INVESTIGAÇÃO, CONSTRUÇÃO E DIFUSÃO DO CONHECIMENTO EM MATEMÁTICA

JOSÉ ELYTON BATISTA DOS SANTOS
(ORGANIZADOR)

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APRESENTAÇÃO

A coletânea “Investigação, Construção e Difusão do Conhecimento em Matemática” é uma obra composta por 27 artigos que tem como foco principal a difusão de conhecimentos na dimensão matemática perante a uma diversidade de trabalhos. O livro apresenta produções científicas do âmbito nacional e internacional em formato de relatos de casos, estudos bibliográficos e experimentais com temáticas relevantes para a comunidade científica, para professores em exercício e aos que estão aperfeiçoando seus conhecimentos acerca do que está sendo pesquisado, debatido e proposto no ensino da educação básica, bem como no ensino superior.

A relevância da matemática nos diferentes níveis educacionais é imensurável. Em todo canto e em toda situação a matemática está presente. Perante esse contexto, esta obra fomenta as pesquisas na área da educação matemática, dissemina os conhecimentos científicos a partir das diferentes visões teóricas e estudos contemplados pela referida área, a saber: etnomatemática, tecnologias, recursos didáticos, formação de professores e modelagem matemática. Também se insere nessa dimensão da difusão do conhecimento, as propostas interdisciplinares e conteudista para a educação básica e ensino superior, que visa primordialmente a aprendizagem com qualidade e de acordo com as exigências da sociedade contemporânea, isto é, um ensino próximo ao contexto do aluno.

Debruçar nessa coletânea permite ao leitor se aventurar por diferentes conhecimentos científicos. Ampliará seus conhecimentos teóricos, bem como, enriquecerá sua prática docente a partir dos relatos com materiais concretos, tecnológicos e problemas contextualizados. Todavia, desejo que esta obra contribua significativamente não apenas para o enriquecimento teórico e prático, mas como meio motivador para novas investigações e consequentemente para a difusão do conhecimento científico matemático.

José Elyton Batista dos Santos

SUMÁRIO

CAPÍTULO 24

A NUMERICAL APPROXIMATION FOR SOLUTIONS OF FREDHOLM FUNCTIONAL-INTEGRAL EQUATIONS BY CHEBYSHEV TAU METHOD

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ABSTRACT: In this work we consider the general functional-integral equation:

$$u(x) = f\left(x, \int_0^1 \kappa(x,y)g\big(y,u(y)\big)dy\right) + h(x), \qquad x \in [0,1]$$

We use Chebyshev Tau method to determine the approximations of the solutions and provide a convergence analysis in the space of square-integrable function. We added some numerical experiments to illustrate the results of this work.

KEYWORDS: Fredholm functional-integral equation, Chebyshev Tau method.

UMA APROXIMAÇÃO NUMÉRICA PARA SOLUÇÕES DE EQUAÇÕES INTEGRAL-FUNCIONAIS DE FREDHOLM PELO MÉTODO DE TAU CHEBYSHEV

RESUMO: Neste trabalho consideramos a equação integral-funcional geral:

$$u(x) = f\left(x, \int_0^1 \kappa(x,y)g\big(y,u(y)\big)dy\right) + h(x), \qquad x \in [0,1]$$

Usamos o método de Tau Chebychev para determinar as aproximações das soluções e fornecemos uma análise de convergência no espaço das funções quadrado integráveis. Adicionamos alguns exemplos numéricos para ilustrar os resultados deste trabalho.

PALAVRAS-CHAVE: Equação integral-

funcional de Fredholm, Método de Tau Chebyshev.

1 I INTRODUCTION

Functional-integral equations of several type have been studied in a vast literature (Browder, 1971, Sheng et al., 2016, Rocha et al., 2018) and references therein. Motivated by these works, we consider the following nonlinear functional-integral

$$u(x) = f\left(x, \int_0^1 \kappa(x,y)g\big(y,u(y)\big)dy\right) + h(x), \qquad x \in [0,1]. \qquad (1.1)$$

Equations of this type arise very often in many applications in engineering, economics and mathematical physics (Banaś; Knap, 1989, Emmanuele, 1991), justifying the importance of your study.

The main purpose is to implement the Tau method for nonlinear functional-integral equation and produce an error analysis which theoretically justifies an exponential-like rate of convergence. The numerical discretization is done with the Chebyshev Tau method (Ansari and Mokhtary, 2019), resulting in a system of nonlinear equations that are subsequently solved by using a Picard iterative method (Rocha et al., 2018). In this step we show that the approximate solution converges, under particular conditions, to an exact solution in space of square-integrable function, and finally analyze the rate of its convergence.

In the following section, we present preliminary concepts and highlight that, under certain conditions, eq. (1.1) has a unique solution in L^2 ([0,1]) space, which can be obtained as the limit of successive approximations.

2 I PRELIMINARIES

Let $h: [0,1] \to \mathbb{R}$ be a Lebesgue measurable function. We say that h is square integrable in [0,1] or that h belongs to L^2 ([0,1]) space if

$$\int_0^1 |h(x)|^2 dx < \infty.$$

Moreover, if $h \in L^2([0,1])$, we define

$$||h||_2 = \left(\int_0^1 |h(x)|^2 dx\right)^{1/2}.$$

In what follows, we assume that $h: \mathbb{R} \to \mathbb{R}$ is square integrable in [0.1] and the function $f:[0,1]\times\mathbb{R}\to\mathbb{R}$ in (1.1) satisfies the Caratheodory conditions, that is,

i. $f(x,y)$ is continuous in x for each fixed y;

ii. $f(x,y)$ is measurable in x for each fixed y;

iii. .There is a non-negative Lebesgue-integrable function $m:[0,1]\to\mathbb{R}$ such that

$$|f(x,y)|\leqslant m(x), \text{ for all } (x,y)\in[0,1]\times\mathbb{R}.$$

Theorem 2.1. Assume that the following conditions are satisfied:

(A1) There are a non-negative function $h_1\in L^2([0,1])$ and a non-negative constant b_1 such that

$$|f(x,y)|\leqslant h_1(x)+b_1|y|,$$

for almost every $x\in[0,1], y\in\mathbb{R}$.

(A2) The kernel $k\,(x,.)$ is measurable, belongs to the space for all L^2 ([0,1]), x ϵ [0,1], and

$$\left(\int_0^1|\kappa(x,y)|^2dy\right)^{1/2}\leq M_1(x), \quad \text{for any } x\in[0,1],$$

where M_1 is a non-negative function in $L^2([0,1])$.

(A3) The function $g:[0,1]\times\mathbb{R}\to\mathbb{R}$ satisfies Caratheodory conditions and

$$|g(s,z)|\leqslant b_0(s)+c_0|z|,$$

where b_0 is a non-negative function in L^2 ([0,1]) and C_0 is a non-negative constant.

(A4) The function $f:[0,1]\times\mathbb{R}\to\mathbb{R}$ satisfies Lipschitz condition in the second variable, i.e., there is $M>0$ such that

$$|f(x,y_1)-f(x,y_2)|\leqslant M|y_1-y_2|, \quad \text{for any } x\in[0,1] \text{ and } y_{1,}\, y_2\in\mathbb{R}.$$

(A5) The function $g:[0,1]\times\mathbb{R}\to\mathbb{R}$ satisfies Lipschitz condition in the second variable, i.e., there is $L>0$ such that

$$|g(s,z_1)-g(s,z_2)|\leqslant L|z_1-z_2|, \quad \text{for any } s\in[0,1] \text{ and } z_{1,}\, z_2\in\mathbb{R}.$$

Under such hypotheses, the successive approximation

$$\begin{cases} u_0(x)=0, \\ u_{n+1}(x)=f\left(x,\int_0^1\kappa(x,y)g(y,,u_n(y))dy\right)+h(x), \quad n=0,1,2,3,\ldots \end{cases}$$

converges almost everywhere to the exact solution of (1.1) provided

$$\int_0^1 C^2|M_1(x)|^2dx := \Gamma^2 < 1, \quad \text{where } C = ML. \qquad (2.1)$$

Therefore, eq. (1.1) has a unique solution in $L^2([0,1])$, which can be obtained as the limit of successive approximations.

Proof. The ideas to prove this result follow from (Afonso et al., 2019). Basically, we consider the operator

$$(Au)(x) = f\left(x, \int_0^1 \kappa(x,y)g\big(y,u(y)\big)dy\right) + h(x), \qquad x \in [0,1],$$

and we show that it is a map from $L^2([0,1])$ into $L^2([0,1])$.

By virtue of condition (2.1), $A: L^2([0,1]) \to L^2([0,1])$ is a contractive map. We apply Banach's fixed point theorem and obtain a unique solution to the abstract equation $Au = u$. Consequently, it is possible to guarantee the unique solvability of (1.1), where the method of successive approximation converges for starting point $u_0 \equiv 0$.

3 I NUMERICAL METHOD

Let us suppose that

$$u(x) = \sum_{j=1}^{\infty} a_j\phi_j(x), \qquad (3.1)$$

and $u_N(x)$ represents the Tau approximation, i.e.,

$$u(x) \approx u_N(x) = \sum_{j=1}^{N} a_j\phi_j(x) = \boldsymbol{u}_N^T\boldsymbol{\Phi}(x), \qquad (3.2)$$

where

$$\boldsymbol{u}_N = (a_1, a_2, \dots, a_N, 0, \dots)^T \text{ and } \boldsymbol{\Phi}(x) = (\phi_1(x), \phi_2(x), \dots, \phi_N(x), 0, \dots)^T$$

are shifted Chebyshev polynomials of degree defined on interval [0,1].

Assume also that

$$h(x) = \sum_{j=1}^{\infty} h_j\phi_j(x) = \boldsymbol{h}_N^T\boldsymbol{\Phi}(x), \qquad (3.3)$$

with $\boldsymbol{h}_N = (h_1, h_2, \dots, h_N, 0, \dots)^T$.

Substituting relations (3.2) and (3.3) in (1.1), we get

$$\boldsymbol{u}_N^T\boldsymbol{\Phi}(x) = \boldsymbol{h}_N^T\boldsymbol{\Phi}(x) + f\left(x, \int_0^1 \kappa(x,y)g\big(y, \boldsymbol{u}_N^T\boldsymbol{\Phi}(y)\big)dy\right). \qquad (3.4)$$

Let $\boldsymbol{x} = (x_1, x_2, \dots, x_N)^T$ be the Chebyshev nodes and the vector that contains values of the integral (3.4) at the same nodes. Following (Driscoll, 2010), the integral into (3.4) can be approximated in matrix form

$$\boldsymbol{V} = \left[\begin{pmatrix} \kappa(x_1,x_1) & \kappa(x_1,x_2) & \cdots & \kappa(x_1,x_N) \\ \kappa(x_2,x_1) & \kappa(x_2,x_2) & \cdots & \kappa(x_N,x_N) \\ \vdots & \vdots & \ddots & \vdots \\ \kappa(x_N,x_1) & \kappa(x_N,x_2) & \cdots & \kappa(x_N,x_N) \end{pmatrix}\right] \begin{bmatrix} w_1 & & & \\ & w_2 & & \\ & & \ddots & \\ & & & w_N \end{bmatrix} g\left(\boldsymbol{x}, \boldsymbol{u}_N^T \boldsymbol{\Phi}(\boldsymbol{x})\right), \quad (3.5)$$

where

$$g\left(\boldsymbol{x}, \boldsymbol{u}_N^T \boldsymbol{\Phi}(\boldsymbol{x})\right) = \left(g\left(x_1, \boldsymbol{u}_N^T \boldsymbol{\Phi}(x_1)\right), g\left(x_2, \boldsymbol{u}_N^T \boldsymbol{\Phi}(x_2)\right), \dots, g\left(x_N, \boldsymbol{u}_N^T \boldsymbol{\Phi}(x_N)\right)\right)^T$$

and the set $\{w_i\}$ for fixed i are quadrature weights relative to integration interval [0,1], with the integration nodes $\{x_i\}$. For more details on the numerical implementation of this integral see (Trif; Ionescu, 2011).

Note that expression (3.5) can be written as

$$\boldsymbol{V} = \mathbf{K}\boldsymbol{W} g\left(\boldsymbol{x}, \boldsymbol{u}_N^T \boldsymbol{\Phi}(\boldsymbol{x})\right). \qquad (3.6)$$

Using collocation about N Chebyshev nodes and substituting relation (3.6) into (3.4), we obtain

$$\boldsymbol{u}_N \boldsymbol{\Phi}(\boldsymbol{x}) = \boldsymbol{h}_N^T \boldsymbol{\Phi}(\boldsymbol{x}) + f\left(\boldsymbol{X}, \mathbf{K}\boldsymbol{W} g\left(\boldsymbol{x}, \boldsymbol{u}_N^T \boldsymbol{\Phi}(\boldsymbol{x})\right)\right).$$

Due to the nonlinearity of this expression, we resort to an iterative procedure. In this case we choose the Picard iterative method that consists of determining a sequence (u_N^{k+1}) in which the following relation of recurrence holds

$$\boldsymbol{u}_N^{(\mathbf{k+1})} \boldsymbol{\Phi}(\boldsymbol{x}) = \boldsymbol{h}_N^T \boldsymbol{\Phi}(\boldsymbol{x}) + f\left(\boldsymbol{x}, \mathbf{K}\boldsymbol{W} g\left(\boldsymbol{x}, \left(\boldsymbol{u}_N^{(K)}\right)^{\boldsymbol{T}} \boldsymbol{\Phi}(\boldsymbol{x})\right)\right), \quad k \geq 0.$$

The procedure is terminated provided that one of the following criteria is satisfied

$$k > k_{max} \qquad \text{or} \qquad r_k = ||\boldsymbol{u}_N^{(k)} - \boldsymbol{u}_N^{(k-1)}|| < tol_1.$$

Once we get the solution vector $\boldsymbol{u}_N^{(k)}$ we have to relate it to the basis functions $\{\phi_j\}_{j=1}^N$ as follows

$$\boldsymbol{u}_N^{(k)} \boldsymbol{\Phi}(x) = \sum_{j=1}^{N} a_j^{(k)} \phi_j(x) = u_N^{(k)}(x).$$

Finally, we obtain a solution $u_N^{(k)}$ that must satisfies

$$u_N^{(k)}(x) = h(x) + f\left(x, \int_0^1 k(x,y) g\left(y, u_N^{(k)}(y)\right) dy\right) + h(x), \forall x \in [0,1].$$

4 I CONVERGENCE ANALYSIS

We provide a convergence analysis of the proposed method for the numerical solution of (1.1) In our subsequent analysis, some definitions and results are needed.

Firstly, we introduce the orthogonal projection

$$\Pi_N : L^2([a,b]) \to \mathcal{P}_N$$

which is a mapping that, for any $v \in L^2([0,1])$, satisfies

$$(v - \Pi_N v, \phi) = 0, \qquad \forall \phi \in \mathcal{P}_N,$$

where P_N is the space of all algebraic polynomials of degree up to N. Concerning the truncation error of a Chebyshev interpolation, the following estimates holds (Atkinson; Han, 2009)

$$\|v - \Pi_N v\|_2 \leq N^{-m} |v|_m, \qquad |v|_m = \left\|\frac{\partial^m v}{\partial x^m}\right\|_2 \qquad (4.1)$$

for $m > 0$.

The next result is devoted to provide a convergence analysis for the numerical scheme showing that the rate of convergence is exponential. Consider integral operator

$$(Ku)(x) = \int_0^1 \kappa(x,y) g\big(y, u(y)\big) dy$$

and Nemytskii operator

$$(Fu)(x) = f\big(x, u(x)\big).$$

Theorem 4.1. Let u_N be the Tau approximation $e_N(x) = u(x) - u_N(x)$ of the exact solution of the Fredholm nonlinear functional-integral equations and consider the error function $e_N(x) = u(x) - u_N(x)$ and the operator

$$(FKu_N)(x) = f\left(x, \int_0^1 \kappa(x,y) g\big(y, u_N(y)\big) dy\right).$$

If the functions k, f and g are sufficiently smooth and satisfy assumptions (A2), (A4) and (A5), respectively, moreover condition is satisfied, then we have

$$\|e_N\|_2 \leq D\left(N^{-k_1}|h|_{k_1} + N^{-k_2}|(FKu_N)(x)|_{k_2} + N^{-k_3}|\kappa|_{k_3}\|u_N\|_2\right),$$

where $k_i \geq 0, i = 1,2,3$, given that is sufficiently large and D is a constant independent of N.

Proof. The ideas of this proof come from (Ansari; Mokhtary, 2019). Let

$$\begin{aligned} e_N(x) &= h(x) - \Pi_N h(x) \\ &\quad + f\left(x, \int_0^1 \kappa(x,y) g\big(y, u(y)\big) dy\right) - \Pi_N f\left(x, \int_0^1 \Pi_N \kappa(x,y) g\big(y, u(y)\big) dy\right) \\ &= (FKu)(x) - (FKu_N)(x) + J_0 + J_1 + J_2 \end{aligned}$$

and consider the difference

$$J_0 = h(x) - \Pi_N h(x),$$

with

$$J_1 = f\left(x, \int_0^1 \kappa(x,y) g\big(y, u_N(y)\big) dy\right) - \Pi_N f\left(x, \int_0^1 \kappa(x,y) g\big(y, u_N(y)\big) dy\right),$$

and

$$J_2 = \Pi_N\left(f\left(x, \int_0^1 \kappa(x,y) g\big(y, u_N(y)\big) dy\right) - f\left(x, \int_0^1 \Pi_N \kappa(x,y) g\big(y, u_N(y)\big) dy\right)\right).$$

Since f and g are Lipschitz in the second variable, from conditions (A4) and (A5), we have

$$\begin{aligned} |e_N(x)| &\leq |(FKu)(x) - (FKu_N)(x)| + |J_0| + |J_1| + |J_2| \\ &\leq ML\left|\int_0^1 \kappa(x,y) e_N(y) dy\right| + |J_0| + |J_1| + |J_2| \\ &\leq ML\left(\int_0^1 |\kappa(x,y)|^2 \, dy\right)^{1/2} \left(\int_0^1 |e_N(y)|^2 \, dy\right)^{1/2} + |J_0| + |J_1| + |J_2| \\ &\leq MLM_1(x)\left(\int_0^1 |e_N(y)|^2 \, dy\right)^{1/2} + |J_0| + |J_1| + |J_2|, \end{aligned}$$

where the last inequality was obtained using condition (A2).

Applying the L^2 norm for both members of the above inequality we have

$$\|e_N\|_2 \leq \Gamma\|e_N\|_2 + \|J_0\|_2 + \|J_1\|_2 + \|J_2\|_2,$$

whence we have

$$(1 - \Gamma)\|e_N\|_2 \leq \|J_0\|_2 + \|J_1\|_2 + \|J_2\|_2.$$

By (2.1), $\Gamma < 1$, thus we obtain $C_1 > 0$ such that

$$\|e_N\|_2 \leq C_1 \left(\|J_0\|_2 + \|J_1\|_2 + \|J_2\|_2\right). \tag{4.2}$$

Employing inequality (4.1), an estimate for J_0 is given below

$$\|J_0\|_2 \leq N^{-k_1}|h|_{k_1}, \tag{4.3}$$

where $K_2 >$ exists by virtue of smoothness of

Using again , we get

$$\|J_1\|_2 \leq N^{-k_2}|(FKu_N)(x)|_{k_2}, \tag{4.4}$$

where $k_2 > 0$ exists due to smoothness of f.

Since Π_N is an orthogonal projection, then $\|\Pi_N\|_2 = 1$. Moreover, as f and g are Lipschitz in the second variable and , we get

$$\begin{aligned}
\|J_2\|_2 \quad &\leq \left\| f\left(x, \int_0^1 \kappa(x,y) g\big(y, u_N(y)\big) dy\right) - f\left(x, \int_0^1 \Pi_N \kappa(x,y) g\big(y, u_N(y)\big) dy\right)\right\|_2 \\
&\leq M \left\| \int_0^1 |\kappa(x,y) - \Pi_N \kappa(x,y)| \left| g\big(y, u_N(y)\big) - g(0,0)\right| dy \right\|_2 \\
&\leq ML \left\| \int_0^1 |\kappa(x,y) - \Pi_N \kappa(x,y)| \, |u_N(y)| dy \right\|_2 \qquad\qquad (4.5) \\
&\overset{(*)}{\leq} C \left(\int_0^1 \left(\int_0^1 \big(\kappa(x,y) - \Pi_N \kappa(x,y)\big)^2 dy \right) \left(\int_0^1 \big(u_N(y)\big)^2 dy \right) dx \right)^{1/2} \\
&\leq CN^{-k_3} |\kappa|_{k_3} \|u_N\|_2,
\end{aligned}$$

where $k_3 > 0$ exists thanks to smoothness of k, and $C = ML$ as in (2.1). We note that to obtain inequality $(*)$ it was used Hölder's inequality (Folland, 1999).

Combining (4.3), (4.4) and (4.5) in (4,2) we have

$$\|e_N\|_2 \leq C_1 \big(N^{-k_1}|h|_{k_1} + N^{-k_2}|(FKu_N)(x)|_{k_2} + CN^{-k_3}|\kappa|_{k_3}\|u_N\|_2\big). \tag{4.6}$$

Let $C_2 \geqslant max\{1, C\}$ be a constant such that

$$N^{-k_1}|h|_{k_1} \leq C_2 N^{-k_1}|h|_{k_1}, \tag{4.7}$$

$$N^{-k_2}|(FKu_N)(x)|_{k_2} \leq C_2 N^{-k_2}|(FKu_N)(x)|_{k_2} \tag{4.8}$$

and

$$CN^{-k_3}|\kappa|_{k_3}\|u_N\|_2 \leq C_2 N^{-k_3}|\kappa|_{k_3}\|u_N\|_2. \tag{4.9}$$

Therefore, $D = C_1 C_2$ taking , we conclude, by (4.6), (4.7), (4.8) and (4.9), that

$$\|e_N\|_2 \leq D\left(N^{-k_1}|h|_{k_1} + N^{-k_2}|(FKu_N)(x)|_{k_2} + N^{-k_3}|\kappa|_{k_3}\|u_N\|_2\right),$$

which completes the proof.

5 I NUMERICAL EXAMPLES

For the numerical application, we use Picard iterative process and admit that the convergence is achieved when the stopping criterion has tolerance $tol = 1e - 12$ in L^2 in norm and $tol = 1e - 12$ in L^2 norm and $k_{max} = 1000$.The error was computed using the following relative measures

$$err = \frac{\left\|u - u_N^{(k)}\right\|_2}{\|u\|_2}. \tag{5.1}$$

Here we employ the Chebyshev series and in the computation MATLAB package Chebpack available at the Mathworks website:

https://www.mathworks.com/matlabcentral/fileexchange/32227-chebpack as a stand-alone algorithm for solving nonlinear systems and investigating the performance of the numerical solution.

Example 1. Consider the nonlinear functional-integral equation with exact solution $u(x) = sin(x)$ Take

$$\kappa(x,y) = sin(x+y), \quad f(x,y) = sin(y)$$

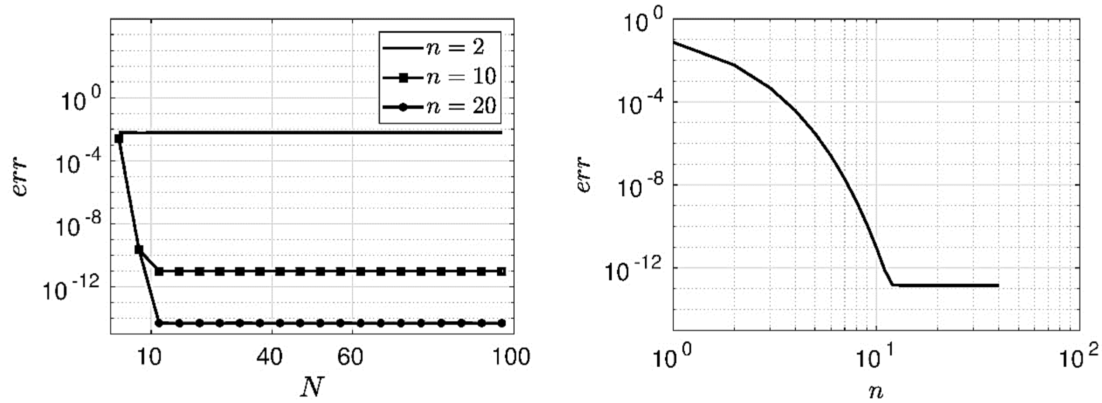
and

$$g(x,y) = \frac{|y|}{10}.$$

The function is defined by

$$h(x) = -\sin\left(\frac{(\cos(x) - \cos(1+x)\sin(1))}{20}\right) + \sin(x).$$

We point out that all of the hypotheses of Theorem 2.1 are satisfied and there exists a unique solution for this equation. Furthermore, the hypotheses of Theorem 4.1 are also satisfied and we can guarantee the convergence for this solution.



(a) (b)

Figure 1. Relative error corresponding to Example 1: (a) of the numerical solution in relation to number of integration points N putting the iterations $n = 2, 10, 20$ using semi-log scale; (b) of the numerical solution on iterations number from to 0 using log-log scale.

Figure 1(a) establishes the minimum number of integration points in terms of error on L^2 norm. We noticed that 10 integration points are enough to have convergence as the number of iterations increases. It suggests us to conclude that few integration points are sufficient to preserve the convergence of the method when the hypotheses of Theorem 4.1 are satisfied.

Figure 1(b) depicts the exponential decay of the relative error considering a variation in the iterations number n, from 1 to 40. Recalling Fig. 1(a), note that for $n > 10$ this decay becomes almost imperceptible.

Example 2. Let us consider the nonlinear functional-integral equation:

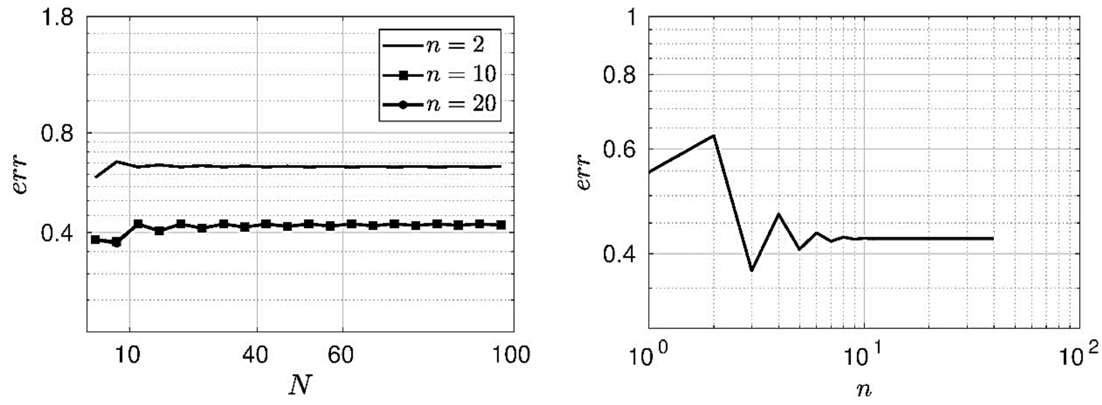
$$u(x) = x^2 - \sin\left(\frac{51+8x}{192}\right) - \sin\left(\int_0^1 \kappa(x,y)u(y)\,dy\right),$$

with kernel

$$\kappa(x,y) = \begin{cases} 1+x-y & 0 < y \leq 1/2, \\ -1 & \frac{1}{2} < y \leq 1. \end{cases}$$

Here the exact solution in is given by

$$u(x) = x^2.$$



(a) (b)

Figure 2. Relative error corresponding to Example 2: (a) of the numerical solution in relation to number of integration points N putting the iterations $n = 2, 10, 20$ using semi-log scale; (b) of the numerical solution on iterations number n, from 1 to 40 using log-log scale.

In this example we can see that the kernel is discontinuous, contrary to the hypotheses of Theorem 4.1. Consequently, disagreeing with the previous experiment, Fig. 2(a) shows that the increase in the number of integration points does not contribute satisfactorily to the convergence of the method. This is confirmed by Fig. 2(b), whose exponential convergence is violated.

CONCLUSION

This paper proposes a numerical method for Fredholm functional-integral equations based on Chebyshev basis functions considering the Tau method. The hypotheses of Theorem 2.1 are sufficient for conclude that the functional-integral equation (1.1) has a unique solution in $L^2([0,1])$ space. Moreover, Example 1 shows that assumptions from Theorem 4.1 were satisfactory to ensure the convergence of the Tau method for a moderate number of iterations. On the other hand, through Example 2 we can infer that piecewise continuous kernel causes damage to exponential convergence, restricting the results to a sufficiently regular kernel.

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ÍNDICE REMISSIVO

A

B

C

D

E

V

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