applying Gronwall's inequality in (3.64), we deduce by using (3.30) that

$$u_{\epsilon 1}(t) = u_{\epsilon 2}(t) \quad \text{and} \quad p_{\epsilon 1}(t) = p_{\epsilon 2}(t) \quad \forall t \in [0, T] \quad \text{and} \quad d = 2.$$

Now supposing $d = 3$ we have $H_0^1 \hookrightarrow L^6(\Omega)$. Using lemma 3 with $s = 4$ and $r = 6$, remark 4 and Young's inequality we have

$$\begin{aligned} |\tilde{b}(z, u_{\epsilon 1}, z)| + |\tilde{b}(u_{\epsilon 2}, z, z)| &\leq |b(z, u_{\epsilon 1}, z)| + |b_1(z, u_{\epsilon 1}, z)| \\ &+ |b(u_{\epsilon 2}, z, z)| + |b_1(u_{\epsilon 2}, z, z)| \leq c \|z\|_{L^6(\Omega)} \|u_{\epsilon 1}\| |z|^{1/2} \|z\|^{1/2} \\ &\quad + c \|u_{\epsilon 2}\|_{L^6(\Omega)} \|z\| |z|^{1/2} \|z\|^{1/2} \\ &\leq \frac{\nu}{2} \|z\|^2 + c_\nu \|u_{\epsilon 1}\|^4 |z|^2 + \frac{\nu}{2} \|z\|^2 + c_\nu \|u_{\epsilon 2}\|^4 |z|^2. \end{aligned}$$

It follows from (3.61) that

$$\frac{1}{2} \frac{d}{dt} (|z|^2 + \epsilon |q|^2) \leq c(\|u_{\epsilon 1}\|^4 + \|u_{\epsilon 2}\|^4)(|z|^2 + \epsilon |q|^2).$$

Integrating from 0 to t we obtain

$$|z(t)|^2 + |q(t)|^2 \leq c \int_0^t (\|u_{\epsilon 1}(s)\|^4 + \|u_{\epsilon 2}(s)\|^4)(|z(s)|^2 + \epsilon |q(s)|^2) ds. \quad (3.64)$$

Applying Gronwall's inequality in (3.64), we deduce again by using (3.30)

$$u_{\epsilon 1}(t) = u_{\epsilon 2}(t) \quad \text{and} \quad p_{\epsilon 1}(t) = p_{\epsilon 2}(t) \quad \forall t \in [0, T] \quad \text{and} \quad d = 3.$$

Theorem 2.3 has been proved.

Proof of Theorem 2.1.

By a similar argument employed in the estimates of Theorem 2.3, we obtain that when $\epsilon \rightarrow 0$

$$(u_\epsilon) \text{ is bounded in } L^\infty(I; \mathbf{L}^2(\Omega)), \quad (3.65)$$

$$(u_\epsilon) \text{ is bounded in } L^4(I; \mathbf{H}_0^1(\Omega)), \quad (3.66)$$

$$(\sqrt{\epsilon}p_\epsilon) \text{ is bounded in } L^\infty(I; \mathbf{L}^2(\Omega)). \quad (3.67)$$

Now we will obtain some estimate to u_ϵ by using fractional derivatives. We denote by $\hat{\psi}$ the Fourier transform of function ψ . Denoting \tilde{u}_ϵ and \tilde{p}_ϵ to extension of u_ϵ and p_ϵ respectively, by zero outside interval $[0, T]$ we obtain from the (3.24)

$$\left\{ \begin{aligned} & \frac{d}{dt}(\tilde{u}_\epsilon, \varphi_r) + \nu a(\tilde{u}_\epsilon, \varphi_r) + \nu_0(\mathcal{K}\tilde{u}_\epsilon, \varphi_r) + \langle \tilde{B}\tilde{u}_\epsilon, \varphi_r \rangle - (\tilde{p}_\epsilon, \nabla \cdot \varphi_r) \\ & = \langle \tilde{f}, \varphi_r \rangle + (u_{\epsilon 0}, \varphi_r)\delta_0 - (u_\epsilon(T), \varphi_r)\delta_T, \\ & \epsilon \frac{d}{dt}(\tilde{p}_\epsilon, q_r) + (\nabla \cdot \tilde{u}_\epsilon, q_r) = \epsilon(p_{\epsilon 0}, q_r)\delta_0 - \epsilon(p_\epsilon(T), q_r)\delta_T. \end{aligned} \right. \quad (3.68)$$

Talking the Fourier transform in (3.68) and next adding equations of (3.68) with $\varphi_r = \tilde{u}_\epsilon$ and $q_r = \tilde{p}_\epsilon$, we derive

$$\begin{aligned} 2\pi i\tau|\hat{u}_\epsilon|^2 + 2\pi i\tau\epsilon|\hat{p}_\epsilon|^2 &= \langle \hat{f}, \hat{u}_\epsilon \rangle - \nu(A\hat{u}_\epsilon, \hat{u}_\epsilon) - \nu_0(\mathcal{K}\hat{u}_\epsilon, \hat{u}_\epsilon) \\ &\quad - \langle \tilde{B}\hat{u}_\epsilon, \hat{u}_\epsilon \rangle + (u_{\epsilon 0}, \hat{u}_\epsilon) - (u_\epsilon(T), \hat{u}_\epsilon)e^{-2\pi i\tau T} \\ &\quad + \epsilon(p_{\epsilon 0}, \hat{p}_\epsilon) - \epsilon(p_\epsilon(T), \hat{p}_\epsilon)e^{-2\pi i\tau T}, \end{aligned} \quad (3.69)$$

where we denoted $\hat{\tilde{u}} = \hat{u}$ and $\hat{\delta}_t = e^{-2\pi i\tau t}$. It follows from (3.69) that

$$\begin{aligned} & |\tau||\hat{u}_\epsilon|^2 + \epsilon|\tau||\hat{p}_\epsilon|^2 \\ & \leq c \left(\|\hat{f}\|_{H^{-1}} + \|A\hat{u}_\epsilon\|_{H^{-1}} + \|\mathcal{K}\hat{u}_\epsilon\|_{H^{-1}} + \|\tilde{B}\hat{u}_\epsilon\|_{H^{-1}} \right) \|\hat{u}_\epsilon\| \\ & + (u_{\epsilon 0}, \hat{u}_\epsilon) - (u_\epsilon(T), \hat{u}_\epsilon)e^{-2\pi i\tau T} + \epsilon(p_{\epsilon 0}, \hat{p}_\epsilon) - \epsilon(p_\epsilon(T), \hat{p}_\epsilon)e^{-2\pi i\tau T}. \end{aligned} \quad (3.70)$$

From the hypothesis about f we have

$$\int_0^T \|f(s)\|_{H^{-1}(\Omega)} ds \leq C.$$

Therefore, $\|\hat{f}\|_{H^{-1}(\Omega)} \leq C$. Besides, we have $|\langle Au_\epsilon, v \rangle| \leq \|u_\epsilon\| \|v\|$, $\forall u_\epsilon, v \in \mathbf{H}_0^1(\Omega)$. So, (3.66) holds

$$(Au_\epsilon) \text{ is bounded in } L^4(I; \mathbf{H}^{-1}(\Omega)) \leftrightarrow L^1(I; \mathbf{H}^{-1}(\Omega)). \quad (3.71)$$

Thus $\|A\hat{u}_\epsilon\|_{H^{-1}(\Omega)} \leq C$. We note that Schwarz's inequality, (1.8) implies that $\forall u_\epsilon, v \in \mathbf{H}_0^1(\Omega)$

$$|\langle \mathcal{K}u_\epsilon, v \rangle| \leq c_2 |e(u_\epsilon)|^3 \|v\| \leq c \|u_\epsilon\|^3 \|v\|.$$

It follows from (3.66)

$$(\mathcal{K}u_\epsilon) \text{ is bounded in } L^{4/3}(I; \mathbf{H}^{-1}(\Omega)) \hookrightarrow L^1(I; \mathbf{H}^{-1}(\Omega)). \quad (3.72)$$

Thus, $\|\mathcal{K}\hat{u}_\epsilon\|_{H^{-1}} \leq C$. Analogously we obtain $\|\tilde{B}\hat{u}_\epsilon\|_{H^{-1}} \leq C$. It follows

$$|\tau|\hat{u}_\epsilon|^2 + \epsilon|\tau|\hat{p}_\epsilon|^2 \leq c(\|\hat{u}_\epsilon\| + \epsilon|\hat{p}_\epsilon|). \quad (3.73)$$

Remark 5 Let $\gamma \in \mathbb{R}$ with $0 < \gamma < \frac{1}{4}$. Then, there exists a positive constant C , such that

$$|\tau|^{2\gamma} \leq C \frac{1 + |\tau|}{1 + |\tau|^{1-2\gamma}} \quad \forall \tau \in \mathbb{R}.$$

Remark 5 and (3.73) implies that

$$\begin{aligned} \int_{\mathbb{R}} |\tau|^{2\gamma} |\hat{u}_\epsilon|^2 dt &\leq C \int_{\mathbb{R}} \frac{1 + |\tau|}{1 + |\tau|^{1-2\gamma}} |\hat{u}_\epsilon|^2 dt \\ &\leq C \int_{\mathbb{R}} \frac{|\hat{u}_\epsilon|^2}{1 + |\tau|^{1-2\gamma}} dt + C \int_{\mathbb{R}} \frac{|\tau| |\hat{u}_\epsilon|^2}{1 + |\tau|^{1-2\gamma}} dt \\ &\leq c \int_{\mathbb{R}} \|\hat{u}_\epsilon\|^2 dt + c \int_{\mathbb{R}} \frac{\|\hat{u}_\epsilon\|^2 + \epsilon|\hat{p}_\epsilon|^2}{1 + |\tau|^{1-2\gamma}} dt. \end{aligned} \quad (3.74)$$

From Plancherel identity we have

$$\int_{\mathbb{R}} \|\hat{u}_\epsilon\|^2 dt = \int_{\mathbb{R}} \|\tilde{u}_\epsilon\|^2 dt \leq c. \quad (3.75)$$

Besides, from Hölder's inequality and (3.66) we can write

$$\int_{\mathbb{R}} \frac{\|\hat{u}_\epsilon\|}{1 + |\tau|^{1-2\gamma}} dt \leq \left(\int_{\mathbb{R}} \|\hat{u}_\epsilon\|^2 dt \right)^{1/2} \left[\int_{\mathbb{R}} \left(\frac{1}{1 + |\tau|^{1-2\gamma}} \right)^2 dt \right]^{1/2} \leq c. \quad (3.76)$$

By using the same argument and (3.67) we obtain

$$\int_{\mathbb{R}} \frac{|\hat{p}_\epsilon|}{1 + |\tau|^{1-2\gamma}} dt \leq c. \quad (3.77)$$

From (3.73), (3.76) and (3.77) we conclude that

$$\int_{\mathbb{R}} |\tau|^{2\gamma} |\hat{u}_\epsilon|^2 dt \leq c, \quad 0 < \gamma < \frac{1}{4}. \quad (3.78)$$

In other words,

$$|\tau|^\gamma \hat{u}_\epsilon \in L^2(\mathbb{R}; \mathbf{L}^2(\Omega)), \quad 0 < \gamma < \frac{1}{4}. \quad (3.79)$$

Combining (3.66) and (3.79) we conclude that \hat{u}_ϵ is bounded in

$$\mathcal{H}(\mathbb{R}; \mathbf{H}_0^1(\Omega), \mathbf{L}^2(\Omega)) = \{\hat{u}_\epsilon; \hat{u}_\epsilon \in L^2(\mathbb{R}; \mathbf{H}_0^1(\Omega)), |\tau|^\gamma \hat{u}_\epsilon \in L^2(\mathbb{R}; \mathbf{L}^2(\Omega))\}.$$

This implies that \tilde{u}_ϵ is bounded in $\mathcal{H}(\mathbb{R}; \mathbf{H}_0^1(\Omega), \mathbf{L}^2(\Omega))$. Thus, u_ϵ is bounded in $\mathcal{H}(I; \mathbf{H}_0^1(\Omega), \mathbf{L}^2(\Omega))$. But the embedding $\mathcal{H}(I; \mathbf{H}_0^1(\Omega), \mathbf{L}^2(\Omega)) \hookrightarrow L^2(I; \mathbf{L}^2(\Omega))$ is compact (see for instance Lions [5]). It follows from this and from the limitations obtained that, there is a subsequence from (u_ϵ) , still denoted by (u_ϵ) , such that

$$\begin{aligned} u_\epsilon &\longrightarrow u \text{ strong in } L^2(I; \mathbf{L}^2(\Omega)) \text{ and a.e. in } Q_T, \\ u_\epsilon &\longrightarrow u \text{ weak star in } L^\infty(I; \mathbf{L}^2(\Omega)), \\ u_\epsilon &\longrightarrow u \text{ weak in } L^4(I; \mathbf{H}_0^1(\Omega)), \\ \mathcal{K}u_\epsilon &\longrightarrow \chi \text{ weak in } L^{4/3}(I; \mathbf{H}^{-1}(\Omega)). \end{aligned} \tag{3.80}$$

From (3.67) we obtain $\epsilon p'_\epsilon \rightarrow 0$ in $\mathcal{D}'(Q)$. Therefore $\nabla \cdot u_\epsilon = -\epsilon p'_\epsilon \rightarrow 0$ in $\mathcal{D}'(Q)$. Combining this and (3.80) we derive $\nabla \cdot u = 0$. Thus, the convergences above permits us to write

$$\left\{ \begin{array}{l} \langle u', v \rangle + \nu a(u, v) + \nu_0 \langle \chi, v \rangle + \langle Bu, v \rangle = \langle f, v \rangle \quad \forall v \in \mathcal{D}(\Omega), \\ \nabla \cdot u = 0, \end{array} \right. \tag{3.81}$$

because $\nabla \cdot u = 0$ implies $\tilde{b}(u, u, v) = b(u, u, v)$. Finally, we can prove that $\mathcal{K}u = \chi$ by using the same argument used in the proof of Theorem 2.1.

Proof of Theorem 2.2.

We employ the method of Faedo-Galerkin again with the basis of eigenvectors from the Stoke's operator (φ_r) . Let $V_m = [\varphi_1, \dots, \varphi_m] \subset \mathbf{H}_0^1(\Omega)$ the subspace generated by $\{\varphi_1, \dots, \varphi_m\}$. So the approximate problem to (2.19) is given by

$$\left\{ \begin{array}{l} (u'_m(t), v) + \nu(Au_m(t), v) + \nu_0(\mathcal{K}u_m(t), v) \\ + \langle Bu_m(t), v \rangle = (f(t), v) \quad \forall v \in V, \\ u_m(0) = u_{0m}, \quad u_{0m} \rightarrow u_0, \quad \text{strong in } \mathbf{L}^2(\Omega). \end{array} \right. \tag{3.82}$$

We know that system (3.82) has a local solution defined on the interval $[0, T]$ and given by

$$u_m(x, t) = \sum_{r=1}^m g_{rm}(t) \varphi_r(x). \tag{3.83}$$

We first show the existence of periodic solutions to system (3.82). For this purpose, we make $v = u_m(t)$ in (3.82) to obtain, by using the same arguments used in the proof of Theorem (2.3) to $\langle \mathcal{K}u_m, v \rangle$

$$\frac{1}{2} \frac{d}{dt} |u_m(t)|^2 + \nu \|u_m(t)\|^2 + \nu_2 \|u_m\|^4 \leq \|f(t)\|_{H^{-1}} \|u_m(t)\|. \quad (3.84)$$

Considering the embedding $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$ and using Young' inequality we derive

$$\frac{1}{2} \frac{d}{dt} |u_m(t)|^2 + c |u_m(t)|^2 + \nu_2 \|u_m\|^4 \leq \nu_2 \|u_m(t)\|^4 + c_{\nu_2} \|f(t)\|_{H^{-1}}^{4/3}. \quad (3.85)$$

That is,

$$\frac{d}{dt} |u_m(t)|^2 + |u_m(t)|^2 \leq c \|f(t)\|_{H^{-1}}^{4/3}. \quad (3.86)$$

Multiplying both sides of (3.86) by e^t we get

$$e^t \frac{d}{dt} |u_m(t)|^2 + e^t |u_m(t)|^2 \leq c \|f(t)\|_{H^{-1}}^{4/3} e^t. \quad (3.87)$$

In other words,

$$\frac{d}{dt} (e^t |u_m(t)|^2) \leq c \|f(t)\|_{H^{-1}}^{4/3} e^t. \quad (3.88)$$

By integrating (3.88) on interval $[0, T]$ we obtain

$$e^T |u_m(T)|^2 \leq |u_m(0)|^2 + c \int_0^T \|f(t)\|_{H^{-1}}^{4/3} e^t dt. \quad (3.89)$$

Therefore

$$|u_m(T)|^2 \leq \theta(T) |u_m(0)|^2 + C, \quad (3.90)$$

where $\theta(T) = e^{-T}$. We have $0 < 1 - \theta(T) < 1$. Now let R be positive constant such that

$$\frac{C}{1 - \theta(T)} < R^2. \quad (3.91)$$

Choosing $u_m(0) \in V_m$ such that $|u_m(0)| < R$ we obtain from (3.90) and (3.91)

$$|u_m(T)|^2 < \theta(T) R^2 + R^2 (1 - \theta(T)) = R^2.$$

Let $\sigma : \mathcal{B}_R(0) \cap V_m \rightarrow \mathcal{B}_R(0) \cap V_m$, be a nonlinear mapping such that $\sigma(u_m(0)) = u_m(T)$ and $\mathcal{B}_R(0) = \{u \in L^2(\Omega); |u| < R\}$. We can establish the continuous dependence of the solution with respect initial data. Therefore, from the Brouwer Fixed-Point Theorem there is $u_{0m} \in V_m$ such that $\sigma(u_{0m}) = u_{0m}$. In other words, $u_m(0) = u_m(T)$.

Therefore, the system (3.82) has a periodic solution u_m . By using the initial data $u_m(0)$ in (3.82) we can obtain as in proof of Theorem 2.3 that there is subsequence of u_m , still denoted by u_m such that

$$u_m \rightarrow u \text{ weak star in } L^\infty(I; \mathbf{L}^2(\Omega)), \quad (3.92)$$

$$u_m \rightarrow u \text{ weak in } L^4(I; \mathbf{H}_0^1(\Omega)), \quad (3.93)$$

$$u'_m \rightarrow u' \text{ weak in } L^{4/3}(I; \mathbf{H}^{-1}(\Omega)). \quad (3.94)$$

We established in Theorem 2.1 that u is a solution to problem (1.6) in the sense of the definition (1). We will show now that $u(0) = u(T)$. In fact, from (3.93) and (3.94)

$$\int_0^T \frac{d}{dt} (u_m(s), v) \theta(s) ds \rightarrow \int_0^T \frac{d}{dt} (u(s), v) \theta(s) ds, \quad (3.95)$$

$\forall \tilde{v} \in \mathbf{H}_1^0(\tilde{\Omega})$ and $\theta \in \mathcal{D}(0, T)$, with $\theta(T) = 0$. In other words,

$$(u_m(0), v) \rightarrow (u(0), v) \quad \forall v \in \mathbf{H}_0^1(\Omega). \quad (3.96)$$

By using the same argument with $\theta \in \mathcal{D}(0, T)$ and $\theta(0) = 0$ we derive

$$(u_m(T), v) \rightarrow (u(T), v) \quad \forall v \in \mathbf{H}_0^1(\Omega). \quad (3.97)$$

It follows from (3.96) and (3.97) that $u(0) = u(T)$. This concludes the proof of the Theorem 2.2.

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ÍNDICE REMISSIVO

A

Alunos cegos 71, 74, 75, 76, 80, 82, 119, 120

Análise combinatória 154, 156, 157, 159

Aprendizagem 1, 2, 5, 10, 13, 16, 17, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 38, 40, 42, 43, 44, 45, 57, 58, 59, 60, 61, 62, 71, 72, 73, 74, 78, 79, 80, 81, 82, 83, 84, 85, 86, 87, 88, 89, 90, 91, 92, 94, 95, 96, 97, 98, 99, 100, 101, 102, 107, 108, 109, 110, 111, 112, 113, 114, 116, 117, 118, 120, 122, 123, 124, 125, 160, 161, 162, 163, 164, 171, 192, 208, 210, 211, 213, 216, 217, 218, 220, 221, 223, 228

Arduíno 1, 3, 4, 6

Arquimedes 154, 155, 156, 157, 159

Atividade remota 18

Atividades exploratórias 85, 86, 87, 91, 92, 95, 97, 98, 108, 109, 112, 116

Auto-similaridade 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 54, 55

B

BNCC 1, 2, 10, 155, 157, 159, 163, 191, 192, 193, 207

C

Curso superior 57, 58

D

Desenvolvimento 5, 12, 13, 16, 19, 22, 24, 26, 27, 28, 29, 30, 31, 32, 37, 42, 43, 46, 49, 58, 60, 61, 73, 75, 85, 86, 88, 91, 92, 95, 101, 102, 106, 110, 115, 118, 120, 121, 126, 139, 142, 143, 151, 152, 153, 154, 159, 163, 164, 165, 192, 208, 209, 213, 217, 218, 221, 222, 228, 230

Desigualdade de Caffarelli-Kohn-Nirenberg (CKN) 63, 65, 66, 67

Desigualdade de Sobolev 63, 64, 67

Desigualdade do tipo Hardy 63

Dificuldade de aprendizagem 24

E

Educação 4, 10, 12, 13, 14, 17, 18, 20, 22, 23, 24, 25, 26, 27, 28, 29, 31, 32, 33, 34, 35, 36, 38, 40, 41, 42, 43, 44, 51, 55, 58, 62, 71, 72, 74, 75, 76, 77, 78, 81, 83, 84, 86, 88, 89, 91, 92, 93, 98, 99, 100, 102, 107, 109, 111, 117, 118, 119, 120, 121, 122, 123, 125, 127, 139, 140, 141, 142, 143, 152, 154, 159, 160, 163, 171, 207, 210, 217, 218, 221, 228, 229, 230

Educação matemática 10, 12, 13, 14, 24, 25, 28, 29, 31, 32, 33, 42, 43, 55, 58, 62, 81, 86, 88, 91, 92, 93, 98, 99, 100, 102, 107, 111, 117, 118, 119, 122, 127, 139, 140, 141, 142, 143, 152, 154, 159, 160, 171, 210, 218, 221, 229, 230

Ensino 1, 2, 3, 4, 10, 12, 13, 14, 16, 17, 18, 19, 20, 22, 23, 24, 25, 26, 27, 29, 30, 31, 32, 33, 34, 35, 36, 38, 40, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 54, 55, 57, 58, 59, 60, 62, 71, 72, 73, 74, 75, 76, 78, 79, 80, 83, 85, 86, 87, 88, 89, 90, 91, 92, 96, 97, 98, 99, 100, 101, 102, 103, 106, 107, 108, 109, 110, 111, 112, 117, 118, 120, 121, 122, 126, 141, 142, 143, 148, 151, 154, 155, 157, 159, 160, 161, 162, 163, 164, 170, 171, 192, 193, 208, 209, 210, 211, 212, 217, 218, 219, 220, 221, 223, 228, 229, 230

Ensino básico 142, 151, 154, 155, 157, 159

Ensino de matemática 13, 30, 33, 57, 143, 229, 230

Ensino fundamental 10, 17, 24, 29, 79, 83, 100, 101, 103, 111, 118, 120, 160, 163, 164, 171, 192, 208, 209, 211, 212, 217, 218, 219, 220, 228, 229

Ensino superior 18, 19, 20, 22, 47, 58, 62, 91, 97, 171, 230

Estatística 5, 10, 108, 109, 111, 112, 113, 114, 116, 117, 118, 143, 230

Estudo orientado 18, 22

Excel 60, 108, 109, 111, 112, 113, 114, 116, 196, 198, 206

Experiência 18, 20, 22, 23, 27, 34, 35, 36, 38, 40, 57, 58, 59, 60, 61, 62, 74, 79, 80, 101, 120, 127, 140, 167, 192, 202, 218, 219, 228

F

Física 1, 4, 10, 64, 121, 170, 171, 192, 229

Fração 208, 210, 212, 213, 214, 215, 216, 218

Fractais 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 54, 55

Função do 1º grau 71, 72, 73, 74, 76

Funções polinomiais 85, 86, 90, 92

G

Geometria 23, 36, 38, 62, 66, 67, 154, 156, 157, 160, 161, 165, 193, 220, 222

Grounded theory 139, 140, 141, 143, 151, 152, 153

H

Hermite 191, 192, 194, 195, 197, 198, 199, 200, 202, 205, 206, 207

História da matemática 154, 155, 159

I

Imunidade coletiva 128, 129, 132, 133, 137

Inclusão 20, 21, 22, 71, 74, 75, 76, 78, 80, 81, 83, 84, 120, 121, 122, 127

Instrumento educativo 100

Instrumentos de pesquisa 139

Interdisciplinaridade 12, 13, 16, 17, 24, 25, 33

Interpolação 67, 68, 191, 192, 193, 194, 199, 206, 207

Itinerário formativo 191, 192, 193

J

Jogos 12, 13, 14, 16, 17, 30, 157, 193

M

Matemática 1, 2, 3, 4, 10, 12, 13, 14, 16, 17, 18, 19, 20, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 40, 41, 42, 43, 44, 46, 47, 48, 49, 51, 55, 57, 58, 59, 60, 61, 62, 63, 64, 71, 72, 73, 74, 76, 77, 78, 79, 80, 81, 83, 84, 85, 86, 87, 88, 89, 90, 91, 92, 93, 95, 96, 97, 98, 99, 100, 101, 102, 103, 106, 107, 108, 109, 110, 111, 112, 113, 114, 116, 117, 118, 119, 120, 122, 124, 126, 127, 129, 132, 138, 139, 140, 141, 142, 143, 148, 150, 151, 152, 154, 155, 156, 157, 159, 160, 161, 162, 163, 164, 167, 170, 171, 172, 191, 192, 193, 207, 210, 218, 219, 221, 222, 228, 229, 230

Material concreto 27, 74, 100, 101, 103, 124

MATLAB 191, 192, 199, 206, 207

Metodologia de pesquisa 91, 111, 139, 153

Metodologias ativas 57, 58, 59, 61, 62

Modelos matemáticos 128, 129

N

Narrativas 119, 120, 122, 123, 124, 125, 127, 230

O

Operações 16, 27, 29, 36, 38, 85, 88, 100, 104, 111, 112, 113, 114, 115, 116, 208, 209, 210, 212, 214, 217

Origami 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 54, 55

P

Papel do professor 24, 30, 32, 57, 109, 148, 217

Pesquisa educacional 139

Pesquisa qualitativa 5, 10, 41, 80, 85, 98, 109, 127, 139, 152, 171

Projeto investigativo 57, 58, 60, 61

R

Resolução de problemas 29, 46, 58, 59, 76, 103, 160, 161, 162, 163, 164, 167, 170, 171, 192, 193, 211, 217, 224

Rigidez 63, 67, 68

Robótica educacional 1, 2, 5, 10

S

Saberes experienciais 85, 87

SEIR 128, 129, 134, 135, 136, 137

Semelhança de triângulos 160, 161, 165, 167, 170, 219, 221, 224, 225, 227, 228

SIR 128, 129, 130, 131, 132, 133, 134, 135, 136, 137, 138

Sistema NODET 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 54, 55

Software GeoGebra 85

Stomachion 154, 155, 156, 157, 158, 159

T

Técnicas 33, 36, 60, 76, 77, 84, 121, 139, 140, 143, 152, 156, 162, 163, 167, 207, 208, 217

Teoria das situações didáticas 111, 118, 208, 209, 210, 211

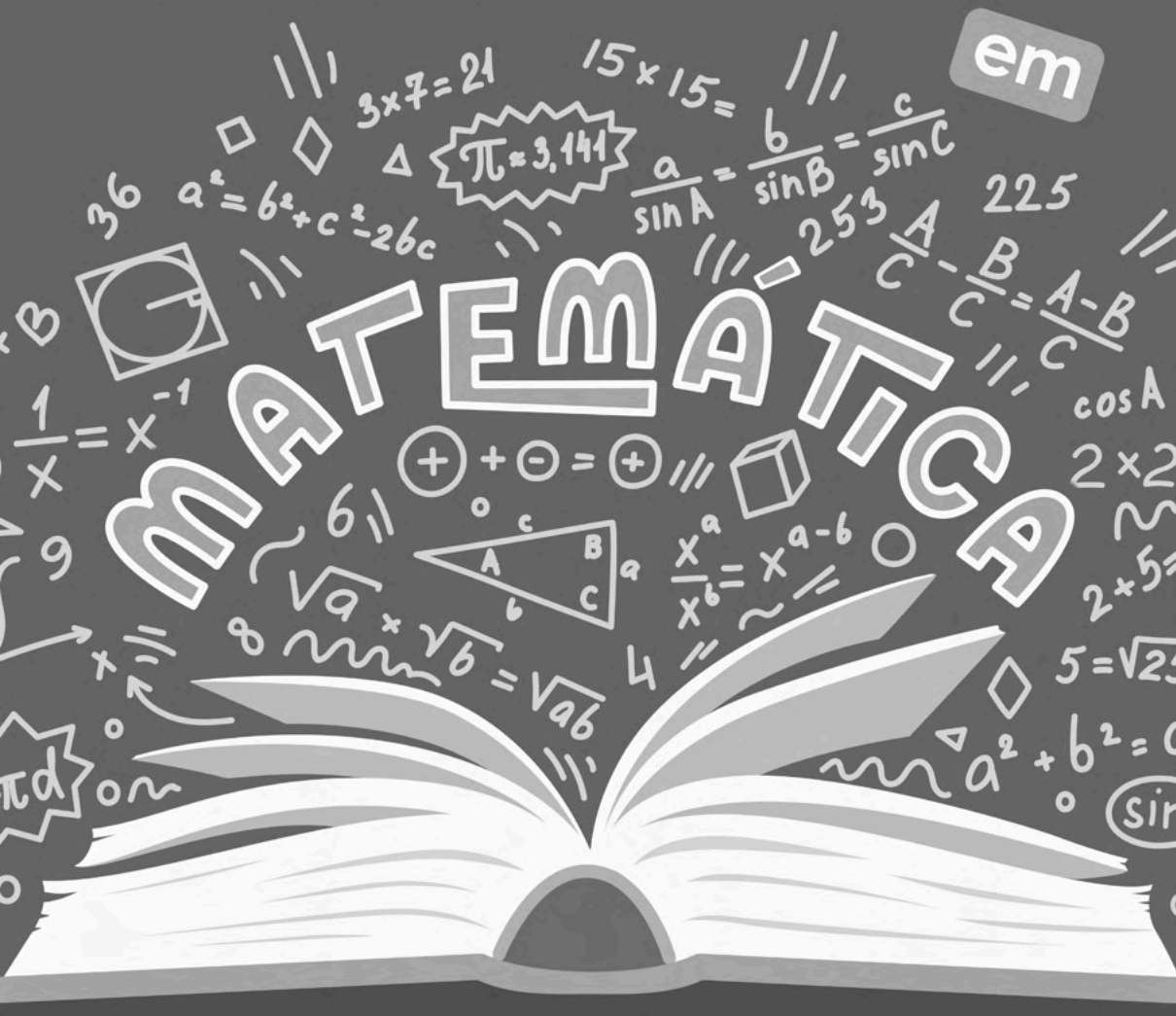
Transposição didática 71, 75, 76, 77, 78, 80, 81

V

Variedades Riemannianas 63, 64, 66, 67, 68

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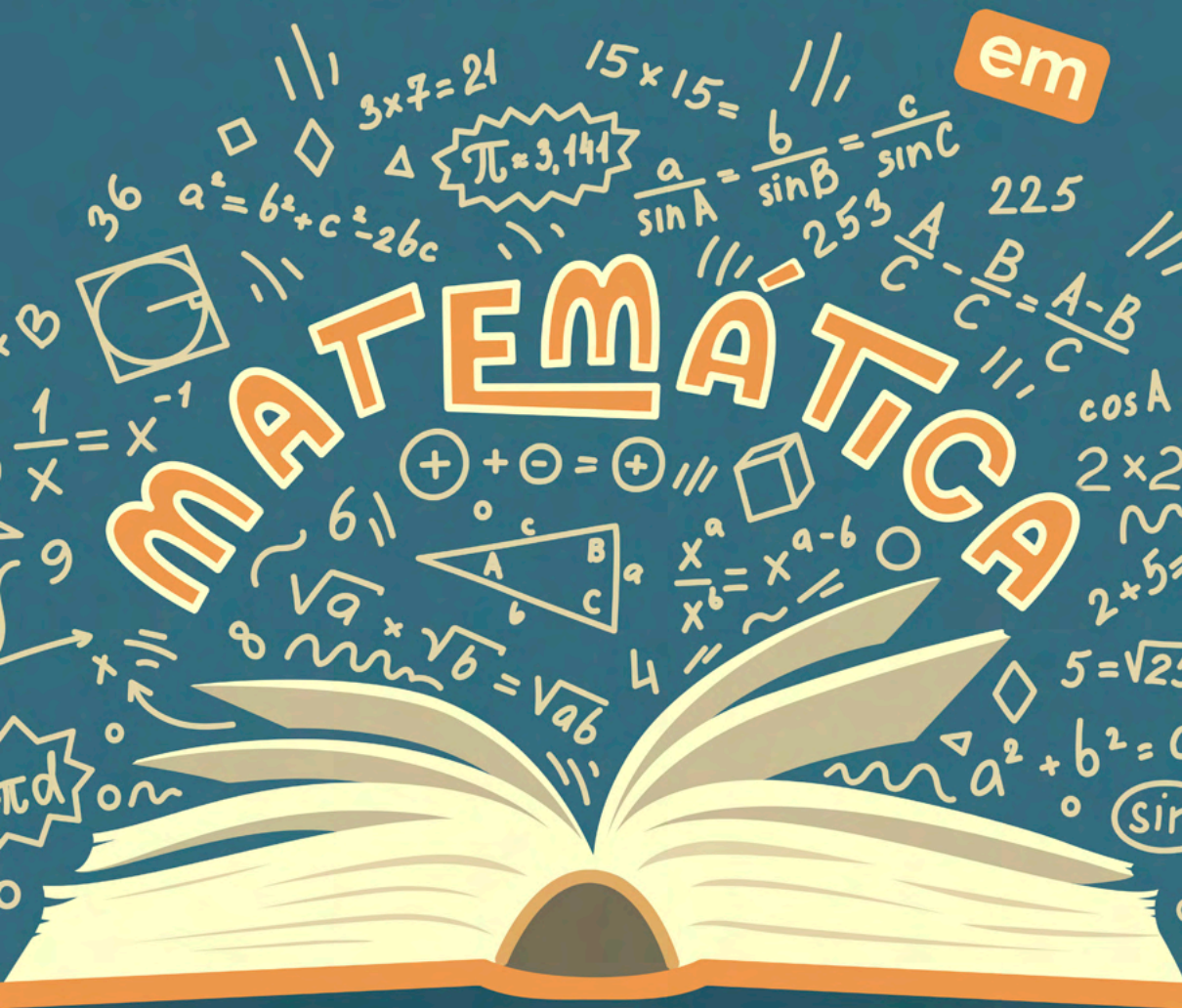


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